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Basic solutions of systems with two max-linear inequalities ☆

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ABSTRACT

We give an explicit description of the basic solutions of max-linear systems $A \otimes x \leqslant B \otimes x$ with two inequalities.

1. Introduction

We consider systems of two max-plus linear inequalities

$$\begin{aligned} a_{11} \otimes x_1 \oplus \cdots \oplus a_{1n} \otimes x_n &\leqslant b_{11} \otimes x_1 \oplus \cdots \oplus b_{1n} \otimes x_n, \\ a_{21} \otimes x_1 \oplus \cdots \oplus a_{2n} \otimes x_n &\leqslant b_{21} \otimes x_1 \oplus \cdots \oplus b_{2n} \otimes x_n. \end{aligned} \tag{1}$$

Here $\otimes := +$, $\oplus := \max$, and $a_{ij}, b_{ij}, x_j \in \mathbb{R} \cup \{-\infty\}$ for $i = 1, 2$ and $j = 1, \ldots, n$.

General systems of max-linear inequalities (equivalently, equalities) were tackled by Butkovič and Hegedüs [3] who established an elimination method for finding basic solutions of such systems, starting with basic solutions of just one equation or inequality and adding all other constraints one by one. This

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algorithm served as a proof that solution sets to max-linear systems have finite bases, and it did not seem to be efficient enough for practical implementation. But at present, Allamegeon et al. [1] have come up with a novel approach to the scheme of Butkovič and Hegedüs [3], in which every step of adding new constraint is dramatically improved by using a max-plus analogue of double description method. Based on a certain criterion of minimality [2,5,6,8], they derive a combinatorial criterion in terms of hypergraphs which allows to efficiently test whether a generator is extremal.

The idea of the present paper is that when the number of inequalities is small, the basic solutions can be written out explicitly. However as shown by Wagneur et al. [11], even in the case of two inequalities (1) the number of generators is large and the problem to establish a systematic classification and to resolve the extremality by writing out explicit conditions is nontrivial. This goal is achieved in the present paper as follows.

Firstly, we represent the set of all solutions as the union of cones generated by certain Kleene stars (Section 2). The same approach is used by Truffet [9], where such decomposition is obtained for solutions to general systems of max-plus linear inequalities. We remark that this is closely related to the Develin–Sturmfels Cellular Decomposition approach in tropical convexity [4,8]. De la Puente [7] uses the same kind of decomposition for precise analysis of tropical linear mappings on the plane.

Secondly, in the main part of our paper we select basic solutions by means of the above mentioned criterion of minimality [2,5,6] (Section 3). This criterion is called call the multiorder principle following [8]. We achieve an explicit description of basic solutions and a procedure which finds all of them in no more than $O(n^3)$ operations.

2. Gathering the generators

2.1. General background in max algebra and particular Kleene stars

We work with the analogue of linear algebra developed over the max-plus semiring $\mathbb{R}_{\max,+}$ which is the set of real numbers with adjoined minus infinity $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$ equipped with the operations of "addition" $a \oplus b := \max(a, b)$ and "multiplication" $a \otimes b := a + b$. Zero $\mathbf{0}$ and unity $\mathbf{1}$ of this semiring are equal, respectively, to $-\infty$ and 0. The operations of the semiring are extended to the nonnegative matrices and vectors in the same way as in conventional linear algebra. That is if $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$ are matrices of compatible sizes with entries from $\overline{\mathbb{R}}$, we write $C = A \oplus B$ if $c_{ij} = a_{ij} \oplus b_{ij}$ for all i, j and $C = A \otimes B$ if $c_{ij} = \bigoplus_k a_{ik} \otimes b_{kj} = \max_k(a_{ik} \otimes b_{kj})$ for all i, j. The notation $\otimes$ will be often omitted.

The main geometrical object of this max-plus linear algebra is a subset $K \subseteq \overline{\mathbb{R}}^n$ closed under the operations of componentwise maximization $\oplus$ and "multiplication" $\otimes$ by scalars (which means addition in the conventional sense). Such subsets are called *max-plus cones* or just *cones* if there is no mix up with the ordinary convexity.

A vector $x \in \overline{\mathbb{R}}^n$ is a (*max-linear*) *combination* of $y_1, \ldots, y^m \in S$ if there exist scalars $\alpha_1, \ldots, \alpha_m \in \overline{\mathbb{R}}$ such that $x = \bigoplus_{i=1}^m \alpha_i y^i$. A max-plus cone $K \subseteq \overline{\mathbb{R}}^n$ is *generated* by $y^1, \ldots, y^m \in S$ if each $x \in S$ is a max-linear combination of $y^1, \ldots, y^m$. When vectors arise as columns (resp. rows) of matrices, it will be convenient to represent them as max-linear combinations of the *column unit vectors*

$$e_i = (\overbrace{\mathbf{0} \ldots \mathbf{0}}^{i-1}\ \mathbf{1}\, \mathbf{0} \ldots \mathbf{0})', \tag{2}$$

respectively, the *row unit vectors* e_i', which are their transpose.

The following series is called the *Kleene star* of A:

$$A^* = I \oplus A \oplus A^2 \ldots, \tag{3}$$

where I is the max-plus unity matrix, which has all diagonal entries $\mathbf{1}$ and all off-diagonal entries $\mathbf{0}$. When A^* has finite entries (in other words, converges) it is easily shown that $A \otimes x \leqslant x$ is equivalent to $A^* \otimes x = x$. We also have the following

Proposition 1. *Let $A \in \overline{\mathbb{R}}^{n\times n}$ be such that A^* has finite entries. Then $\{x \mid A\otimes x \leqslant x\}$ is generated by the columns of A^*.*

The following two particular observations will be most important. In the formulation we use the *row unit vectors* e'_i. We denote by $A_{i\cdot}$, resp. $A_{\cdot i}$, the ith row, resp. the ith column, of A.

Proposition 2. *Given $k \in \{1,\dots,n\}$, let $A \in \overline{\mathbb{R}}^{n\times n}$ have rows*

$$A_{i\cdot} = \begin{cases} e'_k \oplus \bigoplus_{l\neq k} a_{kl}e'_l, & \text{if } i=k,\\ e'_i, \text{ otherwise}, \end{cases} \tag{4}$$

for $i = 1,\dots,n$. Then the set $\{x \mid A\otimes x \leqslant x\}$ is generated by the columns of A.

