1 Introduction

Suppose that $H$ is a subgroup of a finite group $G$ and that $G$ is generated by the conjugates of $H$. In this paper we consider the question:

\textit{how many conjugates of $H$ are needed to generate $G$?}

In order to answer this question we must study chains of subgroups that start with $H$ and end with $G$. The \textit{chain length of $H$ in $G$} is defined by

$$ cl_G(H) = \max \{ n \in \mathbb{N} \mid \text{there is a chain } H = M_0 < M_1 < \ldots < M_n = G \}. $$

We are using the notation $A < B$ to mean that $A$ is a proper subgroup of $B$. It is almost trivial to see that $G$ can be generated by a set consisting of at most $cl_G(H) + 1$ conjugates of $H$. Indeed, suppose that every set consisting of at most $cl_G(H)$ generates a proper subgroup of $G$. Then using the fact that $G$ is generated by the conjugates of $H$ we can construct a chain

$$ H = H_0 < \langle H_0, H_1 \rangle < \langle H_0, H_1, H_2 \rangle < \ldots < \langle H_0, \ldots, H_{cl_G(H)} \rangle \leq G $$

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where each $H_i$ is a conjugate of $H$. This chain has length $\text{cl}_G(H) + 1$ so the last inclusion cannot be proper. Thus $G$ can be generated by $\text{cl}_G(H) + 1$ conjugates of $H$.

If $H$ is a maximal subgroup of $G$ then $\text{cl}_G(H) = 1$ and $\text{cl}_G(H) + 1$ conjugates of $H$ are needed to generate $G$. However, if $H$ is not maximal in $G$ the situation is different. We will prove that fewer than $\text{cl}_G(H) + 1$ conjugates of $H$ are required to generate $G$ unless the structure of $G$ is very restricted. Our main theorem is:

**Theorem A** Let $H$ be a subgroup of a finite group $G$ such that $G = \langle H^G \rangle$ and $\text{cl}_G(H) \geq 2$. Then every set consisting of at most $\text{cl}_G(H)$ conjugates of $H$ generates a proper subgroup of $G$ if and only if $G/H_G$ has the following structure.

(i) $G/H_G$ is a Frobenius group with cyclic complement $H/H_G$;

(ii) There is a prime $p$ such that the Frobenius kernel of $G/H_G$ is an elementary abelian $p$-group;

(iii) Considered as a $\text{GF}(p)H$-module, the kernel of $G/H_G$ is the direct sum of $\text{cl}_G(H)$ irreducible, nontrivial isomorphic $\text{GF}(p)H$-modules.

Theorem A has two immediate corollaries, the latter of which is reminiscent of Baer’s Criterion for a group to have a nontrivial normal $p$-subgroup.

**Corollary B** Let $H$ be a subgroup of a finite group $G$ such that $G = \langle H^G \rangle$, $\text{cl}_G(H) \geq 2$ and $G/H_G$ is insoluble. Then $G$ can be generated by a set consisting of at most $\text{cl}_G(H)$ conjugates of $H$.

**Corollary C** Let $H$ be a subgroup of a finite group $G$ such that $\text{cl}_G(H) \geq 2$ and $G/H_G$ is insoluble. Suppose that every set consisting of at most $\text{cl}_G(H)$ conjugates of $H$ generates a proper subgroup of $G$. Then $G \neq \langle H^G \rangle$.

Corollary B raises the following question for further study:

suppose that $H$ is a subgroup of a finite group $G$ such that $G = \langle H^G \rangle$, $\text{cl}_G(H) \geq 3$, $G/H_G$ is insoluble and every set consisting of at most $\text{cl}_G(H) - 1$ conjugates of $H$ generates a proper subgroup of $G$. What can be said about the structure of $G$?
This situation can happen if \( H \cong \mathbb{Z}_2 \) and \( \operatorname{cl}_G(H) = 3 \). This occurs in \( A_5 \).

The proof of Theorem A uses ideas similar to those used by the author in \([2]\) and \([3]\).

