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The level set method for the two-sided max-plus eigenproblem

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Abstract We consider the max-plus analogue of the eigenproblem for matrix pencils, $A \otimes x = \lambda \otimes B \otimes x$. We show that the spectrum of (A, B) (i.e., the set of possible values of λ), which is a finite union of intervals, can be computed in pseudo-polynomial number of operations, by a (pseudo-polynomial) number of calls to an oracle that computes the value of a mean payoff game. The proof relies on the introduction of a spectral function, which we interpret in terms of the least Chebyshev distance between $A \otimes x$ and $\lambda \otimes B \otimes x$. The spectrum is obtained as the zero level set of this function.

Keywords Max algebra · tropical algebra · matrix pencil · min-max function · nonlinear Perron-Frobenius theory · generalized eigenproblem · mean payoff game · discrete event systems

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1 Introduction

1.1 Motivations and general information

Max-plus algebra is the analogue of linear algebra developed over the max-plus semiring which is the set $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$ equipped with the operations of "addition" $a \oplus b := a \vee b = \max(a, b)$ and "multiplication" $a \otimes b := a + b$. The zero of this semiring is $-\infty$, and the unit of this semiring is 0. Note that a^{-1} in max-plus is the same as $-a$ in the conventional notation. The operations of the semiring are extended to matrices and vectors over $\mathbb{R}_{\max}$. That is if $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$ are matrices of compatible sizes with entries from $\mathbb{R}_{\max}$, we write $C = A \vee B$ if $c_{ij} = a_{ij} \vee b_{ij}$ for all i, j and $C = A \otimes B$ if $c_{ij} = \bigvee_k (a_{ik} + b_{kj})$ for all i, j.

We investigate the two-sided eigenproblem in max-plus algebra: for two matrices $A, B \in \mathbb{R}_{\max}^{m \times n}$, find scalars $\lambda \in \mathbb{R}_{\max}$ called *eigenvalues* and vectors $x \in \mathbb{R}_{\max}^n$ called *eigenvectors*, with at least one component not equal to $-\infty$, such that

$$A \otimes x = \lambda \otimes B \otimes x, \tag{1}$$

where the operations have max-plus algebraic sense. In the conventional notation this reads

$$\max_{j=1}^{n} (a_{ij} + x_j) = \lambda + \max_{j=1}^{n} (b_{ij} + x_j), \quad \text{for } i = 1, \ldots, m. \tag{2}$$

The set of eigenvalues will be called the *spectrum* of (A, B) and denoted by $\operatorname{spec}(A, B)$.

When B is the max-plus identity matrix I (all diagonal entries equal 0 and all off-diagonal entries equal $-\infty$), problem (1) is the max-plus spectral problem. The latter spectral problem, as well as its continuous extension for max-plus linear operators, is of fundamental importance for a wide class of problems in discrete event systems theory, dynamic programming, optimal control and mathematical physics [9, 30, 31].

Problem (1) is related to the Perron-Frobenius theory for the two-sided eigenproblem in the conventional linear algebra, as studied in [34, 35]. When both matrices are nonnegative and depend on a large parameter, it can be shown following the lines of [1, Theorem 1] that the asymptotics of an eigenvalue with nonnegative eigenvector is controlled by an eigenvalue of (1). This argument calls for the development of two-sided analogue of the tropical eigenvalue perturbation theory presented in [3, 2].

A specific motivation to study the two-sided max-plus eigenproblem arises from discrete event systems. In particular, systems of the form $A \otimes x = B \otimes x$ or $A \otimes x \leqslant B \otimes x$ appear in scheduling. Indeed, when $\lambda = 0$, the system of constraints (2) can be interpreted in terms of *rendez-vous*. Here, x_j represents the starting time of a task j (for instance, the availability of a part in a manufacturing system). The expression $\max_{j=1}^{n}(a_{ij} + x_j)$ represents the earliest completion time of a task which needs at least a_{ij} time units to be completed after task j started. Thus, the system $A \otimes x = B \otimes x$ requires to find starting times such that two different sets of tasks are completed at the earliest exactly at the same times. In many situations, such systems cannot be solved exactly, and a natural idea is to calculate the minimal Chebyshev distance between $A \otimes x$ and $B \otimes x$. Theorem 4 below determines this minimal distance. It may be also of interest to solve perturbed problems like

$A \otimes x = \lambda \otimes B \otimes x$, as in (1). Such problems express *no wait* type constraints. Indeed, $y := A \otimes x$ and $z := B \otimes x$ may be thought of as the outputs of two different systems A and B, with a common input x. The time offsets between output events are represented by the differences $y_i - y_j$, for all i, j, where the difference is understood in the usual algebra (these quantities belong to the "second order" max-plus theory, see e.g. [17]). No wait constraint may require that $y_i - y_j$ take prescribed values, for some pair (i, j). The condition that $y = \lambda \otimes z$, i.e., $y = \lambda + z$, for some $\lambda \in \mathbb{R}$, means precisely that the time offsets are the same for the two outputs y and z. Hence, an input x solving $A \otimes x = \lambda \otimes B \otimes x$ has the property of making A and B indistinguishable from the point of view of no wait output constraints. An example of such a situation is demonstrated on Figure 2 in Subsection 3.2.

Problems of a related nature, regarding the time separation between events, arose for instance in the work of Burns, Hulgaard, Amon, and Borriello [13], following the work of Burns on the checking of asynchronous digital circuits [12]. Moreover, systems of the form $A \otimes x \leqslant B \otimes x$ represent scheduling problems with both AND and OR precedence constraints, studied by Möhring, Skutella, and Stork [37].

Similar motivations led to the study of min-max functions by Olsder [39] and Gunawardena [29]. Such functions can be written as finite infima of max-plus linear maps, or finite suprema of min-plus linear maps. They also arise as dynamic programming operators of zero-sum deterministic games. In particular, the fixed points and invariant halflines of min-max functions studied in [16, 23] can be also used to compute values of zero-sum deterministic games with mean payoff [23, 44]. A correspondence between the computation of the value of mean payoff games and two-sided linear systems in max-plus algebra has been established in [5]; we shall exploit here the same correspondence, although in different guises.

In max-plus algebra, a special form of min-max functions appears in Cuninghame-Green [21], under the name of AA^*-products. The same functions appear as nonlinear projectors on max-plus cones playing essential role in the max-plus analogue of Hahn-Banach theorem [18, 33]. The compositions of nonlinear projectors are more general min-max functions, and they appear when one approaches two-sided systems $A \otimes x = B \otimes y$ and $A \otimes x = B \otimes x$ [19], and intersections of max-plus cones [28, 42]. It is immediate to see that (1) is a parametric version of $A \otimes x = B \otimes x$.

In max-plus algebra, partial results for Problem (1) have been obtained by Binding and Volkmer [10], and Cuninghame-Green and Butkovič [15, 20]. In particular, Cuninghame-Green and Butkovič [15, 20] give an interval bound on the spectrum of (1) in the case where the entries of both matrices are real. Besides that, both papers treat interesting special cases, for instance when A and B square, or one of them is a multiple of the other.

The spectrum of (1) is generally a collection of intervals on the real line. By means of projection, this follows from a result of De Schutter and De Moor [40] that solution set to the system of max-plus (in)equalities is a union of convex polyhedra. Note that the approach of [40], related to Develin-Sturmfels cellular decomposition [22], can be also used for solving $A \otimes x = \lambda \otimes B \otimes x$ and more general problems of max-plus linear algebra.

1.2 Contents of the paper

In the present paper, we first show that (1) can be viewed as a fixed-point problem for a family of parametric min-max functions h_λ. Based on this observation, we introduce a spectral function $s(\lambda)$ of (1), defined as the spectral radius of h_λ. The zero level set of $s(\lambda)$ is precisely $\operatorname{spec}(A,B)$. More generally, $s(\lambda)$ has a natural geometric sense, being equal to the inverse of the least Chebyshev distance between $A\otimes x$ and $\lambda\otimes B\otimes x$.

The function $s(\lambda)$ is piecewise-affine and Lipschitz continuous, and it has an affine asymptotics at large and small λ. In an important special case when none of the matrices A and B have an identically $-\infty$ column, the asymptotics is just $\lambda+\alpha_1$ at small λ, and $-\lambda+\alpha_2$ at large λ, in the conventional arithmetics. We also give bounds on the spectrum of two-sided eigenproblem, which improve and generalize the bound of Cuninghame-Green and Butkovič [15,20]. In the case when the entries of A and B are integer or $-\infty$, this allows us to show that all affine pieces of $s(\lambda)$ can be identified in a pseudopolynomial number of calls to an oracle which identifies $s(\lambda)$ at a given point. Importantly, $s(\lambda)$ can be interpreted as the greatest value of the associated parametric mean-payoff game and it can be computed by the policy iteration algorithm of [16,23], as well as by the value iteration of Zwick and Paterson [44] or the subexponential method of Björklund and Vorobyov [11]. This leads to a procedure for computing the whole spectrum of (1). To our knowledge, no such general algorithm for computing the whole spectrum of (1) was known previously. We also believe that the level set method used here, relying on the introduction of the spectral function, is of independent interest and may have other applications.

In some cases the spectral function can be computed analytically. In particular, we will consider an example of [41], where it is shown that any finite system of intervals and points on the real line can be represented as the spectrum of (1).

The paper is organized as follows. In the remaining subsection of Introduction we explain the notation used in the rest of the paper. In Section 2 we consider two-sided systems $A\otimes x=B\otimes y$ and $A\otimes x=B\otimes x$. We relate the systems $A\otimes x=B\otimes x$ to certain min-max functions and show that the spectral radii of these functions are equal to the inverse of the least Chebyshev distance between $A\otimes x$ and $B\otimes x$. In Section 3, we introduce the spectral function of two-sided eigenproblem as the spectral radius of a natural parametric extension of the min-max functions studied in Section 2. We give bounds on the spectrum of two-sided eigenproblem and investigate the asymptotics of $s(\lambda)$. We reconstruct the spectral function and hence the whole spectrum in a pseudopolynomial number of calls to the mean-payoff game oracle.

1.3 Notation

For the sake of simplicity, the sign $\otimes$ will be usually omitted in the remaining part of the paper, or even replaced with $+$ if scalars are involved. In particular we write Ax for $A\otimes x$ and $\lambda+x$ for $\lambda\otimes x$, where $A\in\mathbb{R}_{\max}^{m\times n}$ (a matrix), $x\in\mathbb{R}_{\max}^{n}$ (a vector) and $\lambda\in\mathbb{R}_{\max}$ (a scalar). Moreover in the remaining part of the paper we will prefer conventional arithmetic notation: the four arithmetic operations $a+b$, $a-b$, ab and a/b on the set of real numbers (scalars) will have their usual meaning.

However we often use $\vee$ for max and $\wedge$ for min (also componentwise). The actions of max-plus linear operators, their min-plus linear residuations and nonlinear projectors onto max-plus cones (defined below in Subsection 2.1), which will appear as Ax, $A^\sharp y$, $P_A y$ etc., should not be confused with any conventional linear operator. The notations like $A^\sharp B$ or $P_A P_B$ should be understood as compositions of the corresponding operators rather than any kind of matrix multiplication between them (in the case of $A^\sharp B$ above, B is max-plus linear and $A^\sharp$ is min-plus linear).

Such notation is implied by the methodology of the paper: we consider a problem of max-plus algebra, using nonlinear Perron-Frobenius theory and elementary analysis of piecewise-affine functions, where the max-plus arithmetic notation is not required or is inconvenient. The max-plus matrix product notation $\otimes$, especially when mixed with the dual min-plus $\otimes'$, is also inconvenient when several compositions of such operators appear in the same formula or equation.