Proof. In this case $A^* = A$, after which Proposition 1 is applied. □

Proposition 3. *Given $k, m \in \{1,\dots,n\}$ such that $k \neq m$, let $A \in \overline{\mathbb{R}}^{n\times n}$ have rows*

$$A_{i\cdot} = \begin{cases} e'_k \oplus \bigoplus_{l\in L_1} a_{kl}e'_l, & \text{if } i=k,\\ e'_m \oplus \bigoplus_{l\in L_2} a_{ml}e'_l, & \text{if } i=m,\\ e'_i, \text{ otherwise}, \end{cases} \tag{5}$$

where $L_1 = \{l \neq k \mid a_{kl} \neq \mathbf{0}\}$, $L_2 = \{l \neq m \mid a_{ml} \neq \mathbf{0}\}$.

- *If $a_{km}a_{mk} \leqslant \mathbf{1}$ then $\{x \mid A\otimes x \leqslant x\}$ is generated by the columns of A^*.*
- *If $a_{km}a_{mk} > \mathbf{1}$ then $\{x \mid A\otimes x \leqslant x\}$ is generated by e_i for $i \notin L_1 \cup L_2 \cup \{k\} \cup \{m\}$.*

Proof. In the first case A^* is finite and we apply Proposition 1. For the second case observe that on the one hand, if $x_i \neq \mathbf{0}$ for some $i \in L_1 \cup L_2 \cup \{k\} \cup \{m\}$ then $x_k \neq \mathbf{0}$ and $x_m \neq \mathbf{0}$ which makes $A\otimes x \leqslant x$ impossible. On the other hand, any x such that $x_i = \mathbf{0}$ for all $i \in L_1 \cup L_2 \cup \{k\} \cup \{m\}$ satisfies $A\otimes x \leqslant x$. □

2.2. Extracting generating sets from Kleene stars

Now we write out a generating set for the solution set of (1), which we represent as a union of spans of certain Kleene stars. In the spirit of [11], we will have to introduce several index sets and distinguish between several special cases.

We denote $J_1 := \{i \mid a_{1i} \leqslant b_{1i},\ b_{1i} \neq \mathbf{0}\}$, $J_2 := \{i \mid a_{2i} \leqslant b_{2i},\ b_{2i} \neq \mathbf{0}\}$, $I_1 := \{i \mid a_{1i} > b_{1i}\}$ and $I_2 := \{i \mid a_{2i} > b_{2i}\}$. We also denote $\bar{I}_1 := \{1,\dots,n\}\backslash I_1$, $\bar{I}_2 := \{1,\dots,n\}\backslash I_2$, $K_1 = \{i \mid a_{1i} = b_{1i} = 0\}$ and $K_2 = \{i \mid a_{2i} = b_{2i} = 0\}$. Observe that $\bar{I}_1 = J_1 \cup K_1$ and $\bar{I}_2 = J_2 \cup K_1$, and that $I_1 \cup J_1 \cup K_1 = I_2 \cup J_2 \cup K_2 = \{1,\dots,n\}$. Using the cancellation law

$$ax \oplus b \leqslant cx \oplus d \Leftrightarrow \begin{cases} b \leqslant cx \oplus d, & \text{if } a \leqslant c,\\ ax \oplus b \leqslant d, & \text{if } a > c, \end{cases} \tag{6}$$

system (1) can be rewritten as

$$\begin{aligned} \bigoplus_{i\in I_1} a_{1i}x_i &\leqslant \bigoplus_{i\in J_1} b_{1i}x_i,\\ \bigoplus_{i\in I_2} a_{2i}x_i &\leqslant \bigoplus_{i\in J_2} b_{2i}x_i. \end{aligned} \tag{7}$$

The solution set to (7) is the union of S^{kl} defined by

$$S^{kl} = \{x \mid \bigoplus_{i \in I_1} a_{1i}x_i \leqslant b_{1k}x_k,\ \bigoplus_{i \in I_2} a_{2i}x_i \leqslant b_{2l}x_l\}, \tag{8}$$

for $k \in J_1$ and $l \in J_2$. Further we represent S^{kl} defined by (8) in the form

$$S^{kl} = \{x \mid A^{kl} \otimes x \leqslant x\}, \tag{9}$$

where we have to describe A^{kl}. There are two cases: $k = l$ and $k \neq l$. We denote $\gamma^1_{ki} := b^{-1}_{1k}a_{1i}$ for $k \in J_1$ and $\gamma^2_{ki} := b^{-1}_{2k}a_{2i}$ for $k \in J_2$.

If $k = l$, then the kth row of A^{kl} is

$$e'_k \oplus \bigoplus_{i \in I_1 \cap \bar{I}_2} \gamma^1_{ki}e'_i \oplus \bigoplus_{i \in \bar{I}_1 \cap I_2} \gamma^2_{ki}e'_i \oplus \bigoplus_{i \in I_1 \cap I_2} (\gamma^1_{ki} \oplus \gamma^2_{ki})e'_i. \tag{10}$$

and all other rows are row unit vectors.

If $k \neq l$ then the kth and the lth rows of A^{kl} are given by

$$e'_k \oplus \bigoplus_{i \in I_1} \gamma^1_{ki}e'_i, \quad e'_l \oplus \bigoplus_{i \in I_2} \gamma^2_{li}e'_i, \tag{11}$$

all other rows being row unit vectors.

Now we collect the generators of $\{x \mid A^{kl} \otimes x \leqslant x\}$ considering several special cases:

(1) $k = \ell \in J_1 \cap J_2$.
(2) $k \in J_1 \cap \bar{I}_2, l \in J_2 \cap \bar{I}_1$.
(3) $k \in J_1 \cap I_2, l \in J_2 \cap \bar{I}_1$.
(4) $k \in J_1 \cap \bar{I}_2, l \in J_2 \cap I_1$.
(5) $k \in J_1 \cap I_2, l \in J_2 \cap I_1$.

Case 1. The kth row of A^{kl} is given by (10) and all other rows of A^{kl} are unit vectors. By Proposition 2, S^{kl} is generated by the columns of A^{kl}. These are:

$$\begin{aligned} &e_i,\ \text{for } i \in \bar{I}_1 \cap \bar{I}_2, \\ &\gamma^1_{ki}e_k \oplus e_i,\ \text{for } i \in I_1 \cap \bar{I}_2, \\ &\gamma^2_{li}e_l \oplus e_i,\ \text{for } i \in \bar{I}_1 \cap I_2, \\ &(\gamma^1_{ki} \oplus \gamma^2_{ki})e_k \oplus e_i,\ \text{for } i \in I_1 \cap I_2, \end{aligned} \tag{12}$$

where $k = l \in J_1 \cap J_2$.