## 2 Notation and Quoted Results

Throughout this paper, group means finite group, \( A \leq B \) means \( A \) is a subgroup of \( B \), \( A < B \) means \( A \) is a proper subgroup of \( B \) and \( A \trianglelefteq B \) means \( A \) is a normal subgroup of \( B \). If \( A \) and \( B \) are subgroups of a group \( G \) then

\[
A^B = \{ a^b \mid b \in B \}.
\]

Note that \( A^B \) is a set of conjugates of \( A \), not the subgroup generated by those conjugates, which we shall denote by

\[
\langle A^B \rangle.
\]

If \( H < G \) then

\[
H_G = \bigcap \{ H^g \mid g \in G \} = \text{the core of } H \text{ in } G.
\]

Notice that \( H_G \) is the largest subgroup of \( H \) that is normal in \( G \). If \( H \leq G \) then \( H^G = H - \{1\} \) and

\[
\operatorname{cl}_G(H) = \max \{ n \in \mathbb{N} \mid \text{there is a chain } H = H_0 < H_1 < \ldots < H_n = G \}.
\]

A subgroup \( H \) of a group \( G \) is a Frobenius complement in \( G \) if \( 1 < H < G \) and \( H \cap H^g = 1 \) for all \( g \in G - H \). \( G \) is a Frobenius group if it has a Frobenius complement. If \( H \) is a Frobenius complement in a group \( G \) then the set

\[
K = G - \cup \{ H^{2g} \mid g \in G \}
\]

is called the Frobenius kernel of \( G \).

**Frobenius’ Theorem** Let \( G \) be a Frobenius group with complement \( H \). Then the Frobenius kernel \( K \) of \( G \) is a normal subgroup of \( G \) with order coprime to the order of \( H \), \( G = HK \) and \( H \cap K = 1 \).

[1, (35.24) page 191]

**Thompson’s Theorem** Frobenius kernels are nilpotent.

[1, (40.8) page 207]
3 Preliminary Lemmas

Lemma 3.1 Let $1 < V \trianglelefteq G, H \leq G$ and suppose that $V \cap H = 1$. Set $G^* = G/V$. Then,

(i) $\text{cl}_{G^*}(H^*) \leq \text{cl}_G(H) - 1$;

(ii) If $H$ does not normalize any nontrivial proper subgroups of $V$ then $\text{cl}_{G^*}(H^*) = \text{cl}_G(H) - 1$.

Proof Suppose $H^* = H_0^* < H_1^* < \ldots < H_r^* = G^*$ is a chain of subgroups in $G^*$. Let $H_i$ be the inverse image of $H_i^*$ in $G$. Then $H < HV = H_0 < H_1 < \ldots < H_r = G$ is a chain of subgroups in $G$ of length $r+1$. Thus $r + 1 \leq \text{cl}_G(H)$ and hence $\text{cl}_{G^*}(H^*) \leq \text{cl}_G(H) - 1$.

Now suppose that $H$ does not normalize any nontrivial proper subgroups of $V$. Let $c = \text{cl}_G(H)$. Then there exists a chain $H = H_0 < H_1 < \ldots < H_c = G$ of length $c$ in $G$. Thus we have

\[ H^* = H_0^* \leq H_1^* \leq \ldots \leq H_r^* = G^* \quad (1) \]

and since $\text{cl}_{G^*}(H^*) \leq c - 1$, there exists $i$ such that $H_i^* = H_{i-1}^*$. Choose $i$ minimal with this property. Then $H_iV = H_{i+1}^*V$ so $H_i < H_{i+1} \leq H_iV$ whence $H_{i+1} = H_i(H_{i+1}\cap V)$. Now $H_i \neq H_{i+1}$ so $H_{i+1}\cap V \neq 1$ and as $H \leq H_{i+1}$, we have that $H_{i+1}\cap V = V$ so $V \leq H_{i+1}$. Since $H_{i+1} < \ldots < H_c$ this implies that $H_{i+1}^* < H_{i+2}^* < \ldots < H_c^*$ so all except one of the containments in (1) is strict. We deduce that $\text{cl}_{G^*}(H^*) = c - 1$.

Lemma 3.2 Let $1 < H < G$ and suppose that $N_G(P) \leq H$ whenever $1 < P \leq H$. Then $H$ is a Frobenius complement in $G$.

Proof This is well known.

Lemma 3.3 Let $H$ be a Frobenius complement in a group $G$. Let $g \in G$. Then $\langle H, g \rangle = \langle H, H^g \rangle = H\langle g^H \rangle$. 