The notation used in our main subsections is not new. Being different from Baccelli et al. [9] or Cuninghame-Green [21], it basically follows Gaubert and Gunawardena [25].

2 Two-sided systems and min-max functions

2.1 Max-plus linear systems and nonlinear projectors

Consider the m-fold Cartesian product $\mathbb{R}^m_{\max}$ equipped with operations of taking supremum $u \vee v$ and scalar "multiplication" (i.e., addition) $\lambda \otimes v := \lambda + v$. This structure is an example of semimodule over the semiring $\mathbb{R}_{\max}$ defined in the introduction. The subsets of $\mathbb{R}^m_{\max}$ closed under these two operations are its subsemimodules. We will call them *max-plus cones* or just *cones*, by abuse of language. Indeed, there are important analogies and links between max-plus cones and convex cones [18,22,27,42]. We also need the operation of taking infimum which we denote by inf or $\wedge$.

With a max-plus cone $\mathcal{V} \subseteq \mathbb{R}^m_{\max}$ we can associate an operator $P_{\mathcal{V}}$ defined by its action

$$P_{\mathcal{V}} z = \bigvee \{ y \in \mathcal{V} \mid y \leqslant z \}. \tag{3}$$

Consider the case where $\mathcal{V} \subseteq \mathbb{R}^m_{\max}$ is *generated* by a set $S \in \mathbb{R}^m_{\max}$, which means that it is the set of bounded max-plus linear combinations

$$v = \bigvee_{y \in S} \lambda_y + y. \tag{4}$$

In this case

$$\begin{aligned} &P_{\mathcal{V}} z = \bigvee_{y \in S} z / y + y, \text{ where} \\ &z / y = \max\{\gamma \mid \gamma + y \leqslant z\} = \bigwedge_{j \in \operatorname{supp}(y)} (z_j - y_j) = \bigwedge_{j=1}^{m} (z_j - y_j), \end{aligned} \tag{5}$$

with the convention $(-\infty) + (+\infty) = +\infty$. Here and in the sequel $\operatorname{supp}(y) := \{i \mid y_i \neq -\infty\}$ denotes the support of y. Note that $z / y = \infty$ if and only if $y = -\infty$.

Further we are interested only in the case when $\mathcal{V}$ is finitely generated. Let $S = \{y^1, \ldots, y^n\}$, and T_i denote the set of indices where the minimum in z / y^i is attained. The following result is classical.

Proposition 1 ([9, 14, 30]) *Let a cone $\mathcal{V} \subseteq \mathbb{R}^m_{\max}$ be generated by $y^1, \ldots, y^n$ and let $z \in \mathbb{R}^m_{\max}$. The following statements are equivalent.*

1. $z \in \mathcal{V}$.
2. $P_{\mathcal{V}} z = z$.
3. $\bigcup_{i=1}^m T_i = \operatorname{supp} z$.

We note that the set covering condition 3. has been generalized to the case of Galois connections [6].

By this proposition, the operator $P_{\mathcal{V}}$ is a projector onto $\mathcal{V}$. It is an isotonic and $+$-homogeneous operator, meaning that $z^1 \leqslant z^2$ implies $P_{\mathcal{V}} z^1 \leqslant P_{\mathcal{V}} z^2$, and that $P_{\mathcal{V}}(\lambda + z) = \lambda + P_{\mathcal{V}} z$. However, in general it is neither $\vee$- nor $\wedge$-linear.

A finitely generated cone can be described as a max-plus column span of a matrix $A \in \mathbb{R}^{m\times n}_{\max}$:

$$\operatorname{span}(A) := \{\bigvee_{i=1}^n \lambda_i + A_{\cdot i} \mid \lambda_i \in \mathbb{R}_{\max},\ i = 1, \ldots, n\}. \tag{6}$$

In this case we denote $P_A := P_{\operatorname{span}(A)}$, and there is an explicit expression for this operator which we recall below.

We denote $\overline{\mathbb{R}}_{\max} := \mathbb{R}_{\max} \cup \{+\infty\}$ and view $A \in \overline{\mathbb{R}}^{m\times n}_{\max}$ as an operator from $\overline{\mathbb{R}}^m_{\max}$ to $\overline{\mathbb{R}}^n_{\max}$. The *residuated operator* $A^\sharp$ from $\overline{\mathbb{R}}^n_{\max}$ to $\overline{\mathbb{R}}^m_{\max}$ is defined by

$$(A^\sharp y)_j = y \not/ A_{\cdot j} = \bigwedge_{i=1}^m (-a_{ij} + y_i), \tag{7}$$

with the convention $(-\infty) + (+\infty) = +\infty$. Note that this operator, also known as *Cuninghame-Green inverse*, sends $\mathbb{R}^n_{\max}$ to $\mathbb{R}^m_{\max}$ whenever A does not have columns equal to $-\infty$. The term "residuated" refers to the property

$$Ax \leqslant y \Leftrightarrow x \leqslant A^\sharp y, \tag{8}$$

where $\leqslant$ is the partial order on $\mathbb{R}^m_{\max}$ or $\mathbb{R}^n_{\max}$. Using (5) we obtain

$$P_A(z) = \bigvee_{i=1}^n (z \not/ A_{\cdot i}) + A_{\cdot i} = AA^\sharp z. \tag{9}$$

In this form (9), the nonlinear projectors were studied by Cuninghame-Green [21] (as AA^*-products).

Finitely generated cones are closed in the topology induced by the metric

$$d(x, y) = \max_i |e^{x_i} - e^{y_i}|, \tag{10}$$

which coincides with Birkhoff's order topology. It is known [18, Theorem 3.11] that the projectors onto such cones are continuous.

The intersection of two finitely generated cones can be expressed in terms of two-sided max-plus linear systems with separated variables $Ax = By$, by the following proposition.

Proposition 2 *Let $A \in \mathbb{R}^{m\times n_1}_{\max}$ and $B \in \mathbb{R}^{m\times n_2}_{\max}$.*

1. *If (x, y) satisfies $Ax = By \neq -\infty$ then $z = Ax = By$ belongs to* $\operatorname{span}(A) \cap \operatorname{span}(B)$. *Equivalently, $P_A P_B z = P_B P_A z = z$.*
2. *If $P_A P_B z = z \neq -\infty$ then there exist x and y such that $Ax = By = z$.*

This approach to two-sided systems is also useful in the case of systems with non-separated variables $Ax = Bx$, which is of greater importance for us here. This system is equivalent to

$$\begin{gathered} Cx = Dy, \text{ where} \\ C = \begin{pmatrix} A \\ B \end{pmatrix}, \quad D = \begin{pmatrix} I_m \\ I_m \end{pmatrix}, \end{gathered} \tag{11}$$

and $I_m = (\delta_{ij}) \in \mathbb{R}_{\max}^{m\times m}$ denotes the max-plus $m \times m$ identity matrix with entries

$$\delta_{ij} = \begin{cases} 0, & \text{if } i = j, \\ -\infty, & \text{if } i \neq j. \end{cases} \tag{12}$$

In this case we have the following version of Proposition 2.

Proposition 3 *Let $A, B \in \mathbb{R}_{\max}^{m\times n}$.*

1. *If x satisfies $Ax = Bx \neq -\infty$, then $v = (z\ z)^T$, where $z = Ax = Bx$, belongs to* $\operatorname{span}(C) \cap \operatorname{span}(D)$. *Equivalently, $P_C P_D v = P_D P_C v = P_C v = v$.*
2. *If $v = (z\ z)^T \neq -\infty$ and $P_C v = v$, then there exist x such that $Ax = Bx = v$.*

Pairs $(x, y) \neq -\infty$ such that $Ax = By = -\infty$ are described by: $x_i \neq -\infty \Leftrightarrow A_{\cdot i} = -\infty$ and $y_j \neq -\infty \Leftrightarrow B_{\cdot j} = -\infty$. Analogously, vectors $x \neq -\infty$ such that $Ax = Bx = -\infty$ are described by $x_i \neq -\infty \Leftrightarrow A_{\cdot i} = B_{\cdot i} = -\infty$. Any such pair of vectors can be added to any other pair (x', y') or, respectively, vector x', and the resulting pair of vectors will satisfy the system if and only if so does (x', y') or, respectively, x'. Therefore, we can assume in the sequel without loss of generality that there are no such solutions, i.e., that 1) A and B do not have $-\infty$ columns in the case of separated variables, 2) A and B do not have common $-\infty$ columns in the case of non-separated variables.

2.2 Projectors and Perron-Frobenius theory

Suppose that a function $f : \mathbb{R}_{\max}^n \to \mathbb{R}_{\max}^n$ is homogeneous, isotone and continuous in the topology induced by (10). As $x \mapsto \exp(x)$ yields a homeomorphism with $\mathbb{R}_+^n$ endowed with the usual Euclidean topology, we can use the spectral theory for homogeneous, isotone and continuous functions in $\mathbb{R}_+^n$. We will use the following important identities, which follow from the results of Nussbaum [38], see [5, Lemma 2.8] for the proof.

Theorem 1 (Coro. of [38],[5, Lemma 2.8]) *Let f denote an order-preserving, additively homogeneous and continuous map from $(\mathbb{R} \cup \{-\infty\})^n$ to itself. Then it has a largest eigenvalue*

$$r(f) := \max\{\lambda \mid \exists x \in \mathbb{R}_{\max}^n,\ x \not\equiv -\infty,\ \lambda + x = f(x)\},$$

which coincides with

$$r(f) = \max\{\lambda \mid \exists x \in \mathbb{R}_{\max}^n,\ x \not\equiv -\infty,\ \lambda + x \leqslant f(x)\}, \tag{13}$$

$$r(f) = \inf\{\lambda \mid \exists x \in \mathbb{R}^n,\ \lambda + x \geqslant f(x)\}. \tag{14}$$

Note that (14) is nonlinear generalization of the classical Collatz-Wielandt formula [36]. Equations (13) and (14) are useful in max-plus algebra, since they work for max-plus matrix multiplication as well as for compositions of nonlinear projectors. For (14) it is essential that it is taken over vectors with real entries, and that the infimum may not be reached. Using (14) we obtain that the spectral radius of such functions is isotone: $f(x) \leqslant g(x)$ for all $x \in \mathbb{R}^n$ implies $r(f) \leqslant r(g)$. We next recall an application of (14) to the metric properties of compositions of projectors, which appeared in [28]. The *Hilbert distance* between $u, v \in \mathbb{R}^n_{\max}$ such that $\operatorname{supp}(u) = \operatorname{supp}(v)$ is defined by

$$d_{\mathrm{H}}(u,v) = \max_{i,j \in \operatorname{supp}(v)} (u_i - v_i + v_j - u_j). \tag{15}$$

If $\operatorname{span}(u) \neq \operatorname{span}(v)$ then we set $d_{\mathrm{H}}(u,v) = +\infty$. Using (15) we define the Hilbert distance between cones $\operatorname{span}(A)$ and $\operatorname{span}(B)$, for $A \in \mathbb{R}^{m\times n_1}_{\max}$ and $B \in \mathbb{R}^{m\times n_2}_{\max}$:

$$d_{\mathrm{H}}(A,B) := \min\{d_{\mathrm{H}}(u,v) \mid u \in \operatorname{span}(A),\ v \in \operatorname{span}(B),\ \operatorname{supp}(u) = \operatorname{supp}(v)\}. \tag{16}$$

Theorem 2 (cp. [28], Theorem 25) *Let $A \in \mathbb{R}^{m\times n_1}_{\max}$ and $B \in \mathbb{R}^{m\times n_2}_{\max}$. Then*

$$r(P_A P_B) = r(P_B P_A) = -d_{\mathrm{H}}(A,B). \tag{17}$$

If $d_{\mathrm{H}}(A,B)$ is finite then it is attained by any eigenvector $\overline{u}$ of $P_A P_B$ with eigenvalue $r(P_A P_B)$, and its image $\overline{v}$ by P_B.

