MEAN-PAYOFF GAMES AND PARAMETRIC TROPICAL TWO-SIDED SYSTEMS

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ABSTRACT. We propose a general theory of parametric two-sided tropical linear systems based on the connections with zero-sum deterministic mean-payoff games. It is shown how this general theory specializes to particular tropical linear systems with parameter, arising from 1) the two-sided tropical eigenproblem $Ax = \lambda + Bx$ and 2) the tropical linear programming. This involves 1) the problem of finding the spectrum of $Ax = \lambda + Bx$ solved in pseudopolynomial time by the spectral function approach, and 2) bisection and Newton iteration methods for minimizing tropical linear functionals over tropical convex sets defined as solution sets of two-sided systems $Ax \oplus c \leq Bx \oplus d$. Particular new results include a) explicit formula for the unique eigenvalue of symmetric two-sided igenproblem, b) generalized bound for bisection method for the fractional tropical linear programming, c) comparison between bisecton and Newton methods solving tropical linear programming problem.

1. INTRODUCTION

1.1. **Tropical mathematics, tropical two-sided systems and mean-payoff games.** Tropical algebra is defined over the tropical (max-plus) semiring, which is the set $(\mathbb{R} \cup \{-\infty\}) = \mathbb{R} \cup \{-\infty\}$ equipped with tropical addition $a \oplus b := \max(a, b)$ and multiplication $a \otimes b = a + b$. The element $-\infty$ plays the role of zero, and 0 becomes the unity. This *tropical arithmetics* is extended to matrices and vectors in the usual way: $(A \oplus B)_{ij} = \max(A_{ij}, B_{ij})$ and $(A \otimes B)_{ij} = \max_k(A_{ik} + B_{kj})$ giving rise to tropical subspaces and other geometric structures of $(\mathbb{R} \cup \{-\infty\})^d$. Functional analysis over tropical semiring is known as idempotent analysis [KM97].

Tropical mathematics, which emerged in the 60's in the works of R.A. Cuninghame-Green and N.N. Vorobyev [CG62, Vor67], has seen rapid development over last two decades, see e.g. collections of papers [MS92, Gun98, LM05, LS09]. In the 80's and 90's, the Russian group [KM97, MS92, LM94] and the French group [CDQV85, BCOQ92, GP97] independently observed that certain problems in discrete optimization, optimal

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control, Hamilton-Jacobi-Bellman PDE and quasiclassical asymptotics in quantum mechanics are linear in terms of tropical operations. This tropical linearity gave rise to a new systematic approach to such problems. Moreover, it was observed that one can exploit the following limit passage from the ordinary linearity. Take $\mathbb{R} \cup \{-\infty\}$ and equip it with $a \otimes b = a + b$ like in the tropical case, and $a \oplus_h b = h \log(e^{a/h} + e^{b/h})$ where h is a positive parameter. This new semiring is isomorphic to nonnegative numbers with the usual operations. However as h tends to zero it "becomes" the tropical semiring. In some cases this **Maslov dequantization** can be used directly. If this is not the case, then the **tropical/idempotent correspondence principle,** as formulated by Litvinov and Maslov [LM94], says that "There exists a heuristic correspondence between interesting, important and useful constructions and results over the field of real or complex numbers and similar constructions and results over idempotent semirings".

Tropical convexity started with the tropical Hahn-Banach theorem proved by Zimmermann [Zim77], which was generalized to idempotent functional spaces by Litvinov et al. [LMS01] and Cohen et al. [CGQS05]. However, the whole tropical area gained new impetus after Develin and Sturmfels represented tropical polyhedra as cellular complexes [DS04]. This opened the new era in tropical convexity meaning precise analysis of tropical polyhedra and their combinatorics. There are also contributions coming from **abstract convex analysis** by Martínez-Legaz, Rubinov and Singer [MLRS02], Nitica and Singer [NS08], Briec, Horvath and Rubinov [BH04], with motivations in global optimization theory.

Tropical convexity is intimately related to the **tropical matrix algebra** which is analogous to nonnegative matrix theory, since the role of zero is played by $-\infty$. The link to tropical two-sided systems like $Ax = By$, $Ax \leq Bx$ is especially strong, via the Hahn-Banach theorem. Such systems have been of special importance for R.A. Cuninghame-Green and P. Butkovič who developed new algorithms for 1) finding a solution of two-sided systems [CGB03, BZ06] and 2) describing the whole set of solutions of two-sided systems [BH84]. Other related contributions of this group include studies of strong regularity, linear independence and rank in tropical algebra, see [But03] and [But10] for detailed account.

The equations in tropical two-sided systems can be interpreted as *rendez-vous* constraints in **scheduling**. Motivations of this nature arose for instance in the work of Burns [Bur91] applied to the checking of asynchronous digital circuits. Systems of the form $Ax \leq Bx$ have also been studied in relation to scheduling problems with both AND and OR precedence constraints by Möhring et al. [MSS04].

As shown by Akian et al. [AGG09a], solving some tropical two-sided system is equivalent to solving some zero-sum deterministic repeated **mean-payoff game**. Shapley operator for this game is a **min-max function** (involving min, max and + arithmetics), see e.g. Zwick and Paterson [ZP96]. Such functions were also studied by Gunawardena, Olsder [Gun94, Ols91] and Cochet-Terrasson et al. [CTGG99] with above mentioned motivations from scheduling and design of asynchronous circuits. In the tropical convexity they appear as nonlinear projections onto tropical cones studied by Cuninghame-Green and Cohen et al. [CG79, CGQS05], and also as compositions of such projections in the **tropical cyclic projections** method of Gaubert and Sergeev [GS08, GS10, Ser09]. It is known that solving mean-payoff games is in **NP∩coNP**: this is due to Zwick and Paterson [ZP96] and was foreseen by Gurvich, Karzanov and Khachiyan [GKK88]. The same NP∩coNP result is true of the (equivalent) problem for tropical two-sided systems, which was obtained independently by Bezem, Nieuwenhuis and Rodríguez-Carbonell [BNgC08] who introduced the novel concept of **max atoms**.

1.2. **New applications: the domain of tropical polyhedra.** The goal of static analysis of computer programs is automatical determination of invariants, i.e., properties which are valid for all executions. Such properties represent, e.g., the impossibility of forbidden memory access.All possible states of the program are captured by the *collecting semantics* of the program. The Rice theorem implies that checking given conditions by means of the collecting semantics of a program is *undecidable*.

The approach of abstract interpretation was invented by Cousot and Cousot [CC77], see also [CC92]. The main idea is to create a computable abstract semantics, which over-approximates the set of all possible environments of the program. In the case of numerical invariants, the abstract semantics is based on certain numerical domains of a well-defined geometric nature. Domains include abstract elements and abstract primitives. The latter are operations with abstract elements, and they must over-approximate all possible results of certain operations with program variables. By means of these primitives various over-approximations of the collecting semantics of the program can be defined.

One of the key developments in static analysis by abstract interpretation is the domain of convex polyhedra by Cousot and Halbwachs [CH78]. The abstract elements are closed convex polyhedra, which represent invariants in the form of affine inequalities. It is very precise, but the operations include transitions between internal and external representations, which are exponential in dimension in the worst case.

Many other abstract domains are known in the literature: signs, intervals, zones, octagons, congruence relations and equalities, octahedra, to mention a few.