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Proof We may assume that \( g \not\in H \). Let \( M = \langle H, H^g \rangle \). Then \( 1 < H < M \) so \( H \) is a Frobenius complement in \( M \), as is \( H^g \). Let \( K \) be the Frobenius kernel of \( M \). Then \( K \) is a nilpotent Hall subgroup of \( M \) and \( H \) and \( H^g \) are complements to \( K \) in \( M \). The Schur-Zassenhaus Theorem implies that there exists \( m \in M \) such that \( H^g = H^m \). Then \( H^{gm^{-1}} = H \) forcing \( gm^{-1} \in H \leq M \) whence \( g \in M \). We deduce that \( \langle H, g \rangle = \langle H, H^g \rangle \). Clearly \( \langle H, g \rangle = H \langle g^H \rangle \).

Lemma 3.4 Let \( p \) be a prime; \( H \) a cyclic \( p' \)-group; \( X \) a faithful irreducible \( GF(p)H \)-module; \( n \) a natural number; \( V \) a \( GF(p) \)\( H \)-module that is the direct sum of \( n \) submodules each isomorphic to \( X \) and let \( G = HV \), the semidirect product of \( V \) considered as an abelian group with \( H \). Then \( G = \langle H^G \rangle, \text{cl}_G(H) = n \) and every set consisting of at most \( n \) conjugates of \( H \) generates a proper subgroup of \( G \).

Proof Let \( V = U_1 \oplus \ldots \oplus U_n \) where each \( U_i \) is a submodule isomorphic to \( X \). Since each \( U_i \) is nontrivial and irreducible we have \( U_i = [U_i, H] \) and so \( V = [V, H] \leq \langle H^G \rangle \). Now \( H \leq \langle H^G \rangle \) hence \( G = \langle H^G \rangle \).

Since \( X \) is irreducible, we know that \( \text{End}_H(X) \) is a field. This field contains \( H \) since \( H \) is abelian. Let \( F \) be the subfield of \( \text{End}_H(X) \) generated by \( H \). Then \( X \) is an \( F \)-vectorspace. The irreducibility of \( X \) implies that \( \dim_F(X) = 1 \). Also, \( V \) and each \( U_i \) are \( F \)-vectorspaces, \( \dim_F(U_i) = 1 \) and \( \dim_F(V) = n \).

Suppose that \( H = H_0 < H_1 < \ldots < H_r = G \) is a chain of subgroups. Since \( G = HV \), for each \( i \) we have \( H_i = HW_i \) where \( W_i = H_i \cap V \). Each \( W_i \) is \( H \)-invariant and hence an \( F \)-vectorspace. Moreover, \( 0 = W_0 < W_1 < \ldots < W_r = V \) so as \( \dim_F(V) = n \) we deduce that \( r \leq n \). Thus \( \text{cl}_G(H) \leq n \).

The chain \( H < HV_1 < H(U_1 \oplus U_2) < \ldots < H(U_1 \oplus \ldots \oplus U_n) = G \) has length \( n \) so \( \text{cl}_G(H) = n \).

Let \( g_1, \ldots, g_n \in G \). We will show that \( \langle H^{g_1}, \ldots, H^{g_n} \rangle \neq G \). By conjugating by \( g_1^{-1} \) there is no loss of generality in supposing that \( g_1 = 1 \). For each \( i \), choose \( h_i \in H \) and \( v_i \in V \) such that \( g_i = h_i v_i \). Then

\[
\langle H^{g_1}, \ldots, H^{g_n} \rangle = \langle H, H^{v_2}, \ldots, H^{v_n} \rangle \leq \langle H, v_2, \ldots, v_n \rangle = HW
\]

where

\[
W = \langle \langle v_2, \ldots, v_n \rangle^H \rangle.
\]
Now $W$ is a subgroup of $V$ that is $H$-invariant. Hence it is $F$-invariant and therefore an $F$-subspace of $V$. Considered as an $F$-subspace, we see that every member of $W$ can be written as a linear combination of the $v_i$ with coefficients in $H$. Thus $\{v_2, \ldots, v_n\}$ is an $F$-spanning set for $W$. Hence $\dim_F(W) \leq n - 1 < \dim_F(V)$. We deduce that $W \neq V$ and finally that $\langle H^{g_1}, \ldots, H^{g_n} \rangle \neq G$.

4 The Minimal Counterexample

Throughout the remainder of this paper, we assume the following hypothesis, which is satisfied by a minimal counterexample to Theorem A.