Proof As $\operatorname{supp}(P_A P_B u) \subseteq \operatorname{supp}(P_B u) \subseteq \operatorname{supp}(u)$, it follows that $P_A P_B$ and also $P_B P_A$ may have finite eigenvalue only if $\operatorname{span}(A)$ and $\operatorname{span}(B)$ have vectors with common support. This shows the claim for the case $d_{\mathrm{H}}(A,B) = +\infty$.

Now let $d_{\mathrm{H}}(A,B)$ be finite. We show that $-d_{\mathrm{H}}(\overline{u},\overline{v}) = -d_{\mathrm{H}}(A,B) = r(P_A P_B)$. Take arbitrary vectors $u \in \operatorname{span}(A)$ and $v \in \operatorname{span}(B)$ with $\operatorname{supp}(u) = \operatorname{supp}(v)$, and let P_u, resp. P_v, be projectors onto the rays $U = \{\lambda + u,\ \lambda \in \mathbb{R}_{\max}\}$, resp. $V = \{\lambda + v,\ \lambda \in \mathbb{R}_{\max}\}$. As $U \subseteq \operatorname{span}(A)$ and $V \subseteq \operatorname{span}(B)$, we have that $P_u \leqslant P_A$ and $P_v \leqslant P_B$, hence $P_u P_v \leqslant P_A P_B$ and, by the monotonicity of the spectral radius, $r(P_u P_v) \leqslant r(P_A P_B)$. It can be shown that $-d_{\mathrm{H}}(u,v)$ is the only finite eigenvalue of $P_u P_v$, hence $-d_{\mathrm{H}}(u,v) = r(P_u P_v)$, and consequently $-d_{\mathrm{H}}(u,v) \leqslant r(P_A P_B)$ and $-d_{\mathrm{H}}(A,B) \leqslant r(P_A P_B)$. Now observe that $-d_{\mathrm{H}}(\overline{u},\overline{v}) = r(P_{\overline{u}} P_{\overline{v}})$ is equal to the eigenvalue $r(P_A P_B)$. This completes the proof. □

In the case of the systems with non-separated variables, we will be more interested in the Chebyshev distance. For $u, v \in \mathbb{R}^m_{\max}$ with $\operatorname{supp}(u) = \operatorname{supp}(v)$ it is defined by

$$d_\infty(u,v) = \max_{i \in \operatorname{supp}(v)} |u_i - v_i|. \tag{18}$$

There is an important special case when Hilbert and Chebyshev distances coincide.

Lemma 1 *Let $u, v \in \mathbb{R}^m_{\max}$ be such that $u \geqslant v$ and $u_i = v_i$ for some $i \in \{1, \ldots, n\}$. Then $d_{\mathrm{H}}(u,v) = d_\infty(u,v)$.*

Proof First note that both $d_{\mathrm{H}}(u,v)$ and $d_\infty(u,v)$ are finite if and only if $\mathrm{supp}(u) = \mathrm{supp}(v)$. Only this case has to be considered.

If $u \geqslant v$ then $|u_j - v_j + v_l - u_l| \leqslant \max(u_j - v_j, u_l - v_l)$ for any j and l, hence $d_{\mathrm{H}}(u,v) \leqslant d_\infty(u,v)$.

Fixing $l = i$ (assuming that $u_i = v_i$) we obtain $u_j - v_j + v_l - u_l = u_j - v_j$ and fixing $j = i$ we obtain $u_j - v_j + v_l - u_l = v_l - u_l$. Taking maximum over such terms only yields $d_\infty(u,v)$, hence $d_\infty(u,v) \leqslant d_{\mathrm{H}}(u,v)$. □

Theorem 3 *Let $A, B \in \mathbb{R}_{\max}^{m\times n}$, and let C and D be defined as in* (11). *Then*

$$r(P_C P_D) = r(P_D P_C) = -\min_{x\in\mathbb{R}_{\max}^m} d_\infty(Ax, Bx). \tag{19}$$

Proof Theorem 2 implies that

$$r(P_C P_D) = -\min\{d_{\mathrm{H}}(u,v) \mid u \in \mathrm{span}(C), v \in \mathrm{span}(D).\} \tag{20}$$

Let $u \in \mathrm{span}(C)$ and denote by P_u the projector onto $U := \{\lambda + u \mid \lambda \in \mathbb{R}_{\max}\}$. Then u is an eigenvector of $P_u P_D$ which corresponds to the spectral radius of this operator, and applying Theorem 2 to the max cones U and $\mathrm{span}(D)$ we see that

$$d_{\mathrm{H}}(u, P_D u) = \min\{d_{\mathrm{H}}(u,v) \mid v \in \mathrm{span}(D)\}. \tag{21}$$

Note that (21) also holds if there is no $v \in \mathrm{span}(D)$ with $\mathrm{supp}(u) = \mathrm{supp}(v)$, in which case $d_{\mathrm{H}}(u, P_D u) = +\infty$. This implies

$$r(P_C P_D) = -\min\{d_{\mathrm{H}}(u, P_D u) \mid u \in \mathrm{span}(C)\}. \tag{22}$$

Observe that

$$u = \begin{pmatrix} Ax \\ Bx \end{pmatrix}, \quad P_D u = \begin{pmatrix} Ax \wedge Bx \\ Ax \wedge Bx \end{pmatrix} \tag{23}$$

for some $x \in \mathbb{R}_{\max}^m$, and also that u and $P_D u$ satisfy the conditions of Lemma 1 unless $P_D u = -\infty$. Hence $d_{\mathrm{H}}(u, P_D u) = d_\infty(u, P_D u) = d_\infty(Ax, Bx)$. Conversely, $d_\infty(Ax, Bx)$ equals $d_{\mathrm{H}}(u, P_D u)$ for $u = (Ax\ Bx)^T$. Hence the r.h.s. of (19) is the same as the r.h.s. of (22), which completes the proof. □

2.3 Min-max functions and Chebyshev distance

Let $A \in \mathbb{R}_{\max}^{m\times n_1}$ and $B \in \mathbb{R}_{\max}^{m\times n_2}$. In order to find a point in the intersection of $\mathrm{span}(A)$ and $\mathrm{span}(B)$ (or equivalently, solve $Ax = By$), one can compute the action of $(P_A P_B)^l$, for $l = 1, 2, \ldots$, on a vector $z \in \mathbb{R}_{\max}^m$. Dually one can start with a vector $x^0 \in \mathbb{R}_{\max}^{n_1}$ and compute

$$x^k = A^\sharp B B^\sharp A x^{k-1}, \quad k \geqslant 1. \tag{24}$$

We can assume that A and B do not have columns equal to $-\infty$ so that $A^\sharp z \in \mathbb{R}_{\max}^{n_1}$ and $B^\sharp z \in \mathbb{R}_{\max}^{n_2}$ for any $z \in \mathbb{R}_{\max}^m$.

If at some stage $x^k = x^{k-1} \neq -\infty$ then we can stop, x^k is a solution of the system. If all coordinates of x^k are less than those of x^0 then we can stop, the system has no solution. More details on this simple algorithm called *alternating method* can be found in [19] and [42], see also [7]. In particular, it converges to a solution with all components finite in a finite number of steps, if such a solution exists.

Let $A, B \in \mathbb{R}_{\max}^{m\times n}$. A system $Ax = Bx$ can be written equivalently as $Cx = Dy$ with C and D as in (11). Applying alternating method (24) to this system, i.e., substituting C and D for A and B in (24) we obtain $x^k = g(x^{k-1})$, where

$$g(x) = A^\sharp Ax \wedge B^\sharp Bx \wedge A^\sharp Bx \wedge B^\sharp Ax. \tag{25}$$

As it is assumed that A and B do not have common $-\infty$ columns and hence C (and D) do not have $-\infty$ columns, $g(x) \in \mathbb{R}_{\max}^n$ for all $x \in \mathbb{R}_{\max}^n$.

It can be shown that (see also [19])

$$r(g) = 0 \Leftrightarrow Ax = Bx \text{ is solvable.} \tag{26}$$

In particular, if x is a fixed point of g then it satisfies $Ax = Bx$. For the function

$$f(x) = x \wedge A^\sharp Bx \wedge B^\sharp Ax \tag{27}$$

which appears in [23], it is also true the other way around, since

$$\begin{aligned} Ax = Bx \Leftrightarrow Ax \geqslant Bx \;\&\; Bx \geqslant Ax \Leftrightarrow \\ \Leftrightarrow B^\sharp Ax \geqslant x \;\&\; A^\sharp Bx \geqslant x \Leftrightarrow \\ \Leftrightarrow x \wedge A^\sharp Bx \wedge B^\sharp Ax = x. \end{aligned} \tag{28}$$

We also introduce the function h:

$$h(x) := A^\sharp Bx \wedge B^\sharp Ax. \tag{29}$$

Although f, g and h are different functions, they have the same spectral radius, equal to the inverse minimal Chebyshev distance between Ax and Bx. To show this, we use the following identity.

$$-d_\infty(u, v) = \max\{\lambda \colon \lambda + u \leqslant v \;\&\; \lambda + v \leqslant u\}. \tag{30}$$

Theorem 4 *Let $A, B \in \mathbb{R}_{\max}^{m\times n}$. For C, D defined by* (11)*, and f, g and h defined by* (27)*,* (25) *and* (29)*,*

$$r(P_C P_D) = r(P_D P_C) = r(f) = r(g) = r(h) = -\min_{x\in\mathbb{R}_{\max}^m} d_\infty(Ax, Bx). \tag{31}$$

Proof If v is an eigenvector of $P_D P_C$ with a finite eigenvalue, then $C^\sharp v$ is an eigenvector of g and $P_C v$ is an eigenvector $P_C P_D$, both with the same eigenvalue. The other way around, if x is an eigenvector of g with a finite eigenvalue, then $(Ax\ Bx)^T$ is an eigenvector of $P_D P_C$ with the same eigenvalue. This argument shows that 1) either the spectral radii of $P_D P_C$, $P_C P_D$ and g are all finite or they all equal $-\infty$, 2) the equality $r(g) = r(P_D P_C) = r(P_C P_D)$ holds true both in finite and in infinite case.

We show the remaining equalities. By (13), $r(h)$ is the maximum of λ which satisfy

$$\exists x \in \mathbb{R}_{\max}^n \colon \lambda + x \leqslant A^\sharp Bx \wedge B^\sharp Ax. \tag{32}$$

This is equivalent to

$$\exists x \in \mathbb{R}_{\max}^n \colon \lambda + Ax \leqslant Bx \quad \& \quad \lambda + Bx \leqslant Ax \tag{33}$$

Using (30) we obtain

$$r(h) = \max_{x \in \mathbb{R}^n_{\max}} -d_\infty(Ax, Bx) = -\min_{x \in \mathbb{R}^n_{\max}} d_\infty(Ax, Bx). \tag{34}$$

It follows in particular that $r(h) \leqslant 0$ and moreover, $\lambda \leqslant 0$ for any x satisfying (33). Applying (13) to f and g we obtain that both $r(f)$ and $r(g)$ are equal to the maximum of λ which satisfy

$$\exists x \in \mathbb{R}^m_{\max}: \quad \lambda \leqslant 0 \quad \& \quad \lambda + Ax \leqslant Bx \quad \& \quad \lambda + Bx \leqslant Ax \tag{35}$$

As the first inequality follows from the other two, we obtain $r(f) = r(g) = r(h)$. ∎

Functions f, g and h as well as projectors onto finitely generated max-plus cones and their compositions, belong to the class of *min-max functions*.Such functions were originally considered by Olsder [39] and Gunawardena [29]. See [16] for a formal definition. In a nutshell, these are additively homogeneous and order preserving maps, every coordinate of which can be represented as a minimum of a finite number of max-plus linear forms, or as a maximum of a finite number of min-plus linear forms. It is important that any min-max function $q : \mathbb{R}^n_{\max} \to \mathbb{R}^n_{\max}$ can be represented as infimum of finite number of max-plus linear maps $Q^{(p)}$ meaning that

$$q(x) = \bigwedge_p Q^{(p)} x, \tag{36}$$

in such a way that the following *selection property* is satisfied:

$$\forall x \; \exists p : \; q(x) = Q^{(p)} x. \tag{37}$$

Note that taking infimum or supremum of vectors does not necessarily select one of them, and that selection property (37) is useful, e.g., for the policy iteration algorithm of [23]).