Case 2. Rows k and l of A^{kl} are given by (11), all other rows being the unit vectors. As $l \in \bar{I}_1$ and $k \in \bar{I}_2$, we obtain $A^{kl}_{kl} = \gamma^1_{kl} = \mathbf{0}$ and $A^{kl}_{lk} = \gamma^2_{lk} = \mathbf{0}$ and hence $(A^{kl})^* = A^{kl}$. Taking transpose of (11), we obtain the columns of $(A^{kl})^* = A^{kl}$. By Proposition 3 part 1. they generate S^{kl}:

$$\begin{aligned} &e_i,\ \text{for } i \in \bar{I}_1 \cap \bar{I}_2, \\ &\gamma^1_{ki}e_k \oplus e_i,\ \text{for } i \in I_1 \cap \bar{I}_2, \\ &\gamma^2_{li}e_l \oplus e_i,\ \text{for } i \in \bar{I}_1 \cap I_2, \\ &\gamma^1_{ki}e_k \oplus \gamma^2_{li}e_l \oplus e_i,\ \text{for } i \in I_1 \cap I_2, \end{aligned} \tag{13}$$

where $k \in J_1 \cap \bar{I}_2$ and $l \in J_2 \cap \bar{I}_1$.

Case 3. Rows k and l of A^{kl} are given by (11). However, $(A^{kl})^* \neq A^{kl}$, since $k \in I_2$ implies that $A^{kl}_{lk} = \gamma^2_{lk} \neq \mathbf{0}$. Note that $(A^{kl})^*$ is always finite, since $A^{kl}_{kl} = \mathbf{0}$ implies that the digraph associated with A^{kl} does not contain any cycles with nonzero weight except for the loops (i, i). For $i \in I_1$, we obtain $(A^{kl})^*_{li} = \gamma^2_{li} \oplus \gamma^2_{lk}\gamma^1_{ki}$. More precisely, $(A^{kl})^*_{li} = \gamma^2_{lk}\gamma^1_{ki}$ for $i \in I_1 \cap \bar{I}_2$, and $(A^{kl})^*_{li} = \gamma^2_{li} \oplus \gamma^2_{lk}\gamma^1_{ki}$ for $i \in I_1 \cap I_2$. The lth row of $(A^{kl})^*$ is given by

$$e'_l \oplus \bigoplus_{i\in\bar{I}_2\cap I_1} \gamma^2_{lk}\gamma^1_{ki}e'_i \oplus \bigoplus_{i\in I_2\cap I_1} (\gamma^2_{li} \oplus \gamma^2_{lk}\gamma^1_{ki})e'_i \oplus \bigoplus_{i\in\bar{I}_1\cap I_2} \gamma^2_{li}e'_i. \tag{14}$$

The kth row of $(A^{kl})^*$ is the same as in (11) and all other rows are unit vectors. We obtain the columns of $(A^{kl})^*$:

$$\begin{aligned}
&e_i, \text{ for } i \in \bar{I}_1 \cap \bar{I}_2,\\
&e_i \oplus \gamma^1_{ki}e_k \oplus \gamma^2_{lk}\gamma^1_{ki}e_l, \text{ for } i \in \bar{I}_2 \cap I_1,\\
&e_i \oplus \gamma^1_{ki}e_k \oplus (\gamma^2_{li} \oplus \gamma^2_{lk}\gamma^1_{ki})e_l, \text{ for } i \in I_1 \cap I_2,\\
&e_i \oplus \gamma^2_{li}e_l, \text{ for } i \in \bar{I}_1 \cap I_2,
\end{aligned} \tag{15}$$

where $k \in J_1 \cap I_2$ and $l \in J_2 \cap \bar{I}_1$.

Case 4. Rows k and l are given by (11), and by analogy with Case 3 we obtain that the lth row of $(A^{kl})^*$ is the same as in (11), but the kth row is given by

$$e'_k \oplus \bigoplus_{i\in\bar{I}_1\cap I_2} \gamma^1_{kl}\gamma^2_{li}e'_i \oplus \bigoplus_{i\in I_2\cap I_1} (\gamma^1_{ki} \oplus \gamma^1_{kl}\gamma^2_{li})e'_i \oplus \bigoplus_{i\in\bar{I}_2\cap I_1} \gamma^1_{ki}e'_i. \tag{16}$$

We obtain the columns of $(A^{kl})^*$:

$$\begin{aligned}
&e_i, \text{ for } i \in \bar{I}_1 \cap \bar{I}_2,\\
&e_i \oplus \gamma^2_{li}e_l \oplus \gamma^1_{kl}\gamma^2_{li}e_k, \text{ for } i \in \bar{I}_1 \cap I_2,\\
&e_i \oplus \gamma^2_{li}e_l \oplus (\gamma^1_{ki} \oplus \gamma^1_{kl}\gamma^2_{li})e_k, \text{ for } i \in I_1 \cap I_2,\\
&e_i \oplus \gamma^1_{ki}e_k, \text{ for } i \in \bar{I}_2 \cap I_1.
\end{aligned} \tag{17}$$

where $k \in J_1 \cap \bar{I}_2$ and $l \in J_2 \cap I_1$.

Case 5. If $\gamma^2_{lk}\gamma^1_{kl} \leqslant \mathbf{1}$, then the lth row of $(A^{kl})^*$ is given by (14) and the kth row of $(A^{kl})^*$ is given by (16). By Proposition 3 part 1 the columns of $(A^{kl})^*$ generate S^{kl}. If $\gamma^2_{lk}\gamma^1_{kl} > \mathbf{1}$, then by Proposition 3 part 2, S^{kl} is generated by e_i for $i \in \bar{I}_1 \cap \bar{I}_2$. If $\gamma^2_{lk}\gamma^1_{kl} \leqslant \mathbf{1}$, then $(A^{kl})^*$ is finite and its columns are:

$$\begin{aligned}
&e_i, \text{ for } i \in \bar{I}_1 \cap \bar{I}_2,\\
&e_l \oplus \gamma^1_{kl}e_k, \ e_k \oplus \gamma^2_{lk}e_l,\\
&e_i \oplus \gamma^2_{li}e_l \oplus \gamma^1_{kl}\gamma^2_{li}e_k, \text{ for } i \in \bar{I}_1 \cap I_2,\\
&e_i \oplus (\gamma^2_{li} \oplus \gamma^2_{lk}\gamma^1_{ki})e_l \oplus (\gamma^1_{ki} \oplus \gamma^1_{kl}\gamma^2_{li})e_k, \text{ for } i \in I_1 \cap I_2,\\
&e_i \oplus \gamma^1_{ki}e_k \oplus \gamma^2_{lk}\gamma^1_{ki}e_l, \text{ for } i \in \bar{I}_2 \cap I_1,
\end{aligned} \tag{18}$$

where $k \in J_1 \cap I_2$ and $l \in J_2 \cap I_1$.

In equations (12), (13), (15), (17), (18), we identified all the generators of the max-plus cone of solutions to (1). In section 3 below we show how to identify the subset of *independent* generators.