Hypothesis

(i) $H$ is a subgroup of $G$ and $n = \text{cl}_G(H)$;

(ii) $G = \langle H^G \rangle$ and $n \geq 2$;

(iii) Every set consisting of at most $n$ conjugates of $H$ generates a proper subgroup of $G$;

(iv) $H_G = 1$;

(v) If $Y < X$ with $|X| < |G|$, $X = \langle Y^X \rangle$, $\text{cl}_X(Y) \geq 2$ and if every set consisting of at most $\text{cl}_X(Y)$ conjugates of $Y$ generates a proper subgroup of $X$ then $X$ has the structure given in the conclusion of Theorem A.

We will eventually prove that $G$ has the structure given in the conclusion of Theorem A. This, together with Lemma 3.4 will prove Theorem A.

Definition An $r$-tuple $(H_1, \ldots, H_r)$ of conjugates of $H$ is good if

$$H_i \nleq N_G(H_1) \text{ for all } i, 2 \leq i \leq r$$

and

$$H_1 < \langle H_1, H_2 \rangle < \langle H_1, H_2, H_3 \rangle < \ldots < \langle H_1, H_2, H_3, \ldots, H_r \rangle.$$
Lemma 4.1 Let \((H_1, \ldots, H_r)\) be a good \(r\)-tuple of conjugates of \(H\), with \(r < n + 1\). Then there exist \(H_{r+1}, \ldots, H_{n+1} \in H^G\) such that \((H_1, \ldots, H_{n+1})\) is good.

**Proof** Since \(r \leq n\) we have \(H_1 \leq \langle H_1, \ldots, H_r \rangle < G = \langle H^G \rangle\) so there exists a conjugate \(D\) of \(H_1\) that does not normalize \(\langle H_1, \ldots, H_r \rangle\). In particular, \(D \not\leq \langle H_1, \ldots, H_r \rangle\). In the case that \(D\) does not normalize \(H_1\) we observe that since \(D\) does not normalize \(\langle H_1, \ldots, H_r \rangle\), there exists \(d \in D\) such that \(H_i^d \not\leq \langle H_1, \ldots, H_r \rangle\). Since \((H_1, \ldots, H_r)\) is good it follows that \(H_i\) does not normalize \(H_1\), but as \(d\) does normalize \(H_1\) we see that \(H_i^d\) cannot normalize \(H_1\). Let \(H_{r+1} = H_i^d\). In both cases \((H_1, \ldots, H_{n+1})\) is good.

Repeated application of the above procedure completes the proof of this lemma.

Lemma 4.2 Let \((H_1, \ldots, H_{n+1})\) be a good \((n + 1)\)-tuple of conjugates of \(H\). Then the following hold.

(i) \(G = \langle H_1, \ldots, H_{n+1} \rangle\);

(ii) If \(\sigma\) is any permutation of \(\{2, \ldots, n+1\}\) then \((H_1, H_{2\sigma}, \ldots, H_{(n+1)\sigma})\) is good;

(iii) \(\langle H_1, \ldots, H_i \rangle\) is a maximal subgroup of \(\langle H_1, \ldots, H_{i+1} \rangle\) for each \(i, 1 \leq i \leq n\);

(iv) If \(Y = \langle H_1, \ldots, H_n \rangle\) then \(\text{cl}_Y(H_1) = n - 1\).

**Proof** Trivial.

Lemma 4.3 Let \((H_1, \ldots, H_r)\) be a good \(r\)-tuple of conjugates of \(H\), then

\[\langle H_1, \ldots, H_r \rangle = \langle H_1^{\langle H_1, \ldots, H_r \rangle} \rangle.\]

**Proof** Let \(2 \leq i \leq r\). Then \((H_1, H_i)\) is a good 2-tuple. Using Lemmas 4.1 and 4.2(ii),(iii) we see that \(H_1\) is a maximal subgroup of \(\langle H_1, H_i \rangle\).
Since $H_i$ does not normalize $H_1$, we have $\langle H_1, H_i \rangle = \langle H_1^i, H_i \rangle$ hence $H_i \leq \langle H_1^i, \ldots, H_r \rangle$ and the result follows.

For the remainder of this paper we fix the following notation:

Let $H_1 = H$ and let $H_2, \ldots, H_{n+1}$ be chosen in accordance with Lemma 4.1, so that $(H_1, \ldots, H_{n+1})$ is good.