In connection with the mean payoff games [23, 5], each matrix $Q^{(p)}$ corresponds to a one player game, where the player Min has chosen her strategy and the player Max is trying to win what he can.

In particular, $f(x), g(x)$ and $h(x)$, respectively, are represented as infima of the max-plus linear maps $F^{(p)}$, $G^{(p)}$ and $H^{(p)}$, whose rows are taken from the max-plus linear forms appearing in (27), (25) and (29), respectively, in the following way:

$$F^{(p)}_{i\cdot} = \begin{cases} I_{i\cdot}, \\ -a_{ki} + B_{k\cdot}, \\ -b_{ki} + A_{k\cdot}. \end{cases} \qquad G^{(p)}_{i\cdot} = \begin{cases} -a_{ki} + A_{k\cdot}, \\ -b_{ki} + B_{k\cdot}, \\ -a_{ki} + B_{k\cdot}, \\ -b_{ki} + A_{k\cdot}. \end{cases} \qquad H^{(p)}_{i\cdot} = \begin{cases} -a_{ki} + B_{k\cdot}, \\ -b_{ki} + A_{k\cdot}. \end{cases} \tag{38}$$

Here $I_{i\cdot}$ denotes the ith row of the max-plus identity matrix, and the brackets mean that any possibility, for any $k = 1, \ldots, m$ and $a_{ki} \neq -\infty$ or $b_{ki} \neq -\infty$, can be taken (assumed that A and B do not have common $-\infty$ columns). Applying Collatz-Wielandt formula (14) we obtain the following, some variants of which appeared in several contexts.

Proposition 4 (Compare with [16, 25, 24, 8]) *Suppose that a min-max function $q : \mathbb{R}^n_{\max} \to \mathbb{R}^n_{\max}$ is represented as infimum of max-plus linear maps $Q^{(l)} \in \mathbb{R}^{n\times n}_{\max}$ so that the selection property is satisfied. Then*

$$r(q) = \min_l r(Q^{(l)}). \tag{39}$$

Proof The spectral radius is isotone, hence $r(q) \leqslant r(Q^{(l)})$ for all l. Using (14) we conclude that for any ϵ there is $x \in \mathbb{R}^m$ such that $q(x) \leqslant r(q)+\epsilon+x$. As $q(x) = Q^{(l)}x$ for some l and there is only finite number of matrices $Q^{(l)}$, there exists l such that

$$r(q) = \inf\{\mu \mid \exists x \in \mathbb{R}^n,\ Q^{(l)}x \leqslant \mu + x\} = r(Q^{(l)}). \tag{40}$$

The proof is complete. □

Proposition 4 can be derived alternatively from the duality theorem in [25, Theorem 19] (see also [24]). It is related to the existence of the value of stochastic games with perfect information [32]. Indeed, the spectral radius can be seen to coincide with the value of a game in which Player Max chooses the initial state, see [5] for more information.

The greatest eigenvalue $r(Q^{(l)})$ of the max-plus matrix $Q^{(l)} = (q^{(l)}_{ij}) \in \mathbb{R}^{n\times n}_{\max}$ can be computed explicitly. It is equal to the maximum cycle mean of $Q^{(l)}$ defined by

$$\max_{1\leqslant k\leqslant n} \max_{i_1,\ldots,i_k} \frac{q^{(l)}_{i_1 i_2} + q^{(l)}_{i_2 i_3} + \ldots + q^{(l)}_{i_k i_1}}{k}. \tag{41}$$

This result is fundamental in max-plus algebra, see [4, 9, 15, 30] for more details.

3 The spectrum and the spectral function

3.1 Construction of the spectral function

Given $A \in \mathbb{R}^{m\times n}_{\max}$ and $B \in \mathbb{R}^{m\times n}_{\max}$, we consider the *two-sided eigenproblem* which consists in finding *eigenvalues* $\lambda \in \mathbb{R}_{\max}$ and *eigenvectors* $x \in \mathbb{R}^n_{\max}$ (which have at least one component not equal to $-\infty$), such that

$$Ax = \lambda + Bx. \tag{42}$$

The set of eigenvalues is called the *spectrum of* (A, B) and denoted by $\operatorname{spec}(A, B)$.

Below we assume that A and B do not have $-\infty$ rows and common $-\infty$ columns. Note that the assumption about $-\infty$ rows can be made without loss of generality when the solvability of (42) is considered. Indeed, if the ith row of B is $-\infty$ then all variables x_j such that $a_{ij} \neq -\infty$ must be equal to $-\infty$. Eliminating these variables as well as the corresponding columns in A and B and the ith equation, we obtain a new system where A or B may have $-\infty$ rows. Proceeding this way we either cancel the whole system in which case it is unsolvable, or we are left with a system where A and B (what remains of them) do not have $-\infty$ rows. This procedure can be run in $O(m^2 n)$ operations.

The case of $\lambda = -\infty$ appears if and only if A has $-\infty$ columns, and the corresponding eigenvectors are described by $x_i \neq -\infty \Leftrightarrow A_{\cdot i} = -\infty$. In the sequel we assume that λ is finite.

Problem (42) is equivalent to $C(\lambda)x = Dy$, where $C(\lambda) \in \mathbb{R}_{\max}^{2m\times n}$ and $D \in \mathbb{R}_{\max}^{2m\times m}$ are defined by

$$C(\lambda) = \begin{pmatrix} A \\ \lambda + B \end{pmatrix}, \quad D = \begin{pmatrix} I_m \\ I_m \end{pmatrix}. \tag{43}$$

As it follows from Theorem 4, $\mathrm{spec}(A, B) = \{\lambda\colon r(P_D P_{C(\lambda)}) = 0\} = \{\lambda\colon r(h_\lambda) = 0\}$, where

$$h_\lambda(x) = (\lambda + A^\sharp Bx) \wedge (-\lambda + B^\sharp Ax). \tag{44}$$

The function h_λ can be represented as infimum of max-plus linear maps so that the selection property (37) is satisfied. Namely,

$$h_\lambda(x) = \bigwedge_p H_\lambda^{(p)} x, \tag{45}$$

where for $i = 1, \ldots, n$

$$(H_\lambda^{(p)})_{i\cdot} = \begin{cases} \lambda - a_{ki} + B_{k\cdot}, & \text{for } 1 \leqslant k \leqslant n,\ a_{ki} \neq -\infty, \\ -\lambda - b_{ki} + A_{k\cdot}, & \text{for } 1 \leqslant k \leqslant n,\ b_{ki} \neq -\infty, \end{cases} \tag{46}$$

the brackets meaning that any listed choice can be taken.

The greatest eigenvalue of H_λ equals the maximum cycle mean of H_λ. Using formula (41), we observe that $r(H_\lambda)$ is a piecewise-affine function, meaning that it is composed of a finite number of affine pieces. More precisely, we have the following.

Proposition 5 *$r(H_\lambda^{(p)})$ is a finite piecewise-affine convex Lipschitz function of λ.*

Proof Using (41) we observe that $r(H_\lambda^{(p)}) = -\infty$ if and only if the associated digraph of $H_\lambda^{(p)}$ is acyclic, which cannot happen when A and B and hence $H_\lambda^{(p)}$ do not have $-\infty$ rows.

If $r(H_\lambda^{(p)})$ is finite, then any finite cycle mean of $H_\lambda^{(p)}$ can be written as $(k\lambda + a)/l$, where l is the length of the cycle and k is an integer number with modulus not greater than l, hence this affine function is Lipschitz. The function $r(H_\lambda^{(p)})$ is pointwise maximum of a finite number of such affine functions, hence it is a convex Lipschitz piecewise-affine function. □

Definition 1 (Spectral Function) We define the *spectral function* of (42) by

$$s(\lambda) := r(h_\lambda) = r(P_D P_{C(\lambda)}). \tag{47}$$

It follows from Theorem 4 that $s(\lambda) \leqslant 0$ and that $s(\lambda) = 0$ if and only if $\lambda \in \mathrm{spec}(A, B)$. In general, $s(\lambda)$ is equal to the inverse minimal Chebyshev distance between Ax and $\lambda + Bx$.

By Proposition 4,

$$s(\lambda) = \bigwedge_p r(H_\lambda^{(p)}). \tag{48}$$

As $r(H_\lambda^{(p)})$ are piecewise-affine and Lipschitz, we conclude the following.

Corollary 1 *$s(\lambda)$ is a finite piecewise-affine Lipschitz function.*

Let us indicate yet another consequence of the fact that $r(H_\lambda^{(p)})$ and $s(\lambda)$ are piecewise-affine.

Corollary 2 *If* $\operatorname{spec}(A,B)$ *is not empty, then it is a finite system of closed intervals and points.*

Note that this also follows, by means of projection, from a result by De Schutter and De Moor [40] that the solution set of a system of polynomial (in)equalities in the max-plus algebra is a (finite) union of polyhedra. The method of De Schutter and De Moor can also offer an alternative (computationally expensive) way to determine the spectrum and the generalized eigenvectors.

Conversely, it is shown in [41] that any system of closed intervals and points in $\mathbb{R}$ can be represented as spectrum of (A,B). See also Subsect. 3.5.

3.2 Bounds on the spectrum of (A,B)

Next we recall a bound on the spectrum obtained by Cuninghame-Green and Butkovič [15,20], extending it to the case when $A=(a_{ij})$ and $B=(b_{ij})$ may have infinite entries. Denote

$$\begin{aligned} \underline{D}(A,B) &= \bigvee_{i:\ A_{i\cdot}\ \text{finite}} A_{i\cdot}\not\phi B_{i\cdot}\,, \\ \overline{D}(A,B) &= -\bigvee_{i:\ B_{i\cdot}\ \text{finite}} B_{i\cdot}\not\phi A_{i\cdot}\,. \end{aligned} \tag{49}$$

We assume that $\bigvee\emptyset=-\infty$ and $-\bigvee\emptyset=+\infty$.

Since $A_{i\cdot}\not\phi B_{i\cdot}=\max\{\gamma \mid A_{i\cdot}\geqslant \gamma+B_{i\cdot}\}$ is finite when $A_{i\cdot}$ is finite and $B_{i\cdot}$ is not $-\infty$, we immediately see the following.

Lemma 2 $\underline{D}(A,B)$ *(resp.* $\overline{D}(A,B)$*) is finite if and only if there exists an* $i\in\{1,\ldots,m\}$ *such that* $A_{i\cdot}$ *is finite (resp.* $B_{i\cdot}$ *is finite).*

When A and B have finite entries only, $\underline{D}(A,B)$ and $\overline{D}(A,B)$ are just like the bounds of [20, Theorem 2.1]:

$$\begin{aligned} \underline{D}(A,B) &= \bigvee_i \bigwedge_j (a_{ij}-b_{ij}), \\ \overline{D}(A,B) &= \bigwedge_i \bigvee_j (a_{ij}-b_{ij}). \end{aligned} \tag{50}$$

Note that $\underline{D}(A,B)$ and $\overline{D}(A,B)$ defined by (49) take infinite values if A or B do not contain any finite rows.