The domain of tropical polyhedra was introduced and studied recently in Allamigeon et al. [All09, AGG09b, AGK10a]. It is more precise than the domain of zones which it subsumes. As shown in [AGG09b], the analysis of memory allocation routines like the well-known `memcpy` function of C, naturally leads to disjunctive invariants. Disjunctions reflect for instance the fact that the length of a string to be copied may not be preserved. Such disjunctive invariants are tough for many other types of domains. The computational complexity of this domain is again dominated by the complexity of transition between internal and external descriptions. The main motivation is that the domain of tropical polyhedra is suitable for scalable and automatic inferring certain disjunctive invariants like $\min(len\_src, n) = \min(len\_dst, n)$ of `memcpy` [All09], which are not recognizable by other types of domains. It also follows that the computations with tropical polyhedra in many respects are easier than those with the ordinary polyhedra.

The method of linear programming templates by Sankaranarayanan et al. [SCSM06] is an important development of the polyhedral domain by Cousot and Halbwachs [CH78]. It implements the idea of over-approximation using polyhedra defined by a prescribed number of affine inequalities, the linear parts of which are taken from a fixed "template", determined by the characteristics of the program to analyze. Hence, the development of tropical linear programming initiated by Butkovič and Aminu [BA08], is crucial for the tropical analogue of linear programming templates.

1.3. **Parametric mean-payoff games.** Dynamic operator of a mean-payoff game is a min-max function $f\colon (\mathbb{R}\cup\{-\infty\})^d \to (\mathbb{R}\cup\{-\infty\})^d$. Components of this function are defined by $f_j(x) := \min_k(-A_{kj} + \max_l(B_{kl} + x_l))$. Certain problems of tropical linear algebra lead to parametric tropical two-sided systems $C(\lambda)x \leq D(|lambda)x$ where all entries of C and D are piecewise linear functions of λ. These include the tropical linear programming [BA08, GKS10] mentioned above, as well as the two-sided tropical eigenproblem $Ax = \lambda + Bx$ studied by Cuninghame-Green and Butkovič [CGB08], Binding and Volkmer [BV07a] and later Gaubert and Sergeev [GS10]. To such systems we can associate min-max functions $f_\lambda(x)$ which depend on a parameter λ, as dynamic operators of certain parametric mean-payoff games. In [GS10] Gaubert and Sergeev introduce a novel concept of *spectral function*, which is subsequently used in [AGK10b, GKS10]. It shows how the greatest eigenvalue $r(f_\lambda)$ of $f_\lambda(x)$ depends on λ. As observed in [GS10], in the case of the two-sided tropical spectral problem $r(f_\lambda)$ equals the inverse minimal Chebyshev distance between Ax and λBx, thus yielding a good approximate solution of the problem. The study of general parametric mean-payoff games and the corresponding tropical linear problems relies on the properties of the spectral function (number of linear slopes, bounds on the zero-level set, extreme values, etc.) Gaubert et al. [AGK10b, GKS10] also observed the role of strategies in the associated mean-payoff game as certificates of unboundedness and optimality, analogous to the role of Lagrange multipliers. Moreover, the results of Akian et al. [AGG09a] indicate that the above approaches can be extended to infinite systems of parametric two-sided tropical (in)equalities, which would contribute to more general theory of zero-sum mean-payoff games.

Min-max functions are in the class of functions to which the nonlinear Perron-Frobenius theory developed by Nussbaum et al. [Nus86, AGLN06] can be applied. Such applications, also in the context of tropical convexity, have been described in [GS08, GS10]. The methods of nonlinear Perron-Frobenius theory can be applied on a wider scale of Shapley operators of general stochastic zero-sum mean-payoff games, in the spirit of Kohlberg, Neyman, Rosenberg and Sorin [NS03].

In the rest of the paper we give an overview of existing results on specific tropical parametric two-sided systems (i.e. parametric mean-payoff games) and associated min-max functions. In Section 2 we describe the main principles of the theory of mean-payoff games which we are going to use, as well as connections with the tropical two-sided systems, from the viewpoint of tropical matrix theory. In Section 3 we propose a unified approach to the theory of parametric tropical two-sided systems with piecewise-linear coefficients. This approach is based on the notion of spectral function, for which the problems of reconstruction and general tropical programming are formulated, and the pseudopolynomiality of these problems is shown. Moreover, bisection and Newton algorithms for the general tropical programming are explicitly described. In Section 4 we consider the generalized tropical eigenproblem $Ax = \lambda + Bx$ as a special case of parametric tropical two-sided system. We recall the main results of [GS10] on reconstruction (Theorem 13), that it is possible to reconstruct the whole spectral function and hence the whole spectrum in pseudopolynomial time. This now follows from the general approach of Section 3. Further we treat the special case of symmetric matrices obtaining an explicit formula for the unique eigenvalue of (A, B), thus precising a result of [BV07a, But10], and evaluate the spectral function in the case of one-matrix eigenproblem ($B = I$). Section 5 is devoted to the tropical linear programming as formulated in [GKS10]. This is viewed from the general perspective of Section 3, being a special case of the general tropical programming problem described there. In particular it is shown how bisection method can be applied to the "fractional" tropical programming formulation of Gaubert et al. [GKS10]. New results include a new bound for bisection method which is a generalization of the bounds obtained by Butkovič and Aminu [BA08], and comparison between bisection and Newton algorithms.

2. Mean-payoff games

2.1. **Two-player mean-payoff games.** Consider a two-player deterministic game, where the players "Max" and "Min" make alternate moves of a pawn on a weighted bipartite digraph $\mathcal{G}$. The set of nodes of $\mathcal{G}$ is the disjoint union of nodes $[m] := \{1, \ldots, m\}$ where Max is active, and nodes $[n] := \{1, \ldots, n\}$ where Min is active. When the pawn is in node $k \in [m]$ of Max, he must choose an arc in $\mathcal{G}$ connecting node k to some node $l \in [n]$ of Min, and while moving the pawn along this arc, he receives payment b_{kl} from Min, which is the weight of the selected arc. When the pawn is in node $j \in [n]$ of Min, she must choose an arc in $\mathcal{G}$ connecting node j to some node $i \in [m]$ of Max, and pays $-a_{ij}$ to Max, where $-a_{ij}$ is the weight of the selected arc. We assume that $b_{kl}, a_{ij} \in \mathbb{R}$. Moreover, certain moves may be prohibited, meaning that the corresponding arcs are not present in $\mathcal{G}$. Then, we set $b_{kl} = -\infty$ and $a_{ij} = -\infty$. Thus, the whole game is equivalently defined by two $m \times n$ matrices $A = (a_{ij})$ and $B = (b_{kl})$ with entries in $\mathbb{R} \cup \{-\infty\}$. We make the following assumptions:

Assumption 1. For all $k \in [m]$ there exists $l \in [n]$ such that $b_{kl} \neq -\infty$.

Assumption 2. For all $j \in [n]$ there exists $i \in [m]$ such that $a_{ij} \neq -\infty$.

This assures that both players have at least one move allowed in each node.

A *general strategy* of a player (Max or Min) is a function that for every finite preceding history of a play ending at a node i selects a successor of i (i.e., move of the player). A *positional strategy* for a player is a mapping that selects a unique successor of every node i independently of the preceding history of the play.