3. Identifying the basic solutions

3.1. Multiorder principle

A set $S \subseteq \overline{\mathbb{R}}^n$ is said to be *independent* if no vector in this set is generated by other vectors in this set. If such independent set generates a cone K then it is called a *basis* of K. It can be shown [2,10] that if a basis of K exists, then it consists of all *extremals* (normalized in some sense): a vector $x \in K$ is an extremal if $x = y \oplus z$ and $y, z \in K$ imply $y = x$ or $z = x$. This also means that the basis of any cone is essentially unique: any two bases are obtained from each other by multiplying their elements by scalars. Actually any finitely generated cone has a basis [2,4,10].

The notion of extremal defined above is a max-plus analogue of the notion of extremal ray (or extremal) of a convex cone. It is also a special case of the join irreducible element of a lattice.

The notion of extremal is most conveniently expressed by the following multiorder principle [1,2, 5,6,8] which we formulate here only for the finitely generated case. For any $i = 1, \ldots, n$ we introduce the relation

$$x \leqslant_i y \Leftrightarrow xx_i^{-1} \leqslant yy_i^{-1}, \quad x_i \neq \mathbf{0} \text{ and } y_i \neq \mathbf{0}. \tag{19}$$

A vector $y \in K$ minimal with respect to $\leqslant_i$ will be called *i-minimal*. Define the *support* of $y \in \overline{\mathbb{R}}^n$ by $\operatorname{supp}(y) := \{i \mid y_i \neq \mathbf{0}\}$.

Proposition 4 (Multiorder Principle)**.** *Let $K \subseteq \overline{\mathbb{R}}^n$ be generated by a finite set $S \subseteq \overline{\mathbb{R}}^n$. Then $y \in S$ belongs to the basis of K (equivalently, is an extremal of K) if and only if it is i-minimal for some $i \in \{1, \ldots, n\}$.*

Proof. If y is not i-minimal for any i, then for each $i \in \operatorname{supp}(y)$ there exists $z^i \in K$ such that $z^i \leqslant_i y$. Then it can be verified that

$$y = \bigoplus_{i \in \operatorname{supp}(y)} z^i (z_i^i)^{-1} y_i. \tag{20}$$

Conversely if $y = \bigoplus_k \alpha_k z^k$ for some $z^k \in S$, then for each $i \in \operatorname{supp}(y)$ there is $k(i)$ such that $y_i = \alpha_{k(i)} z_i^{k(i)}$ and as $y_j \geqslant \alpha_{k(i)} z_j^{k(i)}$ for all j, it follows that $z^{k(i)} \leqslant_i y$ and y is not i-minimal for any i. $\square$

3.2. Which generators are extremal

Next we classify all generators obtained in (12), (13), (15), (17) and (18) and give procedures for checking whether they are extremal. The proof that these procedures are sound will be given in the next subsection. We start with unit vectors (S_1) and combinations of two unit vectors ($S_{2.}$).

$$\begin{aligned}
&S_1 = \{e_i | i \in \bar{I}_1 \cap \bar{I}_2\}.\\
&S_{2A1} = \{\phi_{ik} = \gamma_{ki}^1 e_k \oplus e_i \,| k \in J_1 \cap \bar{I}_2, i \in I_1 \cap \bar{I}_2\}.\\
&S_{2A2} = \{\phi_{ik} = \gamma_{ki}^2 e_k \oplus e_i \,| k \in J_2 \cap \bar{I}_1, i \in I_2 \cap \bar{I}_1\}.\\
&S_{2B} = \{\phi_{ik} = (\gamma_{ki}^1 \oplus \gamma_{ki}^2) e_k \oplus e_i \,| k \in J_1 \cap J_2\,,\ i \in I_1 \cap I_2\}.\\
&S_{2C} = \{\phi_{lk} = \gamma_{kl}^1 e_k \oplus e_\ell, \phi_{kl} = \gamma_{lk}^2 e_l \oplus e_k \,| k \in J_1 \cap I_2 l \in J_2 \cap I_1,\ \gamma_{kl}^1 \gamma_{lk}^2 \leqslant \mathbf{1}\}.
\end{aligned}$$

All vectors in S_1, S_{2A} and S_{2B} belong to the basis. Vectors in S_{2C} belong to the basis whenever they exist. For this, we determine the sets

$$\begin{aligned}
W &:= \{(k, l) \mid k \in J_1 \cap I_2,\ l \in J_2 \cap I_1,\ \gamma_{kl}^1 \gamma_{lk}^2 \leqslant \mathbf{1}\}\\
\overline{W} &:= \{(k, l) \mid k \in J_1 \cap I_2,\ l \in J_2 \cap I_1,\ \gamma_{kl}^1 \gamma_{lk}^2 > \mathbf{1}\}
\end{aligned} \tag{21}$$

Then, $\phi_{kl}, \phi_{lk} \in S_{2C}$ exist whenever $(k,l) \in W$. Note that if $\gamma^1_{kl}\gamma^2_{lk} = \mathbf{1}$ then ϕ_{kl} and ϕ_{lk} are multiples of each other so that one of them can be removed.

We now consider combinations of three unit vectors. Recall that $K_1 = \{i \mid a_{1i} = b_{1i} = \mathbf{0}\}$ and $K_2 = \{i \mid a_{2i} = b_{2i} = \mathbf{0}\}$, so that $I_1 \cup J_1 \cup K_1 = I_2 \cup J_2 \cup K_2 = \{1, \dots, n\}$.

$S_{3A} = \{\psi_{ikl} = \gamma^1_{ki} e_k \oplus \gamma^2_{li} e_l \oplus e_i \,|k \in J_1 \cap \bar{I}_2\,,\ l \in J_2 \cap \bar{I}_1\,,\ i \in I_1 \cap I_2\}$.

For all $i \in I_1 \cap I_2$ determine the sets

$$\begin{aligned} L_1(i) &:= \{k \in J_1 \cap J_2 \mid \gamma^1_{ki} < \gamma^2_{ki}\}, \\ L_2(i) &:= \{l \in J_1 \cap J_2 \mid \gamma^2_{li} < \gamma^1_{li}\}. \end{aligned} \tag{22}$$

Then, $\psi_{ikl} \in S_{3A}$ belongs to the basis whenever

$$k \in (J_1 \cap K_2) \cup L_1(i),\ l \in (J_2 \cap K_1) \cup L_2(i). \tag{23}$$

$S_{3B1} = \{\psi_{ikl} = \gamma^1_{kl}\gamma^2_{li} e_k \oplus \gamma^2_{li} e_l \oplus e_i \mid k \in J_1 \cap \bar{I}_2\ \ l \in J_2 \cap I_1\,,\ i \in I_2 \cap \bar{I}_1\}$.
$S_{3B2} = \{\psi_{ikl} = \gamma^2_{lk}\gamma^1_{ki} e_l \oplus \gamma^1_{ki} e_k \oplus e_i \mid k \in J_1 \cap I_2\ \ l \in J_2 \cap \bar{I}_1\,,\ i \in \bar{I}_2 \cap I_1\}$.