Proposition 6 *If* $Ax\leqslant\lambda+Bx$ *(resp.* $Ax\geqslant\lambda+Bx$*) has solution* $x>-\infty$*, then* $\lambda\geqslant\underline{D}(A,B)$ *(resp.* $\lambda\leqslant\overline{D}(A,B)$*).*

Proof If there exists i such that $a_{ij}>\lambda+b_{ij}$ for all $j=1,\ldots,m$, then $Ax\leqslant\lambda+Bx$ cannot have solutions. This condition is equivalent to $A_{i\cdot}\not\phi B_{i\cdot}>\lambda$ plus the finiteness of $A_{i\cdot}$. Taking supremum of $A_{i\cdot}\not\phi B_{i\cdot}$ over i such that $A_{i\cdot}$ is finite yields $\underline{D}(A,B)$. This shows that if $Ax\leqslant\lambda+Bx$ then $\lambda\geqslant\underline{D}(A,B)$. The remaining part follows analogously. ◻

The next result is an extension of [20, Theorem 2.1].

Corollary 3 $\mathrm{spec}(A,B) \subseteq [\underline{D}(A,B), \overline{D}(A,B)]$.

We use identity (13) to give a more precise bound. It will be assumed that A and B do not have $-\infty$ columns. Note that this condition is more restrictive than that A and B do not have *common* $-\infty$ columns, and it cannot be assumed without loss of generality.

Theorem 5 *Suppose that* $A = (a_{ij}), B = (b_{ij}) \in \mathbb{R}_{\max}^{m\times n}$ *do not have* $-\infty$ *columns. Then*

$$\mathrm{spec}(A,B) \subseteq [-r(A^\sharp B), r(B^\sharp A)] \subseteq [\underline{D}(A,B), \overline{D}(A,B)]. \tag{51}$$

Proof Let $Ax = \lambda Bx$, then we also have

$$\begin{aligned} &Ax \leqslant \lambda + Bx \Leftrightarrow -\lambda + x \leqslant A^\sharp Bx, \\ &\lambda + Bx \leqslant Ax \Leftrightarrow \lambda + x \leqslant B^\sharp Ax. \end{aligned} \tag{52}$$

As A and B do not have $-\infty$ columns so that $A^\sharp Bx$ and $B^\sharp Ax$ do not have $+\infty$ entries, we can use (13) to obtain from (52) that $\lambda \in [-r(A^\sharp B), r(B^\sharp A)]$. For $\lambda = r(B^\sharp A)$ we can find $y \neq -\infty$ such that $\lambda + y \leqslant B^\sharp Ay$ and hence $\lambda + By \leqslant Ay$. Using Proposition 6 we obtain $\lambda \leqslant \overline{D}(A,B)$. The remaining inequality $\lambda \geqslant \underline{D}(A,B)$ can be obtained analogously. □

By comparison with the finer bounds $-r(A^\sharp B)$ and $r(B^\sharp A)$, the interest of the bounds of Butkovič and Cuninghame-Green, $\underline{D}(A,B)$ and $\overline{D}(A,B)$, lies in their explicit character. However, these bounds become infinite when the matrices A and B do not have any finite rows. We next give different explicit bounds, which turn out to be finite as soon as A and B do not have any identically infinite columns.

Proposition 7 *We have*

$$\mathrm{spec}(A,B) \subseteq \bigcup_{1\leqslant i\leqslant n} [-(A^\sharp B0)_i, (B^\sharp A0)_i] \enspace ,$$

and so

$$\mathrm{spec}(A,B) \subseteq [-\bigvee_i (A^\sharp B0)_i, \bigvee_i (B^\sharp A0)_i],$$

where 0 *is the* n*-vector of all* 0*'s.*

Proof Consider $x := 0$ and $\mu := \bigvee_i [h_\lambda(0)]_i$, so that $h_\lambda(x) \leqslant \mu + x$. Then, the non-linear Collatz-Wielandt formula (14) implies that $r(h_\lambda) \leqslant \mu$. If $\lambda \in \mathrm{spec}(A,B)$, we have $0 \leqslant r(h_\lambda)$, and so, there exists at least one index $i \in \{1,\dots,n\}$ such that

$$0 \leqslant [h_\lambda(0)]_i = (\lambda + (A^\sharp B0)_i) \wedge (-\lambda + (B^\sharp A0)_i) \enspace .$$

It follows that $\lambda \leqslant (B^\sharp A0)_i$ and $\lambda \geqslant -(A^\sharp B0)_i$. □

Remark 1 It follows readily from the Collatz-Wielandt property (14) that

$$[-r(A^\sharp B), r(B^\sharp A)] \subseteq [-\bigvee_i (A^\sharp B0)_i, \bigvee_i (B^\sharp A0)_i]$$

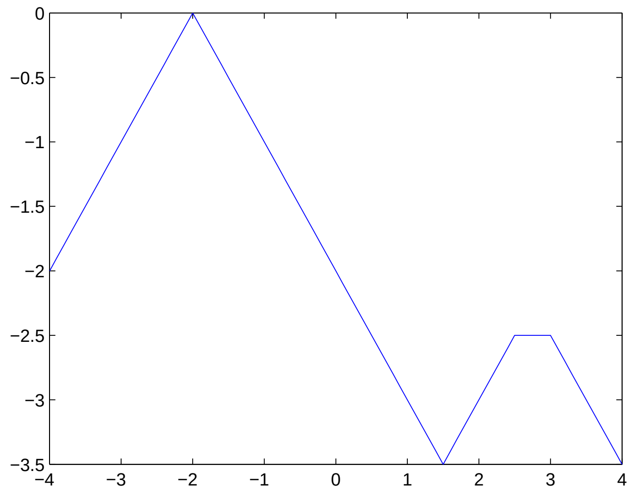

Fig. 1 Spectral function of (53)

Example 1 We next give an example, to compare the bounds of Corollary 3, Theorem 5 and Proposition 7. Consider the following finite matrices of dimension 3×4:

$$A=\begin{pmatrix}-2&3&-3&-3\\-4&1&2&-2\\5&-1&5&-1\end{pmatrix},\quad B=\begin{pmatrix}-4&5&-3&3\\2&0&-1&4\\0&2&-3&-1\end{pmatrix}\tag{53}$$

From the graph of spectral function, Figure 1, it follows that the only eigenvalue is -2 since $s(-2)=0$ and $s(\lambda)<0$ for any $\lambda\neq -2$. The interval $[-r(A^\sharp B), r(B^\sharp A)]$ is in this case $[-2,0.5]$. Bounds (50) of [20, Theorem 2.1] yield the interval $[\underline{D}(A,B),\overline{D}(A,B)]=[-3,2]$, which is less precise. Proposition 7 yields the union of intervals $[3,0]=\emptyset$, $[-2,-2]$, $[3,3]$ and $[-3,-2]$, thus $[-3,-2]\cup\{3\}$. Note that these intervals are incomparable both with $[-r(A^\sharp B), r(B^\sharp A)]$ and $[\underline{D}(A,B),\overline{D}(A,B)]=[-3,2]$.

We remark that the intervals $[-\bigvee_i(A^\sharp B0)_i, \bigvee_i(B^\sharp A0)_i]$ and $[\underline{D}(A,B),\overline{D}(A,B)]$ are also in general incomparable. Also, Subsect. 3.5 will provide an example where the bounds $[-r(A^\sharp B), r(B^\sharp A)]$ are exact.

Example 2 Let us now illustrate the discrete event systems interpretation of the spectral problem of the previous example. For readability, we replace the matrices by

$$A=\begin{pmatrix}-2&3&-\infty&-\infty\\-\infty&1&2&-\infty\\5&-\infty&5&-1\end{pmatrix},\quad B=\begin{pmatrix}-\infty&5&-3&-\infty\\2&-\infty&-\infty&4\\0&2&-\infty&-\infty\end{pmatrix}\tag{54}$$

This pair of matrices can be shown to have the same spectral function (Figure 1) as the previous one, and the same bounds $[-r(A^\sharp B), r(B^\sharp A)]$. Consider now the two discrete event systems

$$y=Ax,\qquad z=Bx\enspace .$$

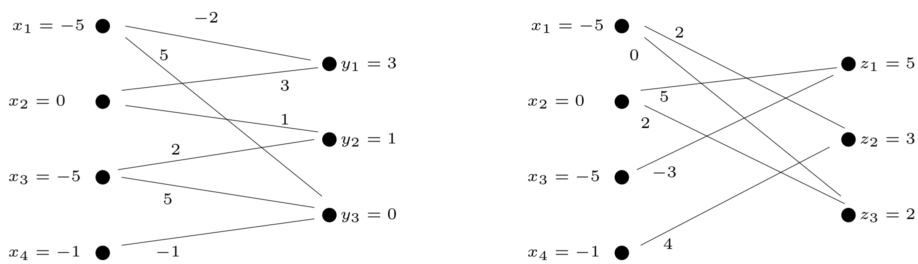

Fig. 2 Finding a common input making the outputs of two discrete event systems indistinguishable, modulo a constant

Here, x_i is interpreted as the starting time of a task i, and y_i and z_i are interpreted as output time. This is illustrated in Figure 2. For instance, the constraint $y_1 = \max(-2 + x_1, 3 + x_2)$ in $y = Ax$ expresses that the first output is released at the earliest, given that it must wait 3 time units after the second input becomes available,and can not be released more than 2 time units before the first input becomes available. We are looking for a common input x such that the time separation between events is the same for both outputs, so that

$$y_i - y_j = z_i - z_j, \qquad \forall i, j \ .$$

This can be solved by finding an eigenvector x, so that $Ax = \lambda + Bx$. By inspection of the spectral function in Figure 1, we see that λ must be equal to -2. Then, computing x reduces to solving a mean payoff game (see the discussion in section 3.4 below for more algorithmic background). In this special example, x can be determined very simply by running the power type algorithm (like the alternating method of [19])

$$x^{(0)} = (0, 0, 0, 0)^T, \qquad x^{(k+1)} = h_{-2}(x^{(k)})$$

where

$$h_{-2}(x) := (-2 + A^\sharp Bx) \ \wedge \ (2 + B^\sharp Ax) \ ,$$

until the sequence x^k converges. Actually,

$$x^{(2)} = x^{(3)} = (-5, 0, -5, -1)^T \ ,$$

and it can be checked that

$$Ax^{(2)} = -2 + Bx^{(2)} = (3, 1, 0)^T \ .$$

3.3 Asymptotics of the spectral function

If A and B do not have $-\infty$ columns, the functions $\lambda + A^\sharp B$ and $-\lambda + B^\sharp A$ are represented as infima of all max-linear mappings $K_\lambda^{(p)}$ and, respectively, $M_\lambda^{(s)}$ such that

$$\begin{aligned} (K_\lambda^{(p)})_{i\cdot} &= \lambda - a_{ki} + B_{k\cdot}, \quad 1 \leqslant k \leqslant n, \ a_{ki} \neq -\infty, \\ (M_\lambda^{(s)})_{i\cdot} &= -\lambda - b_{ki} + A_{k\cdot}, \quad 1 \leqslant k \leqslant n, \ b_{ki} \neq -\infty. \end{aligned} \tag{55}$$

This representation satisfies the selection property.

Matrices $K_\lambda^{(p)}$ and $M_\lambda^{(s)}$ are both instances of $H_\lambda^{(p)}$ which represent h_λ. We will need the following observation on $r(H_\lambda^{(p)})$

Lemma 3 *Denote $\kappa := \min(2m,n)$. The spectral radii $r(H_\lambda^{(p)})$ can be expressed as $\lambda s/l+\alpha$, where $0\leqslant |s|\leqslant l\leqslant \kappa$, and $\alpha\leqslant \Delta(A,B)$, where*

$$\Delta(A,B) := \bigvee_{i,j,k:\ a_{ij}\neq-\infty,\ b_{ik}\neq-\infty} (a_{ij}-b_{ik}) \vee \bigvee_{i,j,k:\ b_{ij}\neq-\infty,\ a_{ik}\neq-\infty} (b_{ij}-a_{ik}). \tag{56}$$

Moreover it is only possible that $s=l-2t$ for $t=0,\ldots,l$.