Let

\[
\begin{align*}
M &= \langle H_1, \ldots, H_{n-1}, H_n \rangle, \\
L &= \langle H_1, \ldots, H_{n-1}, H_{n+1} \rangle, \\
D &= \langle H_1, \ldots, H_{n-1} \rangle, \text{ and} \\
N &= N_G(H).
\end{align*}
\]

Lemma 4.4  
(i) $(H_1, \ldots, H_{n-1}, H_n)$ and $(H_1, \ldots, H_{n-1}, H_{n+1})$ are good;  
(ii) $M$ and $L$ are distinct maximal subgroups of $G$;  
(iii) $D = M \cap L$ and $D$ is a maximal subgroup of $M$ and of $L$.

Proof  Trivial.

5  The Case $n = 2$

In this section we assume that $n = 2$. In particular, we have $D = H$.

Lemma 5.1 Let $k \in G$ be such that $H^k \neq H$. Let $K = \langle H, H^k \rangle$. Let $g \in G$. Then $H^g \leq K$ or $H^g \cap K \leq H_K$.

Proof  Since $G$ cannot be generated by two conjugates of $H$ we have $H < K < G$. Since $n = 2$ it follows that $H$ is maximal in $K$ and that $K$ is maximal in $G$.

Let $x \in G$ be such that $H^x \not\leq K$. Let $E = H^x \cap K$. If $E \not\leq H$ then $K = \langle H, E \rangle$ whence $K \leq \langle H, H^x \rangle$. Since $G$ cannot be generated by
two conjugates of $H$, and as $K$ is a maximal subgroup of $G$, we see that $K = \langle H, H^x \rangle$ hence $H^x \leq K$, a contradiction. Thus $H^x \cap K \leq H$.

Now suppose that $H^g \not\leq K$. Then $H^{g^k} \not\leq K$ for all $k \in K$ so the previous paragraph implies $H^{g^k} \cap K \leq H$ for all $k \in K$. Thus $H^g \cap K \leq \bigcap \{H^{k^{-1}} | k \in K\} = H_K$.

**Lemma 5.2** $H_M = H_L = 1$.

**Proof** Since $M = \langle H, H_2 \rangle \not\leq N$, we may choose $m \in M$ such that $H^m \neq H$. By Lemma 4.4(iii) we have $M \cap L = H$ so $H^m \not\leq L$. Now $L = \langle H, H_3 \rangle$ so Lemma 5.1 implies that $H^m \cap L \leq H_L$. Then $H_M = H^m_M \leq H^m \cap H \leq H^m \cap L \leq H_L$ so $H_M \leq H_L$.

The preceding argument with $L$ in place of $M$ implies that $H_L \leq H_M$, hence $H_M = H_L \leq \langle M, L \rangle$. Lemma 4.4(ii) implies that $\langle M, L \rangle = G$ so $H_M$ is a normal subgroup of $G$ contained in $H$. Thus $H_M \leq H_G = 1$. Hence result.

**Lemma 5.3** Let $1 < P \leq H$. Then $N_G(P) \leq N$.

**Proof** Let $g \in N_G(P)$. Then $1 < P \leq H^g \cap M$ and as $M = \langle H, H_2 \rangle$, Lemmas 5.1 and 5.2 imply that $H^g \leq M$. Similarly $H^g \leq L$ whence $H^g \leq M \cap L = H$. We deduce that $N_G(P) \leq N$.

**Lemma 5.4** Let $g \in G$ and suppose that $H^g \leq N$. Then $g \in N$.

**Proof** Assume false. Then $H^g \neq H$ so $H < N < G$ and since $\text{cl}_G(H) = 2$ we see that $H$ is a maximal subgroup of $N$ and that $N$ is a maximal subgroup of $G$. Since $H \not\leq N$ it follows that $N = HH^g$ and as $g \not\in N$ we also have $G = \langle N, g \rangle$.

Next we consider the subgroup $N^{g^{-1}}$. The factorization $N^{g^{-1}} = H^{g^{-1}}H$ implies that $H$ is not a Hall subgroup of $N^{g^{-1}}$. Thus there exists a Sylow subgroup $P$ of $H$ such that $N_{N^{g^{-1}}}(P) \not\leq H$. Using the previous lemma, we see that $H < \langle H, N_{N^{g^{-1}}}(P) \rangle \leq N \cap N^{g^{-1}} \leq N$. The maximality of $H$ in $N$ forces $N = N^{g^{-1}}$. Since $G = \langle N, g \rangle$ we have that $H \not\leq N \leq G$ whence $G = \langle H^G \rangle \leq N < G$, a contradiction. We deduce that $g \in N$. 