For all $i \in I_2 \cap \bar{I}_1, l \in J_2 \cap I_1$, determine the sets

$$M_1(i,l) := \{t \in J_1 \cap J_2 \mid \gamma^1_{tl}\gamma^2_{li} < \gamma^2_{ti}\}. \tag{24}$$

For all $i \in I_1 \cap \bar{I}_2, k \in J_1 \cap I_2$, determine the sets

$$M_2(i,k) := \{t \in J_1 \cap J_2 \mid \gamma^2_{tk}\gamma^1_{ki} < \gamma^1_{ti}\}. \tag{25}$$

A vector in $\{\psi_{ikl} \in S_{3B1}\}$ (resp. $\{\psi_{ikl} \in S_{3B2}\}$) belongs to the basis if and only if the following two conditions are satisfied:

1. $i \in I_2 \cap K_1$ or $(i,l) \in \overline{W}$ (resp. $i \in I_1 \cap K_2$ or $(k,i) \in \overline{W}$),
2. $k \in M_1(i,l)$ or $k \in J_1 \cap K_2$ (resp. $l \in M_2(i,k)$ or $l \in J_2 \cap K_1$).

$S_{3C1} = \{\psi_{ikl} = (\gamma^1_{ki} \oplus \gamma^1_{kl}\gamma^2_{li}) e_k \oplus \gamma^2_{li} e_l \oplus e_i| > k \in J_1 \cap \bar{I}_2\,,\ l \in J_2 \cap I_1\,,\ i \in I_2 \cap I_1\}$.
$S_{3C2} = \{\psi_{ikl} = (\gamma^2_{li} \oplus \gamma^2_{lk}\gamma^1_{ki}) e_l \oplus \gamma^1_{ki} e_k \oplus e_i \mid k \in J_1 \cap I_2\,,\ l \in J_2 \cap \bar{I}_1\,,\ i \in I_2 \cap I_1\}$.

For all $i \in I_1 \cap I_2, l \in J_2 \cap I_1, k \in J_1 \cap I_2$, determine the sets

$$\begin{aligned} N_1(i,l) :&= \{t \in J_1 \cap J_2 \mid \gamma^1_{ti} \oplus \gamma^1_{tl}\gamma^2_{li} < \gamma^1_{ti} \oplus \gamma^2_{ti}\} \\ &= \{t \in L_1(i) \mid \gamma^1_{tl}\gamma^2_{li} < \gamma^2_{ti}\}, \\ N_2(i,k) :&= \{t \in J_1 \cap J_2 \mid \gamma^2_{ti} \oplus \gamma^2_{tk}\gamma^1_{ki} < \gamma^1_{ti} \oplus \gamma^2_{ti}\} \quad = \{t \in L_2(i) \mid \gamma^2_{tk}\gamma^1_{ki} < \gamma^1_{ti}\}. \end{aligned} \tag{26}$$

Then, $\psi_{ikl} \in S_{3C1}$ (resp. $\psi_{ikl} \in S_{3C2}$) belongs to the basis if and only if $k \in (J_1 \cap K_2) \cup N_1(i,l)$ (resp. $l \in (J_2 \cap K_1 \cup N_2(i,k))$.

We also have the following sets, denoting $Z = \{(k,l))| \ k \in J_1 \cap I_2,\ l \in J_2 \cap I_1,\ \gamma^1_{kl}\gamma^2_{lk} \leqslant \mathbf{1}\}$:
$S_{3D1} = \{\psi_{ikl} = \gamma^1_{kl}\gamma^2_{li} e_k \oplus \gamma^2_{li} e_l \oplus e_i | i \in I_2 \cap\ \bar{I}_1,\ (k,l) \in Z\,\}$.

$S_{3D2} = \{\psi_{ikl} = \gamma^2_{lk}\gamma^1_{ki} e_l \oplus \gamma^1_{ki} e_k \oplus e_i | i \in \bar{I}_2 \cap\ I_1,\ (k,l) \in Z\,\}$.
$S_{3E} = \{\psi_{ikl} = (\gamma^2_{li} \oplus \gamma^2_{lk}\gamma^1_{ki}) e_l \oplus (\gamma^1_{ki} \oplus \gamma^1_{kl}\gamma^2_{li}) e_k \oplus e_i | i \in I_1 \cap I_2,\ (k,l) \in Z\,\}$.

Provided that $(k,l) \in W$, vector $\psi_{ikl} \in S_{3D1}$ (resp. $\psi_{ikl} \in S_{3D2}$) belongs to the basis if and only if $i \in K_1 \cap I_2$ or $(i,l) \in \overline{W}$ (resp. $i \in I_1 \cap K_2$ or $(k,i) \in \overline{W}$), and $\psi_{ikl} \in S_{3E}$ always belongs to the basis.

3.3. Explanation of the procedures stated above

We explain below why the procedures of the previous subsection indeed yield the basis. Recall that S_1 denotes the set of all generators e_i for $i \in \bar{I}_1 \cap \bar{I}_2$, S_2 the set of all 2-generators ϕ_{ik} (and ϕ_{kl}), and S_3 the set of all 3-generators ψ_{ikl}.

S_1, S_2: The supports of all generators in $S_1 \cup S_2$ are different, except for the pairs of generators in S_{2C}, which exist if and only if $\gamma^1_{kl}\gamma^2_{lk} \leqslant \mathbf{1}$, and are multiples of each other if and only if $\gamma^1_{kl}\gamma^2_{lk} = \mathbf{1}$. Removing one vector from every such proportional pair in S_{2C} yields an independent set. Evidently, vectors in $S_1 \cup S_2$ cannot be generated with help of vectors in S_3, and this settles the cases of S_1, S_2.

For the rest of the cases, first note that the supports of all generators in S_3 are different and hence the set S_3 is independent. therefore, dependence may only occur when the vectors in S_3 are linear combinations of the vectors in S_1 and S_2. We now detail all the cases.

S_{3A}: A vector $\psi_{ikl} \in S_{3A}$ may be a combination of vectors in S_1 and S_{2B}, as the supports of some generators in these sets are contained in the support of a vector in S_{3A}. By the minimality principle, a vector ψ_{ikl} is extremal if and only if it is i-, k- or l-minimal. But ψ_{ikl} can be neither k- nor l-minimal since for all $k, l \in \bar{I}_1 \cap \bar{I}_2$ the only minimal generators are e_k and e_l. The i-minimality of $\psi_{ikl} \in S_{3A}$ can be prevented only by $\phi_{ki} \in S_{2B}$ or $\phi_{li} \in S_{2B}$. Condition (23) describes the situation when this does not happen.