Proof According to (41) and (46), $r(H_\lambda^{(p)})$ is a cycle mean of the form

$$(t^{k_1}_{i_1i_2}+\ldots+t^{k_l}_{i_li_1})/l \tag{57}$$

where we use the notation $t^k_{ij} := -a_{ki}+\lambda+b_{kj}$ and $t^{n+k}_{ij} = -\lambda-b_{ki}+a_{kj}$ for $i,j=1,\ldots,n$ and $k=1,\ldots,m$. Actually t^k_{ij} is the (i,j) entry of $H_\lambda^{(p)}$, but here we also need the intermediate index k. Note that it is determined by i.

In (57), only even numbers of $\pm\lambda$ can be cancelled, hence it can be expressed as $\lambda s/l+\alpha$ where $0\leqslant|s|\leqslant l$ with $s=l-2t$ for $t=0,\ldots,l$. The cycle $(i_1,\ldots,i_l)$ is elementary, hence $l\leqslant n$. We also obtain $\alpha\leqslant\Delta(A,B)$ since the arithmetic mean does not exceed maximum.

It remains to show that $l\leqslant 2m$. Indeed if $l>2m$ then there is an upper index which appears at least twice in (57). Assume w.l.o.g. that this is k_1. Then the sum in (57) takes one of the following forms:

$$\begin{aligned}
&-a_{k_1i_1}+[\lambda+b_{k_1i_2}+\ldots-a_{k_1i_r}]+\lambda+b_{k_1i_{r+1}}+\ldots,\\
&-\lambda-b_{k_1i_1}+[a_{k_1i_2}+\ldots-\lambda-b_{k_1i_r}]+a_{k_1i_{r+1}}+\ldots
\end{aligned} \tag{58}$$

Assume w.l.o.g. that we have the first one. Then we can split it into the following two cycles, as indicated by the square bracket in the first line of (58):

$$\begin{aligned}
&t^{k_1}_{i_ri_2}+t^{k_2}_{i_2i_3}\ldots+t^{k_{r-1}}_{i_{r-1}i_r},\\
&t^{k_1}_{i_1i_{r+1}}+t^{k_{r+1}}_{i_{r+1}i_{r+2}}\ldots+t^{k_l}_{i_li_1}.
\end{aligned} \tag{59}$$

Each of these forms is a weight of a cycle in $H_\lambda^{(p)}$. Indeed, (58) (the first expression) indicates that k_1 is chosen by i_r in $H_\lambda^{(p)}$ so that any element $t^{k_1}_{i_rj}$ for $j=1,\ldots,n$ is an entry of $H^{(p)}$. All other elements in (59) are also entries of $H^{(p)}$.

The arithmetic mean for both of the cycles in (59) has to be equal to (57), if this is indeed $r(H_\lambda^{(p)})$. This shows $l\leqslant 2m$. □

We also define

$$\begin{aligned}
\overline{C}(A,B) &:= \bigvee_{i,j,k:\ a_{ij}\neq-\infty,\ b_{ik}\neq-\infty} (a_{ij}-b_{ik}),\\
\underline{C}(A,B) &:= \bigwedge_{i,j,k:\ a_{ik}\neq-\infty,\ b_{ij}\neq-\infty} (a_{ik}-b_{ij}).
\end{aligned} \tag{60}$$

We now study the asymptotics of $s(\lambda)$, both in general case and in some special cases.

Theorem 6 *Suppose that $A, B \in \mathbb{R}_{\max}^{m\times n}$ and denote $\kappa := \min(2m, n)$.*

1. *There exist k_1, l_1, k_2, l_2 such that $0 \leqslant l_1 \leqslant \kappa$, $k_1 = l_1 - 2t_1$ where $0 \leqslant t_1 \leqslant \lfloor l_1/2 \rfloor$, $0 \leqslant l_2 \leqslant \kappa$, $k_2 = l_2 - 2t_2$ where $0 \leqslant t_2 \leqslant \lfloor l_2/2 \rfloor$, and $\alpha_1, \alpha_2 \in \mathbb{R}$ such that*

$$\begin{aligned} s(\lambda) &= \lambda k_1/l_1 + \alpha_1, \quad \text{if } \lambda \leqslant -2\kappa^2 \Delta(A,B), \\ s(\lambda) &= -\lambda k_2/l_2 + \alpha_2. \quad \text{if } \lambda \geqslant 2\kappa^2 \Delta(A,B). \end{aligned} \tag{61}$$

2. *Suppose that A and B do not have $-\infty$ columns. Then there exist $\alpha_1 \leqslant r(A^\sharp B)$ and $\alpha_2 \leqslant r(B^\sharp A)$ such that*

$$\begin{aligned} s(\lambda) &= \lambda + \alpha_1, \quad \text{if } \lambda \leqslant -2\kappa \Delta(A,B), \\ s(\lambda) &= -\lambda + \alpha_2, \quad \text{if } \lambda \geqslant 2\kappa \Delta(A,B). \end{aligned} \tag{62}$$

3. *Suppose that A and B are real. Then*

$$\begin{aligned} s(\lambda) &= \lambda + r(A^\sharp B), \quad \text{if } \lambda \leqslant \underline{C}(A,B), \\ s(\lambda) &= -\lambda + r(B^\sharp A), \quad \text{if } \lambda \geqslant \overline{C}(A,B). \end{aligned} \tag{63}$$

Proof 1: For the proof of this part, we observe that for each λ, the function $s(\lambda)$ is the maximum cycle mean of a representing matrix $H_\lambda^{(p)}$, so that it equals $\lambda k/l + \alpha$ where $0 \leqslant l \leqslant \kappa$, $k = l - 2t$ where $0 \leqslant t \leqslant l$. For any two such terms, difference between coefficients k/l is not less than $1/\kappa^2$, and the difference between the offsets does not exceed $2\Delta(A,B)$, which yields that all intersection points must be in the interval $[-2\kappa^2\Delta(A,B), 2\kappa^2\Delta(A,B)]$. Thus $s(\lambda)$ is just one affine piece for $\lambda \leqslant -2\kappa^2\Delta(A,B)$ and for $\lambda \geqslant 2\kappa^2\Delta(A,B)$. As $s(\lambda) \leqslant 0$ for all λ, the left asymptotic slope is nonnegative, and the right asymptotic slope is non-positive.
2: When A does not have $-\infty$ columns, some of the matrices $H_\lambda^{(p)}$ are of the form $K_\lambda^{(p)}$ and their maximum cycle mean is $\lambda + \alpha$. Taking minimum over all $r(H_\lambda^{(p)})$ of the form $\lambda + \alpha$ yields an offset $\alpha_1 \leqslant r(A^\sharp B)$. The cycle mean $\lambda + \alpha_1$ will dominate at small λ, and the smallest intersection point may occur with a term $\lambda(\kappa-1)/\kappa + \alpha_1'$. Indeed, the difference between coefficients is precisely the smallest possible $1/\kappa$, and the difference $|\alpha_1 - \alpha_1'|$ may be up to $2\Delta(A,B)$. This yields the bound $-2\kappa\Delta(A,B)$. An analogous argument follows when λ is large and B does not have $-\infty$ columns.
3: When A and B are real and $\lambda < \underline{C}(A,B)$, all coefficients in the min-max function $\lambda + A^\sharp B$ are real negative, and all coefficients in the min-max function $-\lambda + B^\sharp A$ are real positive. This implies that $s(\lambda)$ is equal to the minimum over $r(K_\lambda^{(p)})$, which is equal to $\lambda + r(A^\sharp B)$. An analogous argument follows when $\lambda > \overline{C}(A,B)$. ∎

In Proposition 10 we will show by an explicit construction that any slope k/l can be realized as asymptotics of a spectral function.

We next observe that the asymptotics of $s(\lambda)$ can be read off from the spectral function $s^\circ(\lambda)$, which we introduce below. For arbitrary $C = (c_{ij}) \in \mathbb{R}_{\max}^{m\times n}$ define

$$c_{ij}^\circ = \begin{cases} 0, & \text{if } c_{ij} \in \mathbb{R}, \\ -\infty, & \text{if } c_{ij} = -\infty. \end{cases} \tag{64}$$

Let $s^\circ(\lambda)$ be the spectral function of the eigenproblem $A^\circ x = \lambda + B^\circ x$.

Proposition 8 *Suppose that $A, B \in \mathbb{R}_{\max}^{m\times n}$ and that $\lambda k_1/l_1$ where $k_1, l_1 \geqslant 0$ (resp. $-\lambda k_2/l_2$ where $k_2, l_2 \geqslant 0$) is the left (resp. the right) asymptotic slope of $s(\lambda)$. Then*

$$s^\circ(\lambda) = \begin{cases} \lambda k_1/l_1, & \text{if } \lambda \leqslant 0, \\ -\lambda k_2/l_2, & \text{if } \lambda \geqslant 0. \end{cases} \tag{65}$$

Proof Observe that the representing matrices $H_\lambda^{(p\circ)}$ of

$$h_\lambda^\circ := (\lambda + (A^\circ)^\sharp B^\circ x) \ \ \wedge \ \ (-\lambda + (B^\circ)^\sharp A^\circ x) \tag{66}$$

are in one-to-one correspondence with the representing matrices $H_\lambda^{(p)}$ of h_λ. The finite entries $H_\lambda^{\circ(p)}$ equal to $\pm\lambda$, they are in the same places and with the same sign of λ as in $H_\lambda^{(p)}$. Hence the cycle means in $H_\lambda^{\circ(p)}$ have the same slopes as the corresponding cycle means in $H_\lambda^{(p)}$, but with zero offsets. When $s(\lambda) = r(h_\lambda)$ is computed by (45), the asymptotics at large and small λ is determined by the slopes only and yields the same expression as for $s^\circ(\lambda) = r(h_\lambda^\circ)$. □

3.4 Mean-payoff game oracles and reconstruction problems

Here we consider the problem of identifying all affine pieces that constitute the spectral function and computing the whole spectrum of (A, B) in the case when A and B have integer entries.

The result will be formulated in terms of calls to a mean-payoff game oracle (computing the value of a mean payoff game). Let us briefly describe what the mean-payoff games are and how they are related to our problem. For more precise information the reader may consult Akian et al. [5] and Dhingra, Gaubert [23], as well as Björklund, Vorobyov [11] and Zwick, Paterson [44].

It can be observed that the min-max function $A^\sharp B$ is also a dynamic operator of a zero-sum deterministic mean-payoff game, which also corresponds to the system $Ax \leqslant Bx$. A schematic example of such a game is given in Figure 3, left. Two players, named Max and Min, move a pawn on a bipartite digraph, whose nodes belong either to Max (□) or to Min (◯). In the beginning of the game, the pawn is at a node j of Min, and she has to move it to a node i of Max, paying to him $-a_{ij}$ (some real number). Then Max has to choose a node k of Min. While moving the pawn there, he receives b_{ik} from her. The game proceeds infinitely long, and the aim of Max (resp. Min) is to maximize (resp. minimize) the average payment per turn (meaning a pair of consecutive moves of Min and Max). It turns out that the game has a value, which depends on the starting node of Min. Moreover $r(A^\sharp B)$ equals the greatest value over all starting nodes (i.e., all nodes of Min).