A strategy of Max will be usually denoted by σ and a strategy of Min will be usually denoted by τ. Thus a positional strategy of Max is a mapping $\sigma\colon \{1, \ldots, m\} \to \{1, \ldots, n\}$, and a positional strategy of Min is a mapping $\tau\colon \{1, \ldots, n\} \to \{1, \ldots, m\}$.

When Max reveals his positional strategy σ, the play proceeds within the graph $\mathcal{G}_\sigma$ where at each node i of Max all but one edge $(i, \sigma(i))$ are removed. When Min reveals her positional strategy τ, the play proceeds within the graph $\mathcal{G}_\tau$ where at each node j of Min all but one edge $(j, \tau(j))$ are removed.

When Max reveals his positional strategy σ and Min reveals her positional strategy τ, then the play proceeds within the graph $\mathcal{G}_{\sigma,\tau}$ where each node has a unique outgoing edge $(i, \sigma(i))$ or $(j, \tau(j))$. Thus $\mathcal{G}_{\sigma,\tau}$ is a sunflower graph, i.e., such that each node has a unique path to a unique cycle.

Suppose that a play begins at a node j of Min and proceeds p turns (from Min to Max and back). Then the total payment of Min equals

$$\Phi^{(p)}_{A\sharp B}(j,\tau,\sigma)=\sum_{t=1}^{p}-a_{l_t k_{t-1}}+b_{l_t k_t},\ k_0=j,\ k_t=\sigma(l_t),\ l_t=\tau(k_{t-1}), \tag{1}$$

where τ and σ are general (and thus the whole graph $\mathcal{G}$ may be required.

If both σ and τ are positional, then the limit

$$\Phi_{A\sharp B}(i,\tau,\sigma)=\lim_{n\to\infty}\Phi^{(n)}_{A\sharp B}(i,\tau,\sigma)/n \tag{2}$$

exists and is equal to the mean weight (per turn) of the unique cycle of $\mathcal{G}_{\sigma,\tau}$ accessible from j.

When only one strategy is positional the limit in (2) can be replaced by lim inf or lim sup which are within the interval between the smallest and the largest mean of the cycles accessible from j in $\mathcal{G}_\sigma$ or in $\mathcal{G}_\tau$. (The case when neither of the strategies are positional can be handled the same way but we do not require this general case.) The version of (2) with lim sup, or respectively lim inf, will be denoted by $\Phi^{\sup}_{A\sharp B}(i,\tau,\sigma)$, or respectively $\Phi^{\inf}_{A\sharp B}(i,\tau,\sigma)$.

2.2. **One-player mean-payoff games and tropical linear mappings.** We now consider the case when one player has a positional strategy and the other wants to maximize $\Phi^{\sup}_{A\sharp B}(i,\tau,\sigma)$ or to minimize $\Phi^{\inf}_{A\sharp B}(i,\tau,\sigma)$.

For simplicity, consider a game defined by just one matrix $C\in(\mathbb{R}\cup\{-\infty\})^{n\times n}$ such that there is a finite entry in each row. There is just one player Max who wants to maximize

$$\begin{aligned}\Phi^{\sup}_{C}(i,\sigma)&=\limsup_{p\to\infty}(\Phi^{(p)}_C(i,\sigma))/p,\ \text{where}\\ \Phi^{(p)}_C(i,\sigma)&=\sum_{t=1}^{p}c_{l_{t-1}l_t},\ l_0=i,\ l_t=\sigma(l_{t-1}),\end{aligned} \tag{3}$$

where σ is the (non-positional) strategy of Max.

Dually we can consider a game defined by matrix $D\in(\mathbb{R}\cup\{+\infty\})^{n\times n}$ such that there is a finite entry in each row. There is just one player Min who wants to minimize

$$\begin{aligned}\Phi^{\inf}_{D}(i,\tau)&=\liminf_{p\to\infty}(\Phi^{(p)}_D(i,\tau))/p,\ \text{where}\\ \Phi^{(p)}_D(i,\tau)&=\sum_{t=1}^{p}d_{l_{t-1}l_t},\ l_0=i,\ l_t=\tau(l_{t-1}),\end{aligned} \tag{4}$$

where τ is the (non-positional) strategy of Min.

Introducing the max-plus multiplication $C \otimes x$ and the min-plus multiplication $D \otimes' y$

$$(C \otimes x)_i = \max_j c_{ij} + x_j, \quad (D \otimes' y)_i = \min_j d_{ij} + y_j, \tag{5}$$

we note that the repeated actions of C and D on the zero vector 0, denoted shortly $C^k 0$ and $D^k 0$, describe the most of what the players can achieve. It is known (see e.g. [HOvdW05]) that C and D have *cycle-time vectors* meaning that the limits

$$\chi^{\max}(C) = \lim_{k\to\infty} (C^k 0)/k, \quad \chi^{\min}(D) = \lim_{k\to\infty} (D^k 0)/k \tag{6}$$

exist and can be written explicitly.

This explicit expression uses the concept of maximal (minimal) cycle mean. For $C = (c_{ij}) \in (\mathbb{R} \cup \{-\infty\})^{n\times n}$ ($D = (d_{ij}) \in (\mathbb{R} \cup \{+\infty\})^{n\times n}$), it is defined as

$$\begin{aligned} \mu^{\max}(A) &= \bigvee_{p=1}^{n} \bigvee_{i_1,\ldots,i_p} \frac{c_{i_1 i_2} + \cdots + c_{i_p i_1}}{p} \quad (\text{max-plus}) \ , \\ \mu^{\min}(D) &= \bigwedge_{p=1}^{n} \bigwedge_{i_1,\ldots,i_p} \frac{d_{i_1 i_2} + \cdots + d_{i_p i_1}}{p} \quad (\text{min-plus}) \ . \end{aligned} \tag{7}$$

For $C \in (\mathbb{R} \cup \{-\infty\})^{n\times n}$ define the associated digraph $\mathcal{G}_C = \{[n], E\}$ so that $(i,j) \in E$ whenever c_{ij} is finite. Analogously for $D \in (\mathbb{R} \cup \{\infty\})^{n\times n}$. We will say that i *accesses* j if there exists a path from i to j, every edge of which has finite weight. Denote by $\mu_i^{\max}(C)$ ($\mu_i^{\min}(D)$) the maximal (minimal) circuit mean of the component of $\mathcal{G}_C$ (or $\mathcal{G}_D$) to which i belongs. These numbers are given by the same expressions as in (7), but with $i_1, \ldots, i_p$ restricted to that component.

Using $\mu_i^{\max}(C)$ ($\mu_i^{\min}(D)$), we can write explicit expressions for the cycle-time vector of a max-plus linear map $x \mapsto Cx$, or min-plus linear map $x \mapsto Dx$:

$$\begin{aligned} \chi_i^{\max}(C) &= \max\{\mu_j^{\max}(C),\ i \text{ accesses } j\} \quad (\text{max-plus}), \\ \chi_i^{\min}(D) &= \min\{\mu_j^{\min}(D),\ i \text{ accesses } j\} \quad (\text{min-plus}). \end{aligned} \tag{8}$$

See [CTCG$^+$98] or [HOvdW05] for proof.