S_{3B}: A vector $\psi_{ikl} \in S_{3B}$ can be a max combination of vectors in S_1, S_{2A} and S_{2C} due to the inclusion of supports. Again, ψ_{ikl} can be neither k- nor l-minimal, since it can be represented as a combination of e_i and a vector from S_{2A1} (resp. S_{2A2}) in the case of S_{3B1} (resp. S_{3B2}):

$$\begin{aligned}
&\gamma^1_{kl}\gamma^2_{li}e_k \oplus \gamma^2_{li}e_l \oplus e_i = \gamma^2_{li}(\gamma^1_{kl}e_k \oplus e_l) \oplus e_i.\\
&\gamma^2_{lk}\gamma^1_{ki}e_l \oplus \gamma^1_{ki}e_k \oplus e_i = \gamma^1_{ki}(\gamma^2_{lk}e_l \oplus e_k) \oplus e_i.
\end{aligned} \tag{27}$$

We describe the 2-generators which may prevent $\psi_{ikl} \in S_{3B1}$ (resp. $\psi_{ikl} \in S_{3B2}$) to be i-minimal.

1. $\phi_{il}, \phi_{li} \in S_{2C}$ (resp. $\phi_{ki}, \phi_{ik} \in S_{2C}$).
   These 2-generators do not arise only in the following situations:
   - if $i \in K_1$ for S_{3B1} (resp. $i \in K_2$ for S_{3B2}), for in this case there is no vector in S_{2C} whose support is a subset of the support of ψ_{ikl},
   - if the corresponding pair $\phi_{il}, \phi_{li} \in S_{2C}$ (resp. $\phi_{ki}, \phi_{ik} \in S_{2C}$) does not exist meaning $(i, l) \in \overline{W}$ (resp. $(k, i) \in \overline{W}$).
2. $\phi_{ik} \in S_{2A2}$ (resp. $\phi_{il} \in S_{2A1}$).

These vectors do not arise only if $k \in K_2$ (resp. $l \in K_1$), since then $k \notin J_2$ (resp. $l \notin J_1$) unlike in the case of S_{2A2} (resp. S_{2A1}).

Otherwise, ϕ_{ik} (resp. ϕ_{il}) are not $\leqslant_i$ inferior to ψ_{ikl} only if $k \in M_1(i, l)$ (resp. $l \in M_2(i, k)$), see (24) and (25).

S_{3C}: A vector $\psi_{ikl} \in S_{3C}$ can be a max combination of vectors in S_1, S_{2A} and S_{2B}. Again, ψ_{ikl} can be neither k- nor l- minimal. Indeed,

$$\begin{aligned}
&\psi_{ikl} = \gamma^1_{ki}e_k \oplus e_i \oplus \gamma^2_{li}(\gamma^1_{kl}e_k \oplus e_l),\ S_{3C1}\\
&\psi_{ikl} = \gamma^2_{li}e_l \oplus e_i \oplus \gamma^1_{ki}(\gamma^2_{lk}e_l \oplus e_k),\ S_{3C2}
\end{aligned} \tag{28}$$

where the vectors in brackets belong to S_{2A1} and S_{2A2} respectively. The first vector cannot be k-minimal since $k \in \bar{I}_1 \cap \bar{I}_2$, and it cannot be l-minimal as it loses to $\gamma^2_{kl}e_k \oplus e_l \in S_{2A1}$. The second vector cannot be l-minimal since $l \in \bar{I}_1 \cap \bar{I}_2$, and it cannot be k-minimal as it loses to $\gamma^2_{lk}e_l \oplus e_k \in S_{2A2}$. The remaining possibility of being i-minimal may be destroyed by vectors from S_{2B}, and this does not happen if and only if the given conditions are satisfied.

S_{3D}, S_{3E}: A vector $\psi_{ikl} \in S_{3D}$ cannot be a max combination of other vectors of S_2 than those in S_{2C}. It is not a max combination of vectors in S_{2C} only if i is not suitable for existence of vectors in S_{2C}. This happens if $i \in K_1 \cap I_2$ or $(i, l) \in \overline{W}$ for the case $\psi_{ikl} \in S_{3D1}$, and $i \in I_1 \cap K_2$ or $(k, i) \in \overline{W}$ for the case $\psi_{ikl} \in S_{3D2}$. Finally, the vectors in S_{3E} cannot be combinations of vectors in S_2, since only vectors in S_{2C} have relevant supports (and yet not enough). So the vectors in S_{3E} are in the basis whenever they exist.

We note that the complexity of the above procedures is $O(n^3)$, which is due to the computation of the sets $M_1(i, l)$ (24), $M_2(i, k)$ (25), $N_1(i, l)$ and $N_2(i, k)$ (26), and checking conditions for all combinations of three unit vectors (i.e., for all choices of i, k, l). The complexity of the algorithm by Allamegeon et al. [1] is $O(n\alpha(n) \times n^4)$ in the case of two inequalities, adapting [1, Proposition 4.3]. Here $n\alpha(n)$ is the time needed to check the extremality of one generator when adding the second inequality, $\alpha(n)$ being the inverse of Ackermann constant (related to hypergraphs). Further it can be deduced from [11, Proposition 2.4] or the results in [1,3] that the number of generators for the set defined by one inequality is no more than n^2. Hence the multiple n^4, which is a bound on the *squared* maximal number of generators for the set defined by one inequality.

4. Examples

We conclude the paper with two examples. The second example is taken from [11], Example 4.2.