The two-sided eigenproblem $Ax = \lambda + Bx$ can be represented as

$$\begin{pmatrix} A \\ \lambda + B \end{pmatrix} x \leqslant \begin{pmatrix} \lambda + B \\ A \end{pmatrix} x. \tag{67}$$

This is equivalent to $x \leqslant h_\lambda(x)$ where $h_\lambda(x) := (\lambda + A^\sharp Bx) \wedge (-\lambda + B^\sharp Ax)$ as above. Hence the problem $Ax = \lambda + Bx$ corresponds to a parametric mean-payoff game of special kind, with $2m$ nodes of Max and n nodes of Min, whose scheme is

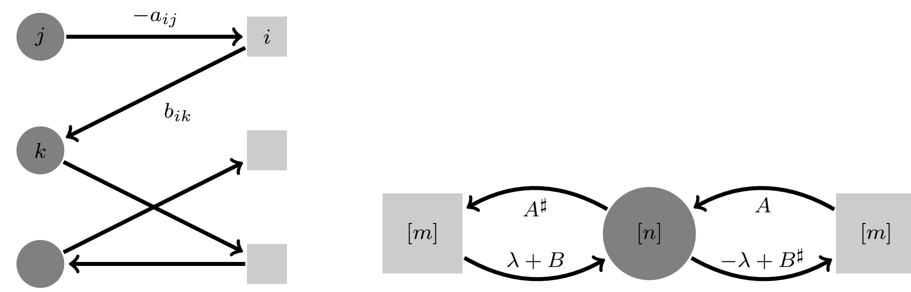

Fig. 3 General mean-payoff game (left) and mean-payoff game corresponding to $Ax = \lambda + Bx$ (right)

displayed on Figure 3, right, where individual nodes of the players are merged in three large groups.

Denoting by $\mathrm{MPG}(m,n,M)$ the worst-case execution time of any mean-payoff oracle computing $r(A^\sharp B)$, where $A, B \in \mathbb{R}_{\max}^{m\times n}$ have $-\infty$ entries and integer entries with the greatest absolute value M, we immediately obtain that for the same A and B we can find $s(0) = r(h)$ by calling that oracle, in no more than $\mathrm{MPG}(2m,n,M)$ operations.

The implementation of a mean-payoff oracle can rely on the policy iteration algorithm of [16,23], as well as the subexponential algorithm of [11] or the value iteration of [44]. Zwick and Paterson [44] showed that $\mathrm{MPG}(m,n,M)$ is pseudo-polynomial. We use this result below to demonstrate that the graph of spectral function $s(\lambda)$ can be reconstructed in pseudo-polynomial time.

Theorem 7 *Let $A, B \in \mathbb{R}_{\max}^{m\times n}$ have only $-\infty$ entries and integer entries with absolute value bounded by M. Denote $\kappa := \min(2m,n)$.*

1. *All affine pieces that constitute the function $s(\lambda)$ and hence the spectrum of (A,B) can be identified in no more than $\Delta(A,B)O(\kappa^6)$ calls to the mean-payoff game oracle, whose worst-case complexity is $\mathrm{MPG}(2m,n,\kappa^2(M+4M\kappa^2))$. In particular, the reconstruction can be done in pseudo-polynomial time.*
2. *When A and B have no $-\infty$ columns, the number of calls needed to reconstruct the function $s(\lambda)$ can be decreased to $\Delta(A,B)O(\kappa^5)$, where each call takes no more than $\mathrm{MPG}(2m,n,\kappa^2(M+4M\kappa)$ operations. When A and B are real, the number of calls is decreased to $(\overline{C}(A,B)-\underline{C}(A,B))O(\kappa^4)$, and the complexity of each call to $\mathrm{MPG}(2m,n,3M\kappa^2)$ operations.*

Proof In all cases we have a finite interval L of reconstruction, determined by the asymptotics of $s(\lambda)$. Using Theorem 6 , we obtain that in case 1 this is

$$L := [-2\kappa^2\Delta(A,B), 2\kappa^2\Delta(A,B)] \subseteq [-4\kappa^2 M, 4\kappa^2 M], \tag{68}$$

In case 2, this is

$$L := [-2\kappa\Delta(A,B), 2\kappa\Delta(A,B)] \subseteq [-4\kappa M, 4\kappa M] \tag{69}$$

when A and B do not have $-\infty$ columns, or

$$L := [\underline{C}(A,B), \overline{C}(A,B)] \subseteq [-2M, 2M] \tag{70}$$

when A and B do not have $-\infty$ entries.

We first compute the asymptotic slopes of $s(\lambda)$ outside L. By Proposition 8, we can do this by computing $s^{\circ}(\pm 1)$ in just two calls to the oracle which computes it in no more than $\mathrm{MPG}(2m, n, 1)$ operations. Then the goal is to reconstruct all affine pieces which constitute $s(\lambda)$ in the interval L.

The affine pieces of $s(\lambda)$ correspond to the maximal cycle means in the matrices from the representation of $h_\lambda(x)$. The points where such affine pieces may intersect are given by

$$\frac{a_1 + k_1\lambda}{n_1} = \frac{a_2 + k_2\lambda}{n_2}, \tag{71}$$

where all parameters are integers and $1 \leqslant |k_1|, |k_2|, n_1, n_2 \leqslant \kappa$ by Lemma 3. This implies

$$\lambda = \frac{a_1 n_2 - a_2 n_1}{k_2 n_1 - k_1 n_2} \tag{72}$$

The denominators of these points range from $-\kappa^2$ to κ^2, hence their number is $|L|O(\kappa^4)$ where $|L|$ is the length of the reconstruction interval L. We reconstruct the whole spectral function by calculating $s(\lambda)$ at these points, since there is only one affine piece of $s(\lambda)$ between them.

Using (68), (69) and (70) we obtain that the absolute value of the entries of A and $\lambda + B$ at each call does not exceed $M + 4\kappa^2 M$ in case 1, and $M + 4\kappa M$ or $M + 2M$ in case 2. Multiplying the entries of A and $\lambda + B$ by the denominator of λ which does not exceed κ^2, we obtain a problem with integer costs, where all maximum cycle means $r(H_\lambda^{(p)})$ get multiplied by that denominator, and hence $s(\lambda)$ gets multiplied by that denominator as well. Thus we can solve this mean-payoff game instead of the initial one. In case 1, the new integer problem can be resolved by the mean-payoff oracle in $\mathrm{MPG}(2m, n, \kappa^2(M + 4\kappa^2 M))$ operations. In case 2, it takes no more than $\mathrm{MPG}(2m, n, \kappa^2(M + 4\kappa M))$ operations when A and B do not have $-\infty$ columns, and no more than $\mathrm{MPG}(2m, n, 3M\kappa^2)$ operations when A and B do not have $-\infty$ entries. The proof is complete. $\square$

Since $\mathrm{spec}(A, B)$ is the zero set of $s(\lambda)$, we can identify $\mathrm{spec}(A, B)$ by reconstructing $s(\lambda)$ in the intervals given by Proposition 7 or more generally, Theorem 6. However, the task of reconstructing spectrum of (A, B) as zero-level set is even more simple, by the following arguments.

Theorem 8 *Let $A, B \in \mathbb{R}_{\max}^{m \times n}$ have only integer or $-\infty$ entries.*

1. *In general, the identification of $\mathrm{spec}(A, B)$ requires no more than $MO(\kappa^3)$ calls to the mean-payoff game oracle, whose worst-case complexity is $\mathrm{MPG}(2m, n, 2\kappa(M + 2M\kappa))$. In particular, $\mathrm{spec}(A, B)$ can be identified in pseudo-polynomial time.*
2. *If A and B have no $-\infty$ columns, then the number of calls to the oracle needed to identify $\mathrm{spec}(A, B)$ does not exceed $(\bigvee_i (B^\sharp A0)_i + \bigvee_i (A^\sharp B0)_i) O(\kappa^2)$, and the complexity of the oracle does not exceed $\mathrm{MPG}(2m, n, 6M\kappa)$ operations.*

Proof We have to reconstruct the zero-level set of $s(\lambda)$, within a finite interval L of reconstruction. In case 1, we notice that the intersection of $s(\lambda$ with zero level can occur only at points with absolute value not exceeding $2M\kappa$ (since $s(\lambda)$ consists of affine pieces $(a + k\lambda)/l$ where $|a| \leqslant 2M\kappa$). Hence in case 1

$$L := [-2M\kappa, 2M\kappa]. \tag{73}$$

In case 2 we use the bounds of Proposition 7:

$$L := [-\bigvee_i (A^\sharp B0)_i, \bigvee_i (B^\sharp A0)_i] \subseteq [-2M, 2M] \tag{74}$$

when A and B do not have $-\infty$ columns. In case 1, we also need to check the asymptotics of $s(\lambda)$ outside the interval, for which we check $s^\circ(\pm 1) = 0$ (i.e., $s^\circ(\pm 1) \geqslant 0$ which takes no more than $\mathrm{MPG}(2m, n, 1)$ operations).

The absolute value of entries of A and $\lambda + B$ does not exceed $M + 2M\kappa$ in case 1 and $M + 2M$ in case 2. We have to check $s(\lambda) = 0$ (i.e., $s(\lambda) \geqslant 0$) at all possible intersections of affine pieces constituting $s(\lambda)$ with zero, i.e., at the points $\lambda = a/k$ within L, such that a and k are integers and $k \leqslant \kappa$. We also may have to check $s(\lambda) = 0$ for one intermediate point between each pair of neighbouring points λ_1 and λ_2 such that $s(\lambda_1) = s(\lambda_2) = 0$. If it holds then $s(\lambda) = 0$ holds for the whole interval, and if it does not then it holds only at the ends. Note that such an intermediate point for a_1/k_1 and a_2/k_2 can be chosen as $(a_1 + a_2)/(k_1 + k_2)$ thus leading to $k \leqslant 2\kappa$.

Multiplying all the entries by k yields a mean-payoff game with integer costs, for which we check whether the value is nonnegative. This takes no more than $\mathrm{MPG}(2m, n, 2\kappa(M + 2M\kappa))$ in Case 1 and $\mathrm{MPG}(2m, n, 2\kappa \times 3M)$ in Case 2, with the number of calls not exceeding $|L|O(\kappa^2)$. ◻

Note that this theorem uses the oracles checking $s(\lambda) \geqslant 0$, not requiring to compute the exact value.

We can also formulate a certificate that λ is an end (left or right) of a spectral interval.

Proposition 9 *Supose that $s(\lambda^*) \geqslant 0$. Then λ^* is the left (resp., the right) end of an interval of $\mathrm{spec}(A, B)$ if and only if there exists a representing matrix $H_\lambda^{(p)}$ where the weights of all cycles are nonpositive, and the slopes of all cycles with zero weight are strictly positive (resp., negative).*

Proof Condition $s(\lambda^*) \geqslant 0$ assures that $\lambda^* \in \mathrm{spec}(A, B)$. We will consider the left end case, for the right end the argument is similar. Recall that $s(\lambda^*) := r(h_{\lambda^*}$ admits an inf-representation (48) with selection property. Since there is only finite number of representing matrices, there exists $H_\lambda^{(p)}$ such that $s(\lambda) = r(H_\lambda^{(p)}$ for $\lambda \in [\lambda^* - \epsilon, \lambda^*]$ (for the right end, we would consider $\lambda \in [\lambda^*, \lambda^* + \epsilon]$). Then λ^* is the left end of a spectral interval if and only if $r(H_{\lambda^*}^{(p)}) = 0$ but $r(H_\lambda^{(p)}) < 0$ for $\lambda \in [\lambda^* - \epsilon, \lambda^*)$. After applying the definition of $H_\lambda^{(p)}$ (46) and the maximum cycle mean formula for $r(H_\lambda^{(p)})$ (41), the claim follows. ◻

Observe that the condition in Proposition 9 can be verified in polynomial time for a given $H^{(p)}$. Namely, it suffices to compute the maximum cycle mean, identify the *critical subgraph* consisting of all cycles with zero cycle mean, and solve the maximum cycle mean problem for that subgraph, with the edges weighted by 1 or -1 according to the choice of λ or $-\lambda$ in (46).