These explicit formulae for $\chi_i^{\max}(C)$ and $\chi_i^{\min}(D)$ lead to the following observation.

Proposition 1. *$\Phi_C^{\max}(i,\sigma) \leq \chi_i^{\max}(C)$ for any i and σ. Dually $\Phi_D^{\min}(i,\tau) \geq \chi_i^{\min}(D)$ for any i and τ.*

Proof. $\Phi_C^{\max}(i,\sigma)$ cannot be greater than the largest mean of a cycle accessible from i in $\mathcal{D}(C)$, which equals $\chi_i^{\max}(C)$. Dually $\Phi_D^{\min}(i,\tau)$ cannot be less than the smallest mean of a cycle accessible from i in $\mathcal{D}(D)$, which equals $\chi_i^{\min}(D)$. $\square$

There is more to say here. Actually there exist positional strategies σ and τ which turn the inequalities in Proposition 1 into equalities. They can be deduced from the existence of invariant halflines, as it will be explained in the next subsection.

The setting in which both max-plus and min-plus matrix multiplications are considered simultaneously has been called minimax algebra by Cuninghame-Green [CG79]. Then, we need to allow the scalars to belong to the enlarged set. Note that in $\overline{\mathbb{R}}_{\max} := \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$, $(-\infty) + (+\infty) = -\infty$ if the max-plus convention is understood, and $(-\infty) + (+\infty) = +\infty$ if the min-plus convention is understood.

Max-plus and min-plus linear maps are mutually adjoint, or *residuated*. Recall that for a max-plus linear map A from $\overline{\mathbb{R}}^n_{\max}$ to $\overline{\mathbb{R}}^m_{\max}$, the *residuated operator* $A^\sharp$ from $\overline{\mathbb{R}}^m_{\max}$ to $\overline{\mathbb{R}}^n_{\max}$ is defined by

$$(A^\sharp y)_j := \bigwedge_{i=1}^{m} (-a_{ij} + y_i) \ , \tag{9}$$

with the convention $(-\infty) + (+\infty) = +\infty$. Note that this operator, also known as *Cuninghame-Green inverse*, sends $\mathbb{R}^m_{\max}$ to $\mathbb{R}^n_{\max}$ whenever A does not have columns identically equal to $-\infty$. The term "residuated" refers to the property

$$Ax \leq y \Leftrightarrow x \leq A^\sharp y \ , \tag{10}$$

where $\leq$ is the partial order on $\mathbb{R}^m_{\max}$ or $\mathbb{R}^n_{\max}$. This residuated operator is crucial for max-plus two-sided systems of inequalities, since

$$Ax \leq Bx \Leftrightarrow x \leq A^\sharp Bx \ . \tag{11}$$

The notation $A^\sharp Bx$ is understood as composition of two operators: Bx performs max-plus product of matrix B and vector x, and $A^\sharp(Bx)$ performs min-plus product of $A^\sharp$ and Bx. Further this composition will be written just as $A^\sharp B$.

2.3. **Min-max functions and invariant halflines.** The dynamic operator of a two-player game described in Subsection 2.1 is precisely $A^\sharp B$ which appears in (11):

$$(A^\sharp Bx)_j = \min_{k\in[m]} (-a_{kj} + \max_{l\in[n]} (b_{kl} + x_l)) \ . \tag{12}$$

This is known as a *min-max function* [CTGG99]. Min-max functions are isotone ($x \leq y \Rightarrow A^\sharp Bx \leq A^\sharp By$) and additively homogeneous ($A^\sharp B(\lambda + x) = \lambda + A^\sharp Bx$). Hence, they are nonexpansive in the sup-norm. Moreover, they are piecewise affine ($\mathbb{R}^n$ can be covered by a finite number of polyhedra on which $A^\sharp B$ is affine). We are again interested in the cycle-time vector:

$$\chi(f) = \lim_{k\to\infty} (f^k 0)/k \ . \tag{13}$$

The existence of $\chi(A^\sharp B)$ follows from a theorem of Kohlberg.

Theorem 2 (Kohlberg [Koh80])**.** *Let $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ be a nonexpansive and piecewise affine map. Then, there exist $v \in \mathbb{R}^n$ and $\chi \in \mathbb{R}^n$ such that*

$$f(v + t\chi) = v + (t+1)\chi \ , \quad \forall t \geq T \ , \tag{14}$$

where T is a large enough real number.

The map $t \mapsto v + t\chi$ is known as an *invariant half-line*. Using the nonexpansiveness of $A^\sharp B$, one deduces that the limit (13) exists, is the same for all $x \in \mathbb{R}^n$ and is equal to the growth rate χ of any invariant halfline. All above said also applies to purely max-plus or purely min-plus functions described in Subsect. 2.2.

In the case of min-max functions (as well as purely max-plus or min-plus maps) there are constructive ways of proving the existence of invariant halflines, which rely on efficient algorithms for their computation. like the policy iteration of Cochet-Terrasson et al. [CTGG99], improved by Dhingra and Gaubert [DG06]. Below we deduce the existence of value, originally due to Ehrenfeucht and Mycielski [EM79], from the existence of invariant halflines.

Theorem 3. *For the two-player (zero-sum, deterministic) mean-payoff game with players Max and Min, where costs are given by matrices $A, B \in (\mathbb{R} \cup \{-\infty\})^{m\times n}$, there exists a pair of positional strategies σ^* and τ^* such that*

1. $\Phi^{\sup}_{A^\sharp B}(i, \tau^*, \sigma) \leq \chi_i(A^\sharp B)$ *for all (not necessarily positional) strategies σ;*
2. $\Phi^{\inf}_{A^\sharp B}(i, \tau, \sigma^*) \geq \chi_i(A^\sharp B)$ *for all (not necessarily positional) strategies τ.*

Proof. First notice that σ^* and τ^* can be defined from v, namely one takes the indices where max and min are attained in $\min_l -a_{li} + (\max_k b_{lk} + v_k)$.

Fix τ^* and let σ vary. We obtain a one-player game described in Subsect. 2.2, with player Max solo. Then χ, v is also an invariant halfline of the pure max-plus linear function $A^\sharp_{\tau^*}B$:

$$(A^\sharp_{\tau^*}Bx)_i = \max_j(-a_{\tau^*(i)i} + b_{\tau^*(i)j} + x_j). \tag{15}$$

This can be also expressed by max-plus matrix C_{τ^*} whose entries are

$$(C_{\tau^*})_{ij} = -a_{\tau^*(i)i} + b_{\tau^*(i)j}. \tag{16}$$

Dually, fix σ^* and let τ vary. We obtain a one-player game described in Subsect. 2.2, with player Min solo. Then χ, v is also an invariant halfline of the pure min-plus linear function $A^\sharp B^{\sigma^*}$:

$$(A^\sharp B^{\sigma^*}x)_i = \min_k(-a_{ki} + b_{k\sigma^*(k)} + x_{\sigma^*(k)}). \tag{17}$$

This can be also expressed by min-plus matrix D_{σ^*} whose entries are

$$(D_{\sigma^*})_{ik} = \min_{l\colon k=\sigma^*(l)}(-a_{li} + b_{lk}). \tag{18}$$

Each strategy σ of Max in the two-player game corresponds to a strategy π in the one-player game defined by C_{τ^*}, defined by $\pi(i) = \sigma\tau^*(i)$. Then $\Phi^{(n)}_{C_{\tau^*}}(i,\pi) = \Phi^{(n)}_{A^\sharp B}(i,\sigma,\tau^*)$, hence also $\Phi^{\sup}_{C_{\tau^*}}(i,\pi) = \Phi^{\sup}_{A^\sharp B}(i,\sigma,\tau^*)$. Applying Proposition 1 to C_{τ^*} and π we obtain that $\Phi^{\sup}_{A^\sharp B}(i,\sigma,\tau^*) \leq \chi_i(A^\sharp B)$.