4.1. A simple example

To illustrate the sets of generators constructed in the paper on a simple example, we consider the following system of two inequalities with four variables:

$$\begin{aligned} &4 \otimes x_3 \oplus 2 \otimes x_4 \leqslant x_1 \oplus 2 \otimes x_2, \\ &3 \otimes x_1 \oplus x_3 \leqslant x_2. \end{aligned} \tag{29}$$

We have $I_1 = \{3, 4\}, J_1 = \bar{I}_1 = \{1, 2\}, I_2 = \{1, 3\}, J_2 = \{2\}, \bar{I}_2 = \{2, 4\}, K_1 = \emptyset, K_2 = \{4\}$. We compute
S_1: just e_2, since $\bar{I}_1 \cap \bar{I}_2 = \{2\}$;
S_{2A1}: just $\gamma_{24}^1 e_2 \oplus e_4 = e_2 \oplus e_4$, since $J_1 \cap \bar{I}_2 = \{2\}$ and $I_1 \cap \bar{I}_2 = \{4\}$;
S_{2A2}: just $\gamma_{21}^2 e_2 \oplus e_1 = 3e_2 \oplus e_1$, since $J_2 \cap \bar{I}_1 = \{2\}$ and $I_2 \cap \bar{I}_1 = \{1\}$;
S_{2B}: $(\gamma_{23}^1 \oplus \gamma_{23}^2)e_2 \oplus e_3 = 2e_2 \oplus e_3$, since $J_1 \cap J_2 = \{2\}$ and $I_1 \cap I_2 = \{3\}$;
S_{2C}: empty, since $J_2 \cap I_1$ is empty;
S_{3A}: trivializes to S_{2B};
S_{3B1}: empty, since $J_2 \cap I_1$ is empty;
S_{3B2}: just $\gamma_{21}^2\gamma_{14}^1 e_2 \oplus \gamma_{14}^1 e_1 \oplus e_4 = 5e_2 \oplus 2e_1 \oplus e_4$, since $J_1 \cap I_2 = \{1\}, J_2 \cap \bar{I}_1 = \{2\}, \bar{I}_2 \cap I_1 = \{4\}$;
S_{3C1}: empty, since $J_2 \cap I_1$ is empty;
S_{3C2}: just $(\gamma_{23}^2 \oplus \gamma_{21}^2\gamma_{13}^1)e_2 \oplus \gamma_{13}^1 e_1 \oplus e_3$, which is $7e_2 \oplus 4e_1 \oplus e_3$, since $J_1 \cap I_2 = \{1\}, J_2 \cap \bar{I}_1 = \{2\}$, $I_2 \cap I_1 = \{3\}$;
S_{3D1}, S_{3D2} and S_{3E}: empty, since $J_2 \cap I_1$ is empty.

In this example, the basis consists of four generators in S_1, S_{2A1}, S_{2A2} and S_{2B}: e_2, $e_2 \oplus e_4$, $3e_2 \oplus e_1$ and $2e_2 \oplus e_3$. Indeed, the remaining two generators in S_3 are redundant: (1) $5e_2 \oplus 2e_1 \oplus e_4$ (S_{3B2}) is a combination of $e_2 \oplus e_4$ (S_{2A1}) and $3e_2 \oplus e_1$ (S_{2A2}), (2) $7e_2 \oplus 4e_1 \oplus e_3$ (S_{3C2}) is a combination of $3e_2 \oplus e_1$ (S_{2A2}) and $2e_2 \oplus e_3$ (S_{2B}).

The redundancy can be also interpreted in terms of the procedures given in Subsection 3.2, see S_{3B2} and S_{3C2}. In the case of S_{3B2} there are two conditions, and the first of them is satisfied: $I_1 \cap K_2 = \{4\}$ and $i = 4$. However, the second condition fails since $M_2(i, k)$ is empty. In the case of S_{3C2}, $N_2(i, k)$ is empty.

In terms of the explanations in Subsection 3.3, which use the multiorder principle (Proposition 4), the vector in S_{3B2} could be 4-minimal, but it is defeated by the vector from S_{2A1}. The vector in S_{3C2} could be 3-minimal, but it is defeated by the vector from S_{2B}.

4.2. An example from [11]

To compare our results with the approach of [11], we consider [11], Example 4.2, which is a system of two inequalities with seven variables:

$$\begin{aligned} &x_4 \oplus 4 \otimes x_5 \oplus 2 \otimes x_6 \oplus 6 \otimes x_7 \leqslant x_1 \oplus 1 \otimes x_2 \oplus 5 \otimes x_3, \\ &5 \otimes x_2 \oplus 6 \otimes x_3 \oplus 2 \otimes x_7 \leqslant 3 \otimes x_1 \oplus x_4 \oplus 2 \otimes x_5 \oplus 4 \otimes x_6. \end{aligned} \tag{30}$$