The reconstruction of spectral function has been implemented in MATLAB, also to generate Figures 1 and 4.

3.5 Examples of analytic computation

In this section we consider two particular situations when the spectral function can be constructed analytically. The first example shows that any asymptotics k/l, where $l = 1, \ldots, m$ and $k = l - 2t$ for $t = 1, \ldots, l$, can be realized. The second example is taken from [41], and it shows that any system of intervals and points on the real line can be represented as spectrum of a max-plus two-sided eigenproblem.

Asymptotic slopes. In our first example we consider pairs of matrices $A^{m,l} \in \mathbb{R}_{\max}^{m\times m}, B^{m,l} \in \mathbb{R}_{\max}^{m\times m}$ with entries in $\{0, -\infty\}$, where $0 \leqslant l \leqslant \lfloor m \rfloor$. An intuitive idea is to make some "exchange" between the max-plus identity matrix and some cyclic permutation matrix. For instance

$$A^{6,2} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 & \cdot \end{pmatrix}, B^{6,2} = \begin{pmatrix} 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 \end{pmatrix}, \tag{75}$$

where the dots denote $-\infty$ entries.

Formally, $A^{m,l} = (a_{ij}^{m,l})$ are defined as matrices with $\{0, -\infty\}$ entries such that $a_{ij}^{m,l} = 0$ for $i = 1$ and $j = m$, or $i = j + 1$ where $2l < i \leqslant m$, or $i = j = 2k$ where $1 \leqslant k \leqslant l$, or $i = 2k + 1$ and $j = 2k$, where $1 \leqslant k < l$, and $a_{ij}^{m,l} = -\infty$ otherwise.

Similarly, $B^{m,l} = (b_{ij}^{m,l})$ are defined as matrices with entries in $\{0, -\infty\}$ such that $b_{ij}^{m,l} = 0$ for $i = j$ where $2l < i \leqslant m$, or $i = j = 2k - 1$ where $1 \leqslant k \leqslant l$, or $i = 2k$ and $j = 2k - 1$, where $1 \leqslant k \leqslant l$, and $b_{ij}^{m,l} = -\infty$ otherwise.

Proposition 10 *The spectral function associated with $A^{m,l}, B^{m,l}$ consists of two linear pieces: $s(\lambda) = \lambda(m - 2l)/m$ for $\lambda \leqslant 0$ and $s(\lambda) = -\lambda(m - 2l)/m$ for $\lambda \geqslant 0$.*

Proof Let us introduce yet another matrix $C^{m,l}(\lambda) = (c_{ij}^{m,l}(\lambda)) \in \mathbb{R}_{\max}^{m\times m}$. Informally, it is a sum of a $\{0, -\infty\}$ permutation (circulant) matrix and its inverse, weighted by $\pm\lambda$. This pattern corresponds to the above mentioned "exchange" in the construction of $A^{m,l}$ and $B^{m,l}$. In particular, (75) corresponds to

$$C^{6,2} = \begin{pmatrix} \cdot & -\lambda & \cdot & \cdot & \cdot & -\lambda \\ \lambda & \cdot & \lambda & \cdot & \cdot & \cdot \\ \cdot & -\lambda & \cdot & -\lambda & \cdot & \cdot \\ \cdot & \cdot & \lambda & \cdot & \lambda & \cdot \\ \cdot & \cdot & \cdot & -\lambda & \cdot & \lambda \\ \lambda & \cdot & \cdot & \cdot & -\lambda & \cdot \end{pmatrix}. \tag{76}$$

Defining formally, $c_{1,m}^{m,l} = -\lambda$, $c_{m,1}^{m,l} = \lambda$, and

$$c_{ij}^{m,l} = \begin{cases} \operatorname{sign}(i,j)\lambda, & \text{if } 1 \leqslant i, j \leqslant m \text{ and } |j - i| = 1, \\ -\infty, & \text{otherwise,} \end{cases} \tag{77}$$

where

$$\operatorname{sign}(i,j) = \begin{cases} 1, & j - 1 = i \geqslant 2l \text{ or } j \pm 1 = i = 2k \leqslant 2l, \\ -1, & i - 1 = j \geqslant 2l \text{ or } i \pm 1 = j = 2k \leqslant 2l. \end{cases} \tag{78}$$

Observe that the pairs (i,j) and (j,i) for $j=i+1$ (and also $(1,m)$ and $(m,1)$) have the opposite sign.

It can be shown that each representing max-plus matrix of the min-max function

$$h_\lambda^{m,l}(x)=(\lambda+(A^{m,l})^\sharp B^{m,l}x)\wedge(-\lambda+(B^{m,l})^\sharp A^{m,l}x) \tag{79}$$

is choosing one of the two entries in each row of $C^{m,l}(\lambda)$. The matrices can be classified according to this choice as follows (see (76) for example):
1. Choose $(m,1)$, and $(i,i+1)$ for $i=1,\ldots,m-1$;
2. Choose $(1,m)$, and $(i,i-1)$ for $i=2,\ldots,m$;
3. Choose both $(m,1)$ and $(1,m)$, or both $(i-1,i)$ and $(i,i-1)$ for some $i=2,\ldots,n$.
The first two strategies give just one matrix each, with the (maximum) cycle means $\lambda(m-2l)/m$ and $-\lambda(m-2l)/m$. The rest of the representing matrices are described by 3., and it follows that their maximum cycle means are always greater than or equal to 0. Hence $s(\lambda)=\lambda(m-2l)/m\wedge-\lambda(m-2l)/m$. □

The spectrum of two-sided eigenproblem. Now we consider an example of [41]. Let us define $A\in\mathbb{R}_{\max}^{2\times 3t}$, $B\in\mathbb{R}_{\max}^{2\times 3t}$:

$$\begin{aligned} A&=\begin{pmatrix}\ldots & a_i & b_i & c_i & \ldots\\ \ldots & 2a_i & 2b_i & 2c_i & \ldots\end{pmatrix},\\ B&=\begin{pmatrix}\ldots & 0 & 0 & 0 & \ldots\\ \ldots & a_i & c_i & b_i & \ldots\end{pmatrix},\end{aligned} \tag{80}$$

where $a_i\leqslant c_i<a_{i+1}$ for $i=1,\ldots,t-1$, where $b_i:=\frac{a_i+c_i}{2}$. The following result describes $\operatorname{spec}(A,B)$.

Theorem 9 ([41]) *With A,B defined by (80),*

$$\operatorname{spec}(A,B)=\bigcup_{i=1}^{t}[a_i,c_i]. \tag{81}$$

To calculate $s(\lambda)$, which is a more general task, one can study the representing matrices like in the previous example. Another way is to guess, for each λ, a finite eigenvector of $P_DP_{C(\lambda)}$ and then $s(\lambda)$ is the corresponding eigenvalue. By this method we obtained that:

$$s(\lambda)=\begin{cases}\lambda-a_1, & \text{if } \lambda\leqslant a_1,\\ 0, & \text{if } a_k\leqslant\lambda\leqslant c_k,\ k=1,\ldots,t,\\ \max(c_k-\lambda,\lambda-a_{k+1}), & \text{if } c_k\leqslant\lambda\leqslant a_{k+1},\ k=1,\ldots,t-1,\\ c_t-\lambda, & \text{if } \lambda\geqslant c_t.\end{cases} \tag{82}$$

More precisely, it can be shown that the following vectors are eigenvectors of $P_DP_{C(\lambda)}$:

$$y^\lambda=\begin{cases}(0\quad a_1\ 0\ a_1), & \text{if } \lambda\leqslant a_1,\\ (0\quad \lambda+b_k-a_k\ 0\ \lambda+b_k-a_k), & \text{if } a_k\leqslant\lambda\leqslant b_k,\ k=1,\ldots,t,\\ (0\quad c_k\ 0\ c_k), & \text{if } b_k\leqslant\lambda\leqslant c_k,\ k=1,\ldots,t,\\ (0\quad \lambda\ 0\ \lambda), & \text{if } c_k\leqslant\lambda\leqslant a_{k+1},\ k=1,\ldots,t-1,,\\ (0\quad c_t\ 0\ c_t)^T, & \text{if } \lambda\geqslant c_t,\end{cases} \tag{83}$$

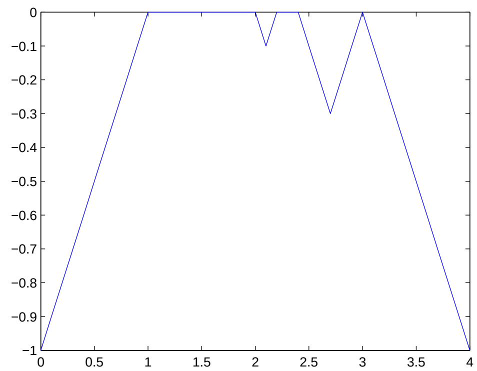

Fig. 4 The spectral function of A and B in (84)

with the eigenvalues expressed by (82).

We can also conclude that in this case $-r(A^\sharp B) = a_1$ and $r(B^\sharp A) = c_t$. Indeed, by (82), $s(\lambda) = \lambda - a_1$ for $\lambda \leqslant a_1$ and $s(\lambda) = c_t - \lambda$ for $\lambda \geqslant c_t$. Comparing this with the result of Theorem 6, part 3, we get the claim.

As a_1 and c_t are eigenvalues, the last result shows that the bounds given in Theorem 5 cannot be improved in general.

For example, take $t = 3$, $[a_1,\ c_1] = [1,\ 2]$, $[a_2,\ c_2] = [2.2,\ 2.4]$ and $[a_3,\ c_3] = [3,\ 3]$. Then

$$
\begin{aligned}
A &= \begin{pmatrix} 1 & 1.5 & 2 & 2.2 & 2.3 & 2.4 & 3 \\ 2 & 3 & 4 & 4.4 & 4.6 & 4.8 & 6 \end{pmatrix}, \\
B &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1.5 & 2.2 & 2.4 & 2.3 & 3 \end{pmatrix}
\end{aligned} \tag{84}
$$

The spectral function is shown on Figure 4. Note that this is the least Lipschitz function with a given zero-level set. The same observation holds for the general case (82).

4 Conclusions

We have developed a new approach to the two-sided eigenproblem $A \otimes x = \lambda \otimes B \otimes x$ in max-plus linear algebra, based on parametric min-max functions. This yields a reduction to mean-payoff games problems, for which a number of algorithms have already been developed. We introduced the concept of spectral function $s(\lambda)$, defined as the greatest eigenvalue of the associated parametric min-max function (or the greatest value of the associated mean-payoff game). We showed that $s(\lambda)$ has a natural geometric sense being equal to the inverse of the least Chebyshev distance between $A \otimes x$ and $B \otimes x$. The spectrum of (A, B) can be regarded as the zero-level set of the spectral function, which is a 1-Lipschitz function consisting of a finite number of affine pieces. These pieces can be reconstructed in pseudopolynomial time, hence the spectrum of (A, B) can be also effectively identified.

A similar approach can be used in max-plus linear programming [26]. Spectral functions of a different type are used in the decision procedure associated with the tropical Farkas lemma in [8], allowing one to check whether a max-plus inequality can be logically deduced from other max-plus inequalities. The present approach can be generalized to the case when the entries of A and B are general piecewise-affine functions of λ [43], but the case of many parameters would be even more interesting. Such development could lead to practical applications in scheduling and design of asynchronous circuits. Also note that the parametric tropical systems are equivalent to parametric mean-payoff games, directing to useful stochastic and infinite-dimensional generalizations.

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