Each strategy τ of Min in the two-player game corresponds to a strategy π in the one-player game defined by D_{σ^*} defined by $\pi(i) = \sigma^*\tau(i)$. Then $\Phi^{(n)}_{D_{\sigma^*}}(i,\pi) \leq \Phi^{(n)}_{A^\sharp B}(i,\sigma^*,\tau)$, hence also $\Phi^{\inf}_{D_{\sigma^*}}(i,\pi) \leq \Phi^{\inf}_{A^\sharp B}(i,\sigma^*,\tau)$. Applying Proposition 1 to D_{σ^*} and π we obtain that $\Phi^{\inf}_{A^\sharp B}(i,\sigma^*,\tau) \geq \chi_i(A^\sharp B)$.

The claims are proved. □

It follows that $\chi_i(A^\sharp B)$ is determined uniquely by

$$\chi_i(A^\sharp B) = \Phi(i,\tau^*,\sigma^*) = \min_\tau \max_\sigma \Phi(i,\tau,\sigma) = \max_\sigma \min_\tau \Phi(i,\tau,\sigma), \tag{19}$$

We will also need the game beginning at a node i of Max. Formally, dynamic operator of such a game is different: $BA^\sharp : \mathbb{R}^m \mapsto \mathbb{R}^m$ defined by

$$(BA^\sharp x)_i = \min_{l\in[n]}(b_{il} + \max_{k\in[m]}(-a_{kl} + x_k)) \ . \tag{20}$$

Further notations related to this game:

$$\begin{aligned} &\Phi^{(n)}_{BA^\sharp}(j,\tau,\sigma) = \sum_{t=1}^n b_{l_{t-1}k_t} - a_{l_tk_t},\ l_0 = i,\ k_t = \sigma^{(t)}(l_{t-1}),\ l_t = \tau^{(t)}(k_t),\\ &\Phi_{BA^\sharp}(j,\tau,\sigma) = \lim_{n\to\infty} \Phi^{(n)}_{BA^\sharp}(j,\tau,\sigma)/n \end{aligned} \tag{21}$$

Evidently $BA^\sharp$ is also a min-max function and all above said including Theorems 2 and 3 applies to this situation too. Next we establish the relation between $\chi(A^\sharp B)$ and $\chi(BA^\sharp)$.

Proposition 4. *Let σ^* and τ^* be optimal strategies of Max and Min for the game starting at a node of Min. Then these strategies are also optimal for the game starting at a node of Max. Further let $i \in [m], j \in [n]$ and $i = \tau^*(j)$ or $j = \sigma^*(i)$. Then $\chi_i(BA^\sharp) = \chi_j(A^\sharp B)$.*

Proof. Let $i=\tau^*(j)$.

Fix the strategy τ^* of Min and let the other strategy σ vary. Then $\Phi_{BA^\sharp}(i,\tau^*,\sigma)=\Phi_{A^\sharp B}(j,\tau^*,\sigma)$ for any σ. Applying Theorem 3 we obtain $\Phi_{BA^\sharp}(i,\tau^*,\sigma)\leq\chi_j(A^\sharp B)$.

Fix the strategy σ^* of Max and let the other strategy τ vary. Notice that $\Phi_{BA^\sharp}(i,\tau,\sigma^*)=\Phi_{A^\sharp B}(j,\tilde{\tau},\sigma^*)$ for any τ, where the (nonpositional) strategy $\tilde{\tau}$ first takes $i=\tau^*(j)$ and then proceeds according to τ. Applying Theorem 3 we obtain $\Phi_{BA^\sharp}(i,\tau,\sigma^*)\geq\chi_j(A^\sharp B)$. Hence $\chi_j(A^\sharp B)$ is the value of the game which starts at the node i of Max. This value also equals $\chi_i(BA^\sharp)$.

The case $j=\sigma^*(i)$ is treated analogously. ∎

2.4. **Mean-payoff games and two-sided systems of inequalities.** The starting point already appeared above as (11): $Ax\leq Bx\Leftrightarrow x\leq A^\sharp Bx$. Moreover, positional strategies $\sigma\colon[m]\mapsto[n]$ and $\tau\colon[n]\mapsto[m]$ correspond to affine mappings B^σ and A_τ defined by

$$(A_\tau)_{ij}=\begin{cases}a_{ij} & \text{if } i=\tau(j),\\ -\infty & \text{otherwise,}\end{cases}\qquad (B^\sigma)_{ij}=\begin{cases}b_{ij} & \text{if } j=\sigma(i),\\ -\infty & \text{otherwise.}\end{cases}\tag{22}$$

The existence of value in mean-payoff games (Theorem 3in the form of (19)) can be expressed in max(min)-plus algebra as follows:

$$\min_{\tau\in\mathcal{T}}\chi(A^\sharp_\tau B)=\chi(A^\sharp B)=\max_{\sigma\in\mathcal{S}}\chi(A^\sharp B^\sigma)\ .\tag{23}$$

where $\mathcal{T}$ (resp. $\mathcal{S}$) is the set of positional strategies of Min (resp. Max). This form of (19) was obtained by Gaubert and Gunawardena [GG98].

The following observation due to Akian et al. [AGG09a] relates the solutions of $Ax\leq Bx$ and the nonnegative coordinates of $\chi(A^\sharp B)$. These coordinates correspond to *winning states* of the game: if the game starts in these states, then Max can ensure nonnegative profit with any positional strategy of Min.

Theorem 5 ([AGG09a, Th. 3.2])**.** *Let $A,B\in(\mathbb{R}\cup\{-\infty\})^{m\times n}$. Then, $\chi_i(A^\sharp B)\geq 0$ if and only if there exists $x\in(\mathbb{R}\cup\{-\infty\})^n$ such that $Ax\leq Bx$ and $x_i\neq-\infty$.*

Proof. The "if" part: If there exists x with $x_i\neq-\infty$ such that $Ax\leq Bx$, then we take any finite $y\in\mathbb{R}^n$ such that $y\geq x$. We have $x\leq A^\sharp Bx\leq A^\sharp By$. Applying $A^\sharp B$ to this inequality k times, we obtain $x\leq(A^\sharp B)^k y$ for all k. Using limit expression (13), we obtain $\chi_i(A^\sharp B)\geq\lim_{k\to\infty}x_i/k=0$.