In this case $I_1 = \{4, 5, 6, 7\}, J_1 = \{1, 2, 3\} = \bar{I}_1, I_2 = \{2, 3, 7\}, J_2 = \{1, 4, 5, 6\} = \bar{I}_2, K_1 = K_2 = \emptyset$. We compute the generators comparing them with those in the table of [11] page 365:
S_1: just e_1, since $\bar{I}_1 \cap \bar{I}_2 = \{1\}$. This is x_1 in the table of [11].
S_{2A1}: Combining $J_1 \cap \bar{I}_2 = \{1\}$ and $I_1 \cap \bar{I}_2 = \{4, 5, 6\}$ we obtain $e_1 \oplus e_4$, $4e_1 \oplus e_5$ and $2e_1 \oplus e_6$. Vector $e_1 \oplus e_4$ corresponds to x_3, and the remaining two vectors are x_5 and x_{10} in the table of [11].
S_{2A2}: Combining $J_2 \cap \bar{I}_1 = \{1\}$ and $I_2 \cap \bar{I}_1 = \{2, 3\}$ we obtain $2e_1 \oplus e_2$ and $3e_1 \oplus e_3$. These correspond to x_4 and x_7 in the table of [11].
S_{2B}: just $6e_1 \oplus e_7$, combining $J_1 \cap J_2 = \{1\}$ with $I_1 \cap I_2 = \{7\}$. This is x_2 in the table of [11].
S_{2C}. To compute these we need to combine $J_1 \cap I_2 = \{2, 3\}$ with $J_2 \cap I_1 = \{4, 5, 6\}$. For each $k = 2, 3$ and $l = 4, 5, 6$ we need to check whether $\gamma^1_{kl}\gamma^2_{lk} \leqslant \mathbf{1}$, and each time this condition is satisfied we have two vectors (or just one vector if $\gamma^1_{kl}\gamma^2_{lk} = \mathbf{1}$). In our case the condition is satisfied only with $k = 3$ and $l = 6$. This yields two vectors $2e_6 \oplus e_3$ and $-3e_3 \oplus e_6$, which are x_6 and x_{11} in the table of [11].
S_{3A}: trivializes to S_{2B}.
S_{3B1}: We need to combine $J_1 \cap \bar{I}_2 = \{1\}, J_2 \cap I_1 = \{4, 5, 6\}$ and $I_2 \cap \bar{I}_1 = \{2, 3\}$. For $i = 2, 3, l = 4, 5, 6$ and $k = 1$, each time when $\gamma^1_{il}\gamma^2_{li} > \mathbf{1}$, we have to verify whether $\gamma^1_{kl}\gamma^2_{li} < \gamma^2_{ki}$ holds. Each time when both conditions are satisfied, a new vector is added. Here it never happens.
S_{3B2}: We need to combine $J_1 \cap I_2 = \{2, 3\}, J_2 \cap \bar{I}_1 = \{1\}$ and $\bar{I}_2 \cap I_1 = \{4, 5, 6\}$. For $k = 2, 3$, $i = 4, 5, 6$ and $l = 1$, each time when $\gamma^1_{ki}\gamma^2_{ik} > \mathbf{1}$, we have to verify whether $\gamma^2_{lk}\gamma^1_{ki} < \gamma^1_{li}$ holds. Each time when both conditions are satisfied, a new vector is added. Here it happens with 1) $l = 1, k = 3$ and $i = 4$ leading to $3e_1 \oplus e_3 \oplus 5e_4$ which corresponds to x_8 of [11], 2) $l = 1, k = 3$ and $i = 5$ leading to $3e_1 \oplus e_3 \oplus 1e_5$, which corresponds to x_9 of [11].
S_{3C1}: Here we combine $J_1 \cap \bar{I}_2 = \{1\}$ with $J_2 \cap I_1 = \{4, 5, 6\}$ and $I_1 \cap I_2 = \{7\}$. Since $\gamma^1_{ki} > \gamma^2_{ki}$ with $k = 1$ and $i = 7$, no vector belongs to the basis in this case.
S_{3C2}: We combine $J_1 \cap I_2 = \{2, 3\}, J_2 \cap \bar{I}_1 = \{1\}$ and $I_1 \cap I_2 = \{7\}$. For each $k = 2, 3, l = 1$ and $i = 7$ we have to verify $\gamma^2_{li} \oplus \gamma^2_{lk}\gamma^1_{ki} < \gamma^2_{li} \oplus \gamma^1_{li}$. This happens for $k = 3$, $l = 1$ and $i = 7$ and yields the vector $4e_1 \oplus 1e_3 \oplus e_7$, which corresponds to x_{12} of [11].
S_{3D1}: We combine $J_1 \cap I_2 = \{2, 3\}, J_2 \cap I_1 = \{4, 5, 6\}, I_2 \cap \bar{I}_1 = \{2, 3\}$. For $k = 2, 3$ and $l = 4, 5, 6$, the condition $\gamma^1_{lk}\gamma^2_{kl} \leqslant \mathbf{1}$ holds only for $k = 3$ and $l = 6$, so it remains to verify $\gamma^1_{il}\gamma^2_{li} > \mathbf{1}$ for $i = 2$ and $l = 6$. This condition holds and we obtain $\gamma^1_{36}\gamma^2_{62}e_3 \oplus \gamma^2_{62}e_6 \oplus e_2$ which is proportional with $2e_2 \oplus e_3 \oplus 3e_6$. Note that the max-linear combination of e_2, e_3, e_6 given for x_{13} in the table of [11] is an error, since $Ax_{13} \not\leq Bx_{13}$.
S_{3D2}: We combine $J_1 \cap I_2 = \{2, 3\}, J_2 \cap I_1 = \{4, 5, 6\}, \bar{I}_2 \cap I_1 = \{4, 5, 6\}$. For $k = 2, 3$ and $l = 4, 5, 6$, the condition $\gamma^1_{lk}\gamma^2_{kl} \leqslant \mathbf{1}$ holds only for $k = 3$ and $l = 6$, so it remains to verify $\gamma^1_{ki}\gamma^2_{ik} > \mathbf{1}$ for $i = 4, 5$ and $k = 3$. This condition holds in both cases and yields $\gamma^2_{63}\gamma^1_{34}e_6 \oplus \gamma^1_{34}e_3 \oplus e_4$ which is proportional with $2e_6 \oplus e_3 \oplus 5e_4$, and $\gamma^2_{63}\gamma^1_{35}e_6 \oplus \gamma^1_{35}e_3 \oplus e_5$ proportional with $2e_6 \oplus e_3 \oplus 1e_5$.
S_{3E}: We combine $J_1 \cap I_2 = \{2, 3\}, J_2 \cap I_1 = \{4, 5, 6\}$ and $I_1 \cap I_2 = \{7\}$. As $\gamma^1_{kl}\gamma^2_{lk} < \mathbf{1}$ only for $k = 3$ and $l = 6$, we have only one generator, namely $3e_6 \oplus 1e_3 \oplus e_7$.

Thus the basis consists of e_1, 8 combinations of two unit vectors and 7 combinations of three unit vectors.

The two-combinations are: $e_1 \oplus e_4$, $4e_1 \oplus e_5$ and $2e_1 \oplus e_6$ (S_{2A1}), $2e_1 \oplus e_2$ and $3e_1 \oplus e_3$ (S_{2A2}), $6e_1 \oplus e_7$ (S_{2B}), $2e_6 \oplus e_3$ and $e_3 \oplus 3e_6$ (S_{2C}).

The three-combinations are: $3e_1 \oplus e_3 \oplus 5e_4$, $3e_1 \oplus e_3 \oplus 1e_5$ (S_{3B2}), $4e_1 \oplus 1e_3 \oplus e_7$ (S_{3C2}), $2e_2 \oplus e_3 \oplus 3e_6$ (S_{3D1}), $2e_6 \oplus e_3 \oplus 5e_4$ and $2e_6 \oplus e_3 \oplus 1e_5$ (S_{3D2}), $3e_6 \oplus 1e_3 \oplus e_7$ (S_{3E}).

We note that all vectors that we have found, are solutions of the system. Moreover, all three-generators turn both inequalities into equalities, which in analogy with the convex analysis also suggests that they must be extremals (the two-generators correspond to the intersections with coordinate planes). Actually vectors in S_{3B2} and S_{3C2} are different from x_8, x_9 and x_{12} from the table of [11] page 365, to which they correspond in terms of supports. For these, $x_8 = 4e_1 \oplus e_3 \oplus 4e_4$ is a combination of $3e_1 \oplus e_3 \oplus 5e_4$ (from S_{3B2}), $3e_1 \oplus e_3$ (from S_{2A2}) and e_1, $x_9 = 4e_1 \oplus 1e_3 \oplus e_5$ is a combination of $3e_1 \oplus e_3 \oplus 1e_5$ (from S_{3B2}) and $3e_1 \oplus e_3$, and $x_{12} = 5e_1 \oplus 1e_3 \oplus e_7$ is a combination of $4e_1 \oplus 1e_3 \oplus e_7$ (from S_{3C2}) and e_1. The remaining generator in the table of [11] is $x_{13} = e_2 \oplus 2e_3 \oplus 1e_6$. This generator is incorrect, since it violates the second inequality of (30), but in terms of support, it corresponds to $2e_2 \oplus e_3 \oplus 3e_6$ from S_{3D1}. Also, there are three combinations which are not in the table of [11], from S_{3D2} and S_{3E}.

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