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Lemma 5.5 G is a Frobenius group with complement H

Proof First we prove that N is a Frobenius complement in G. Suppose not. Then by Lemma 3.2 there exists P such that 1 < P ≤ N and NG(P) ≤ N. Choose g ∈ NG(P) - N and set T = ⟨H, H^g⟩. Then H < T < G so as clG(H) = 2, we see that H is maximal in T and that T is maximal in G.

Observe that P normalizes both H and H^g so P ≤ NG(T). Since H is not contained in any proper normal subgroup of G, we have that NG(T) = T, whence P ≤ T. We have H ≤ T ∩ N ≤ T so either H = T ∩ N or T ∩ N = T. Lemma 5.4 and the fact that g /∈ N imply that H^g ≤ N so as H^g < T, we see that T ∩ N ≠ T. Thus H = T ∩ N, and in particular, P ≤ H. Lemma 5.3 implies that NG(P) ≤ N, a contradiction. We deduce that N is a Frobenius complement in G.

Frobenius' Theorem implies that G contains a normal subgroup K such that G = NK and N ∩ K = 1. But H ≤ N whence HK ≤ G. So as G = ⟨H^G⟩, we have NK = HK. Since N ∩ K = 1, we see that N = H, hence result.

The previous lemmas together with Frobenius’ Theorem imply that G contains a normal subgroup K, the Frobenius kernel of G, such that

G = HK and H ∩ K = 1.

Lemma 5.6 There is a prime p such that K is an elementary abelian p-group. Considered as a GF(p)H-module, K is the direct sum of two irreducible, nontrivial isomorphic GF(p)H-modules. Moreover, H is cyclic.

Proof Let a ∈ K and let U = ⟨a^H⟩. Lemma 3.3 implies that ⟨H, H^a⟩ = HU so using the hypothesis that G cannot be generated by two conjugates of H we see that H < HU < G. Since clG(H) = 2 it follows that H is maximal in HU and that HU is maximal in G. In particular, U is both a maximal and a minimal H-invariant subgroup of K.

Thompson’s Theorem implies that K is nilpotent, hence NK(U) > U and it follows that U ≤ K. Now U is nilpotent and is a minimal H-invariant subgroup of K so we see that U is an elementary abelian p-group for some prime p on which H acts irreducibly.

Now choose b ∈ K - U and let W = ⟨b^H⟩. Then again, W ≤ K and W is an elementary abelian q-group for some prime q on which H acts irreducibly.
If \( p \neq q \) then as \( K \) is nilpotent we have \([a, b] = 1\) so \( ab \) is an element with order \( pq \). However, the previous argument shows that every element of \( K^\sharp \) has prime order, a contradiction. Thus \( p = q \). Since \( U \) and \( W \) are minimal \( H \)-invariant subgroups of \( K \) we have \( U \cap W = 1 \) and as they are maximal \( H \)-invariant subgroups of \( K \) we have \( (U, W) = K \). It follows that \( K = U \times W \). Thus \( K \) is an elementary abelian \( p \)-group.

We are left with the task of proving that \( U \) is isomorphic to \( W \) as an \( H \)-module and that \( H \) is cyclic. Choose \( u \in U^\sharp \) and \( w \in W^\sharp \). Set \( t = uw \) and \( T = \langle t^H \rangle \). The projection maps

\[
\pi_U : T \longrightarrow U \quad \text{and} \quad \pi_W : T \longrightarrow W
\]

are \( H \)-module homomorphisms. Since \( t\pi_U = u \) and \( t\pi_W = w \) it follows that \( \pi_U \) and \( \pi_W \) are nontrivial. As previously we have that \( T \) is an irreducible \( H \)-module, as are \( U \) and \( W \). It follows that \( \pi_U \) and \( \pi_W \) are \( H \)-module isomorphisms. Also, \( \pi_U^{-1}\pi_W \) is an \( H \)-module isomorphism \( U \rightarrow W \) that maps \( u \) to \( w \).

By keeping \( w \) fixed and letting \( u \) range over \( U^\sharp \) we see that \( \text{End}_H(U) \) acts transitively on \( U^\sharp \). Let \( E = \text{End}_H(U) \). Then \( U \) is an irreducible \( E \)-module and so \( \text{End}_E(U) \) is a field. Now \( H \subseteq \text{End}_E(U) \) and as the multiplicative group of a finite field is cyclic, we deduce that \( H \) is cyclic.