The "only if" part: Let v be an invariant halfline of $A^\sharp B$:

$$\min_k(-a_{ki}+\max_l(b_{kl}+v_l+t\chi_l))=v_i+(t+1)\chi_i\ ,\ \ t\geq T\ \ ,\forall i\in[n]\ ,\tag{24}$$

and let $I := \{i\colon \chi_i \geq 0\}$. For all $i \in I$, in all max-linear brackets that contribute to the minimization, there must be at least one $l \in I$ such that b_{kl} is finite, otherwise the l.h.s. of (24) would fall below v_i at sufficiently large t. Also in any such bracket Max will choose $l \in I$ for all sufficiently large t. This shows that for $i \in I$ and large enough t in (24), we can replace v by the reduced vector $\tilde{v}$ such that $\tilde{v}_i = v_i$ for $i \in I$ and $\tilde{v}_i = -\infty$ for $i \notin I$. Taking $x := \tilde{v} + t\chi$ for large enough t, we get $A^\sharp Bx \geq x$ and hence $Ax \leq Bx$. $\square$

Theorem 5 shows that to decide whether $Ax \leq Bx$ can be satisfied by a vector x such that $x_i \neq -\infty$, we can exploit a *mean-payoff oracle*. This oracle will decide whether i is a winning node of the associated mean-payoff game and give a winning strategy of Max. This oracle can be implemented either by using the value iteration method, which is pseudo-polynomial [ZP96], by the approach of Puri (solving an associated discounted game for a discount factor close enough to 1 by policy iteration [Pur95]), by using the policy iteration algorithm for mean payoff games of [CTGG99, GG98, DG06], or the one of [BV07b].

A weaker version of Theorem 5 uses the spectral radius $r(A^\sharp B)$ meaning just the greatest eigenvalue of $A^\sharp B$. The spectral radius of additively homogeneous and isotone function $f\colon (\mathbb{R}\cup\{-\infty\})^n \to (\mathbb{R}\cup\{-\infty\})^n$ satisfies the following identities from the nonlinear Perron-Frobenius theory [Nus86]:

$$
\begin{aligned}
r(f) &= \inf\{\lambda\colon \exists x \in \mathbb{R}^n \quad f(x) \leq \lambda + x\} \\
r(f) &= \max\{\lambda\colon \exists x \in (\mathbb{R}\cup\{-\infty\})^n \quad f(x) \geq \lambda + x\}.
\end{aligned} \tag{25}
$$

Using the first of these identities and the definition of χ(13) we obtain that $r(f) = \overline{\chi}(f) := \max_i \chi_i(f)$ for any $+$-homogeneous and isotone f. Using the second identity we obtain a weaker version of Theorem 5.

Proposition 6. *Let $A, B \in (\mathbb{R}\cup\{-\infty\})^{m\times n}$. Then $r(A^\sharp B) \geq 0$ if and only if there exists nontrivial $x \in (\mathbb{R}\cup\{-\infty\})^n$ such that $Ax \leq Bx$.*

3. The method of spectral function

3.1. **Elementary properties and reconstruction.** We are now interested in the case when $A, B \in (\mathbb{R}\cup\{-\infty\})^{m\times n}$ depend on one parameter λ. Namely, let an entry of A or B be an (L, M) *piecewise linear function*, by which we mean a function $f(\lambda)$ consisting of a finite number of linear pieces $\alpha + \lambda k$, where $|\alpha| \leq M$ and $|k| \leq L$. We also assume that the slopes k are integer. We will also consider the *monomial* case where each entry of A and B is of the form $\alpha + \lambda k$ with the same restrictions on α and k.

Applying the principle $A(\lambda)x \leq B(\lambda)x \Leftrightarrow x \leq A^\sharp(\lambda)B(\lambda)x$ we obtain parametric min-max function $A^\sharp(\lambda)B(\lambda)$ which is dynamic operator of a mean-payoff game with parametric costs. The value of this game starting in a node j of Min will be given by the *spectral function*

$$\phi_j(\lambda) := \chi_j(A^\sharp(\lambda)B(\lambda)) \tag{26}$$

The pointwise maximum of $\phi_j(\lambda)$ yields the *principal spectral function*

$$\overline{\phi}(\lambda) := r(A^\sharp(\lambda)B(\lambda)) \tag{27}$$

Fixing positional strategy $\sigma : \{1,\ldots,m\} \to \{1,\ldots,n\}$ of Max or $\tau : \{1,\ldots,n\} \to \{1,\ldots,m\}$ of Min gives rise to the partial spectral functions

$$\begin{aligned} \phi_j^\sigma(\lambda) &:= \chi_j(A^\sharp(\lambda)B^\sigma(\lambda)), \quad \phi_j^\tau(\lambda) := \chi_j(A_\tau^\sharp(\lambda)B(\lambda)),\\ \overline{\phi^\sigma}(\lambda) &:= r(A^\sharp(\lambda)B^\sigma(\lambda)), \quad \overline{\phi^\tau}(\lambda) := \overline{\chi}(A_\tau^\sharp(\lambda)B(\lambda)) \end{aligned} \tag{28}$$

Collecting the results of Theorem 3, Theorem 5 and Proposition 6 yields the following.

Theorem 7. *System $A(\lambda)x \leq B(\lambda)x$ is solvable if and only if $\overline{\phi}(\lambda) \geq 0$. Moreover it has solution with x_j finite if and only if $\phi_j(\lambda) \geq 0$.*

Theorem 8. *The spectral functions can be represented in terms of partial spectral functions as follows:*

$$\phi_j(\lambda) = \max_\sigma \phi_j^\sigma(\lambda) = \min_\tau \phi_j^\tau(\lambda), \quad \overline{\phi}(\lambda) = \max_\sigma \overline{\phi^\sigma}(\lambda) = \min_\tau \overline{\phi^\tau}(\lambda). \tag{29}$$

We note that partial spectral functions at a given point can be found explicitly by means of Karp's algorithm (8). The same formulae combined with Theorem 8 yield the following observations.

Proposition 9. *Let the entries of $A(\lambda), B(\lambda) \in (\mathbb{R}\cup\{-\infty\})^{m\times n}$ be (L, M) piecewise-linear functions with integer slopes. Then all spectral functions are $(L, 2M)$ piecewise-linear functions, whose slopes are of the form k/p where $1 \leq p \leq \min(m,n)$ and $|k| \leq pL$.*

Proof. Any coordinate of χ of a min-max function or of a purely max-plus(min-plus) map is the mean weight of a certain cycle, see (7) and (8). This shows that any spectral function is of the form $\alpha + \lambda(k/p)$ at any given λ, where $|\alpha| \leq 2M$ (using that cycle mean does not exceed max), and $|k| \leq pL$ (using the given bound L on the slopes of linear functions). The integer p is half of the length of a certain cycle in the bipartite graph $\mathcal{G}$ of the game, hence it is bounded by $\min(m,n)$. □

Proposition 10. *If the entries of $A(\lambda)$ and $B(\lambda)$ are monomial in λ, then $\phi_j^\tau(\lambda)$ and $\overline{\phi^\tau(\lambda)}$ are convex, and partial spectral functions $\phi_j^\sigma(\lambda)$ are concave.*

Proof. Follows from (7) and (8). □

Note that spectral functions $\phi_j(\lambda)$ and also partial spectral functions $\overline{\phi^\sigma}(\lambda)$ are neither concave nor convex in general.

Also the assumption of monomiality can be relaxed. For instance, if the entries of $B(\lambda)$ are max-polynomials and the entries of $A(\lambda)$ are min-polynomials, then $\phi_j^\tau(\lambda)$ and $\overline{\phi^\tau}$ are still convex, and if the entries of $B^(\lambda)$ are min-polynomials and the entries of $A^(\lambda)$ are max-polynomials, then $\phi_j^\sigma(\lambda)$ are still concave. The properties of convexity and concavity mean that the partial functions consist of smaller number of linear pieces and can be reconstructed more quickly, like the characteristic max-polynomials of matrices [But10], Subsect. 5.3.3.

Proposition 11. *Let the entries of $A(\lambda), B(\lambda) \in (\mathbb{R} \cup \{-\infty\})^{m\times n}$ be (L, M) piecewise-linear functions with integer slopes. Then the spectral functions are linear at $\lambda \geq 4M(\min(m, n))^2$ and $\lambda \leq -4M(\min(m, n))^2$.*

Proof. By Proposition 9 the spectral functions are piecewise linear and the linear pieces are of the form $a + (k/p)\lambda$ where in particular $p \leq \min(m, n)$ and $|a| \leq 2M$. Considering the intersection point of one such piece $a_1 + (k_1/p_1)\lambda$ with another piece $a_2 + (k_2/p_2)\lambda$ we deduce from

$$| \frac{k_1}{p_1} - \frac{k_2}{p_2} | \geq \frac{1}{(\min(m, n))^2}, \quad |a_1 - a_2| \leq 4M,$$

that $|\lambda| \leq 4M(\min(m, n)^2)$. This means that at $\lambda \geq 4M(\min(m, n))^2$ and $\lambda \leq -4M(\min(m, n))^2$ any spectral function is linear. □

To determine the linear slopes at $|\lambda| \geq 4M(\min(m, n))^2$ we can take, for each coefficient, only the slopes at $\pm\infty$ and set the offsets to 0. Then we "play" the mean-payoff game at $\lambda = \pm 1$.

Proposition 12. *Let the entries of $A, B \in (\mathbb{R} \cup \{-\infty\})^{m\times n}$ be (L, M) piecewise-linear functions with integer slopes and integer offsets of the linear pieces. Then the interval between breaking points of spectral functions is not less than $1/(\min(m, n))^4$. Hence spectral functions consist of no more than $8M(\min(m, n))^6$ pieces.*

Proof. When the offsets are integer, the difference between two breaking points is not less than the difference between their inverse denominators. The denominators of breaking points do not exceed $(\min(m, n))^2$ (see Proposition 9), and hence the difference between their inverses is not less than $1/(\min(m, n))^4$. □

There exist pseudopolynomial algorithms computing values of mean-payoff games. For instance Zwick and Paterson [ZP96] propose a value iteration algorithm with $O(mn^4M)$

complexity. Using this we now formulate the main result of this subsection, about the pseudopolynomial reconstruction of spectral functions.

Theorem 13. *Let the entries of $A, B \in (\mathbb{R} \cup \{-\infty\})^{m\times n}$ be (L, M) piecewise-linear functions with integer slopes and integer offsets of the linear pieces, and let $C(m, n, M)$ be the complexity of a mean-payoff oracle that computes value of $m\times n$ mean-payoff game such that all costs are integers bounded by M. Then all linear pieces that constitute the graph of $\phi_i(\lambda)$ can be identified in $O(M \min(m, n)^6 \times C(m, n, M \min(m, n)^2 + 4LM \min(m, n)^4))$ operations.*

Proof. It follows from Proposition 12 that we can reconstruct the graph of spectral function in $O(M \min(m, n)^6)$ calls of a mean-payoff oracle. The costs in the mean-payoff games that the oracle works with, are of the form $a + k\lambda$ where $|a| \leq M$, $|k| \leq L$ and λ ia fixed rational number whose denominator D does not exceed $\min(m, n)^2$, and the absolute value of numerator does not exceed $4M \min(m, n)^2 D$. Thus $a + k\lambda$ is a rational number whose numerator does not exceed $MD + 4LM \min(m, n)^2 D$. All costs in the mean-payoff game are rational numbers with equal denominators D. The properties of the game will not change if we multiply all costs by D, and the complexity of mean-payoff oracle will not exceed $C(m, n, MD + 4LM \min(m, n)^2 D)$. This shows the claim. □

The technique of spectral function has been applied in [GS10] to the generalized eigenproblem $Ax = \lambda + Bx$ yielding a method for identifying the whole spectrum of (A, B) in pseudopolynomial time. It can also be applied to the case when $A(\lambda)$ and $B(\lambda)$ are max-polynomial matrices.

3.2. **General tropical programming.** Assume as above that the entries of $A(\lambda)$ and $B(\lambda)$ are (L, M) piecewise linear functions with integer slopes. When $\phi(\lambda)$ is nondecreasing we can pose the following problem.

Problem 1 (General Tropical Programming)**.** *Given $A(\lambda), B(\lambda) \in (\mathbb{R} \cup \{-\infty\})^{m\times n}$ with nondecreasing $\phi_i(\lambda)$, find the minimal λ at which the system $A(\lambda)x \leq B(\lambda)x$ has a solution with $x_i \neq -\infty$.*

The spectral functions are nondecreasing in particular when all entries in A are nonincreasing functions, and all entries in B are nondecreasing functions. For this note in particular that $A^\sharp(\lambda)B(\lambda)x$ is nondecreasing with λ, and that both $\chi(f)$ and $r(f)$ are isotone with respect to f.

In terms of spectral functions, solving Problem 1 is equivalent to finding the least value of λ such that $\phi_i(\lambda) \geq 0$. As $\phi_i(\lambda)$ is a continuous and monotone function and has piecewise-linear structure, Problem 1 can be solved by a variation of bisection method, which we now formulate in more details.

Algorithm 1 (Bisection method)**. Start.** Set $U(0) := 4M\min(m,n)^2$ and $L(0) := -4M\min(m,n)^2$.

Check $\phi_i(L(0)) \leq 0$. If not, then compute $\phi_i(L(0))$ and the slope at $-\infty$. Return its intersection with zero level, if it exists. If not, the problem is unbounded.

Check $\phi_i(U(0)) \geq 0$. If not, then compute $\phi_i(U(0))$ and the slope at $+\infty$. Return its intersection with zero level, if it exists. If not, the problem is infeasible.

Iteration k**.** At each iteration, compute $\mu = (U(k)+L(k))/2$. If $\phi_i(\mu) \geq 0$ then set $U(k+1) = \mu$ and $L(k+1) = L(k)$. Otherwise set $U(k+1) = U(k)$ and $L(k+1) = \mu$.

Stop. After each iteration verify whether $U(k+1) - L(k+1) < 1/(\min(m,n))^4$. If true then return $\lambda = (\phi_i(U(k+1))L(k+1) + \phi_i(L(k+1))U(k+1))/L(k+1) + U(k+1)$.