\section{The Case \( n \geq 3 \)}

In this section we assume that \( n \geq 3 \).

\textbf{Lemma 6.1} \hspace{1em} (i) \( M \) and \( L \) are Frobenius groups with cyclic complement \( H \).

(ii) There is a prime \( p \) such that the Frobenius kernels of \( M \) and \( L \) are elementary abelian \( p \)-groups.

(iii) Considered as a \( GF(p)H \)-module, the kernel of \( M \) is the direct sum of \( n - 1 \) nontrivial, irreducible isomorphic \( GF(p)H \)-modules.

(iv) The kernel of \( L \) satisfies (iii) also.
A there exist $A$ Since $G = \langle M, H_{n+1} \rangle$ and since $G$ cannot be generated by $n$ conjugates of $H$, we see that $M$ cannot be generated by $n - 1$ conjugates of $H$. Then $M/H_M$ is a Frobenius group with complement $H/H_M$. Let $\overline{M} = M/H_M$. Since $n \geq 3$ we have that $D > H$ so $\overline{D}$ is a Frobenius group with complement $\overline{H}$. In particular, the only subgroup of $\overline{H}$ that is normal in $\overline{D}$ is 1. Now $\overline{H}_L \leq \overline{D}$, whence $\overline{H}_L = 1$ forcing $H_L \leq H_M$. A similar argument with the roles of $M$ and $L$ interchanged proves $H_M \leq H_L$. Thus $H_L = H_M \leq \langle M, L \rangle = G$ and as $H_G = 1$, we deduce that $H_L = H_M = 1$.

We have that $M$ is a Frobenius group with cyclic complement $H$ and whose kernel is an elementary abelian $p$-group, which considered as a $\text{GF}(p)H$-module is the direct sum of $n - 1$ irreducible, nontrivial isomorphic $\text{GF}(p)H$-modules. Similarly, there is a prime $q$ such that $L$ is a Frobenius group with complement $H$ whose kernel is an elementary abelian $q$-group etc. Since $n \geq 3$, we have $M \cap L = D > H$ whence $p = q$.

**Lemma 6.2** $G$ is a Frobenius group with complement $H$ and kernel $O_p(G)$. Moreover, $O_p(G) = \langle O_p(M), O_p(L) \rangle$.

**Proof** Lemma 6.1 implies that $M = H O_p(M)$. Since $H \leq D$ we have $D = H(D \cap O_p(M))$ and as $H$ has order coprime to $p$ we see that $D \cap O_p(M) = O_p(D)$. By Lemma 6.1, $O_p(M)$ is abelian so $O_p(D) \leq O_p(M)$ and hence $O_p(D) \leq M$. Similarly, $O_p(D) \leq L$ and it follows that $O_p(D) \leq G$. As $n \geq 3$, we have $H < D$, but as $D = H O_p(D)$ it follows that $O_p(D) \neq 1$.

Let $V$ be a minimal normal subgroup of $G$ that is contained in $O_p(D)$ and set $G^* = G/V$. Since $O_p(M)$ and $O_p(L)$ centralize $O_p(D)$, we have the factorization $G = HC_G(V)$ so $H$ normalizes no nontrivial proper subgroup of $V$. Moreover, $H \cap V = 1$ so Lemma 3.1 implies that $\text{cl}_{G^*}(H^*) = n - 1$. Since $G = \langle H^G \rangle$ we also have $G^* = \langle H^{*G^*} \rangle$.

Suppose that $G^*$ can be generated by $n - 1$ conjugates of $H^*$. Then there exist $A_1, \ldots, A_{n-1} \in H^G$ such that $G = V \langle A_1, \ldots, A_{n-1} \rangle$. Since $V$ is an abelian minimal normal subgroup of $G$ we see that $\langle A_1, \ldots, A_{n-1} \rangle$ is a maximal subgroup of $G$. Since $G = \langle H^G \rangle$, there exists $A_n \in H^G$ such that $A_n \not\leq \langle A_1, \ldots, A_{n-1} \rangle$. Then $G = \langle A_1, \ldots, A_n \rangle$, contrary to hypothesis. Thus $G^*$ cannot be generated by $n - 1$ conjugates of $H^*$.
Next we show that $H^*_G = 1$. Let $E$ be the inverse image of $H^*_G$ in $G$. Then $V \leq E = V(H \cap E) \leq G$ and $E \leq M$. Using a Frattini Argument and the Schur-Zassenhaus Theorem, we have that $M = EN_M(H \cap E)$ and then $M = VN_M(H \cap E)$. But $VH \leq D < M$ so as $H$ is a Frobenius complement in $M$ we must have $H \cap E = 1$. Then $E = V$ and $H^*_G = 1$.

The previous two paragraphs together with the hypothesis on $G$ imply that $G^*$ is a Frobenius group with complement $H^*$ and kernel $O_p(G^*)$ for some prime $q$. Now $M = HO_p(M)$ and as $O_p(M) > V$, we see that $|M^* : H^*|$ is divisible by $p$, so as $G^* = H^*O_p(G^*)$ we have $p = q$. Thus $G = HO_p(G)$ and as $H$ has order coprime to $p$, it follows that $\langle O_p(M), O_p(L) \rangle \leq O_p(G)$. Since $M$ is a maximal subgroup of $G$, we see that $O_p(M)$ is a maximal $H$-invariant subgroup of $O_p(G)$, whence $\langle O_p(M), O_p(L) \rangle = O_p(G)$.

Let $g \in G$ and suppose that $H \cap H^0 \neq 1$. Now $H \cap V = 1$ hence $H^* \cap H^0 \neq 1$ so the fact that $H^*$ is a Frobenius complement in $G^*$ forces $g^* \in H^*$. Then $g \in HV \leq M$ and since $H$ is a Frobenius complement in $M$, we have $g \in H$. We deduce that $G$ is a Frobenius group with complement $H$. Since $H \cap O_p(G) = 1$ and as $G = HO_p(G)$, it follows that $O_p(G)$ is the Frobenius kernel of $G$.

**Lemma 6.3** $O_p(G)$ is an elementary abelian $p$-group that, considered as a $GF(p)H$-module is, the direct sum of $n$ nontrivial, irreducible isomorphic $GF(p)H$-submodules.

**Proof** Let $m \in O_p(M) - O_p(D)$ and $l \in O_p(L) - O_p(D)$. Then $H \neq H^m, m \in \langle H, H^m \rangle \leq M, H \neq H^l$ and $l \in \langle H, H^l \rangle \leq L$. Now $O_p(D) = O_p(M) \cap O_p(L)$ so as $l \notin M$ and $l \in \langle H, H^l \rangle$, it follows that $H^l \notin \langle H, H^m \rangle$. Since $N_G(H) = H$, the 3-tuple $(H, H^m, H^l)$ is good. By Lemma 4.1, there exist $B_4, \ldots, B_{n+1} \in H^G$ such that $(H, H^m, H^l, B_4, \ldots, B_{n+1})$ is a good $(n+1)$-tuple. All the preceding arguments can be carried out using this $(n+1)$-tuple in place of $(H_1, \ldots, H_{n+1})$. Thus $T = \langle H, H^m, H^l, B_4, \ldots, B_n \rangle$ is a Frobenius group with complement $H$ and abelian kernel $O_p(T) = T \cap O_p(G)$. Since $m, l \in T \cap O_p(G)$, it follows that $m$ and $l$ commute. Since $O_p(M)$ and $O_p(L)$ are elementary abelian, we deduce that $O_p(G)$ is elementary abelian.

Now we regard $O_p(G)$ as a $GF(p)H$-module. Maschke’s Theorem, Lemma 6.1(iii) and the fact that $D$ is maximal in $M$ imply that there exist nontrivial isomorphic irreducible $GF(p)H$-modules $V_1, \ldots, V_{n-1}$ such that
\[ O_p(M) = V_1 \oplus \ldots \oplus V_{n-1} \text{ and } O_p(D) = V_1 \oplus \ldots \oplus V_{n-2}. \] Since \( O_p(M) \nleq O_p(L) \), there exists an irreducible GF(\( p \))-submodule \( W \) of \( O_p(L) \) such that \( W \nleq O_p(M) \). Since \( O_p(M) \) is a maximal GF(\( p \))-submodule of \( O_p(G) \), we have \( O_p(G) = O_p(M) \oplus W \). Lemma 6.1(iv) and the fact that \( n \geq 3 \) imply \( W \cong V_1 \). This completes the proof of Lemma 6.3.

References