We can show that the bisection algorithm is pseudoplolynomial using the same arguments as in Theorem 13

Theorem 14. *Let the entries of* $A, B \in (\mathbb{R} \cup \{-\infty\})^{m\times n}$ *be* (L, M) *piecewise-linear functions with integer slopes and integer offsets of the linear pieces, and let* $\tilde{C}(m,n,M)$ *be the complexity of a mean-payoff oracle that verifies that the value of* $m \times n$ *mean-payoff game, such that all costs are integers bounded by* M*, is nonnegative. Then the main phase of the bisection method requires no more than* $O(\log(M\min(m,n)^6)\tilde{C}(m,n,M^2(\min(m,n))^6 + 4LM^2(\min(m,n))^8)$ *operations.*

Proof. Following the proof of Theorem 13 we show that the complexity of one call to a mean-payoff oracle does not exceed $\tilde{C}(m,n,M\tilde{D} + 4LM\min(m,n)^2\tilde{D})$ where $\tilde{D}$ is the greatest denominator of values λ which we may encounter. The denominators of λ are bounded by $2^{\log(M\min(m,n)^6)} = M\min(m,n)^6$. □

Further research. Note that the bound on Bisection is not necessarily worse than that on the general reconstruction: if we use the above mentioned value iteration of Zwick and Paterson [ZP96], then $C(m,n,M) = mn^4M$ and $\tilde{C}(m,n,M) = mn^3M$. Hence Theorem 13 yields $M\min(m,n)^6(mn^3LM\min(m,n)^4) = LM^2mn^3\min(m,n)^10$ while Theorem 14 yields $\log(M\min(m,n)^6)mn^2LM^2\min(m,n)^8$: the factor after logarithm still looks $\min(m,n)^3$ better than in the general reconstruction. Still it would be nice to improve the bisection algorithm by introducing rounding to the points with bounded denominators.

Having in mind (29) we can also formulate Newton iterations. For this we need the concepts of stategies which are *left-(right-)optimal at* λ: positional strategies of Max or Min which are optimal also in some left (right) neighbourhood of λ.

Algorithm 2 (Positive Newton iteration)**. Start.**Set $\lambda_0 := 4M\min(m,n)^2$

Check $\phi_i(\lambda_0) \geq 0$. If not, then compute $\phi_i(\lambda_0)$ and the slope at $+\infty$. Return its intersection with zero level, if it exists. If not, the problem is infeasible.

Iteration k**.** Find a left-optimal strategy $\sigma \in S$ at λ_{k-1} and compute $\lambda_k = \min\{\lambda\colon \phi_j^\sigma(\lambda) \geq 0\}$.

Stop. Verify $\lambda_k = \lambda_{k-1}$ or $\lambda_k = -\infty$.

Algorithm 3 (Negative Newton iteration)**. Start.**Set $\lambda_0 := -4M\min(m,n)^2$.

Check $\phi_i(\lambda_0) \leq 0$. If not, then compute $\phi_i(L(0))$ and the slope at $-\infty$. Return its intersection with zero level, if it exists. If not, the problem is unbounded.

Iteration k**.** Find a (right-)optimal strategy $\tau \in T$ at λ_{k-1} and compute $\lambda_k = \min\{\lambda\colon \phi_i^\tau(\lambda) \geq 0\}$.

Stop. Verify $\phi_i(\lambda_k) = 0$ or $\lambda_k = +\infty$.

In the negative Newton iteration, the strategy can be just optimal (though the right-optimality may be a better option).

Proposition 15. *Algoritms 2 and 3, terminate in a finite number of steps, the number of which does not exceed, respectively, the number of strategies of Max and Min.*

Proof. At different iterations of Algorithm 2 we have different strategies, because $\lambda_k = \min\{\lambda\colon \phi_i^\sigma(\lambda) \geq 0\}$ are different for all k. Similarly for Algorithm 3, with τ instead of σ (where $\lambda_k \neq \lambda_{k-1}$ also follows as long as $\phi_i(\lambda_{k-1}) < 0$). Thus the number of steps is limited by the number of strategies, which is finite. □

Note that the implementation of Newton iterations crucially depends on the properties of partial spectral functions. These algorithms become available in practice when partial spectral functions are convex/concave, see Proposition 10.

The tropical linear programming described below in Section 5 is a special case of Problem 1. In this case both bisection method and Newton iterations can be effectively implemented. In particular, the entries of A are constant and the entries of $B(\lambda)$ are monomial, so that the partial spectral functions do possess convexity/concavity properties.

4. Tropical two-sided eigenproblem

4.1. **Two-sided eigenproblem as a mean-payoff game.** The method of spectral function was applied in [GS10] to the tropical two-sided eigenproblem, where one seeks scalars λ such that $Ax = \lambda + Bx$ where $A, B \in (\mathbb{R}\cup\{-\infty\})^{m\times n}$ can be satisfied by a nontrivial $x \in (\mathbb{R}\cup\{-\infty\})^n$. Such λ and x are then called eigenvalue and eigenvector of (A, B).

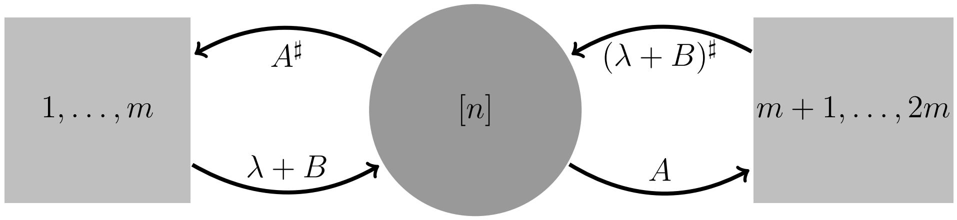

FIGURE 1. Mean-payoff game for $Ax=\lambda+Bx$

The problem $Ax=\lambda+Bx$ can be expressed as $C(\lambda)x\leq D(\lambda)x$ where

$$C(\lambda)=\begin{pmatrix}A\\ \lambda+B\end{pmatrix},\quad D(\lambda)=\begin{pmatrix}\lambda+B\\ A\end{pmatrix} \tag{30}$$

A diagram of the corresponding mean-payoff game is shown in Figure 1

The min-max function is $C^\sharp(\lambda)D(\lambda)$ can be also expressed as

$$g_\lambda(x)=A^\sharp Ax\wedge B^\sharp Bx\wedge A^\sharp(\lambda+B)x\wedge(\lambda+B)^\sharp Ax. \tag{31}$$

It can be shown that the following reduced versions of g_λ have the same spectral radius (or $\overline{\chi}$):

$$\begin{aligned}f_\lambda(x)&=x\wedge A^\sharp(\lambda+B)x\wedge(\lambda+B)^\sharp Ax,\\ h_\lambda(x)&=A^\sharp(\lambda+B)x\wedge(\lambda+B)^\sharp Ax.\end{aligned} \tag{32}$$

More precisely we have the following [GS10]:

$$\overline{\phi}(\lambda):=r(f_\lambda)=r(h_\lambda)=-\min_x|Ax-(\lambda+Bx)|, \tag{33}$$

where || means the max norm (greatest absolute value of coordinates). Now we collect the basic properties of $\overline{\phi}(\lambda)$ which follow from the results of Subsect. 3.1 and (33).

Theorem 16 (Gaubert and Sergeev [GS10])**.** *Let $A,B\in(\mathbb{R}\cup\{-\infty\})^{m\times n}$ and suppose that $|a_{ij}|\leq M$ and $|b_{ij}|\leq M$ for all finite a_{ij} and b_{ij}. Then*

1. *$\overline{\phi}(\lambda)\leq 0$, and the spectrum of (A,B) is precisely the zero-level set of $\overline{\phi}(\lambda)$;*
2. *$\overline{\phi}(\lambda)$ is $(1,2M)$ piecewise linear function. It can change slopes only at $|\lambda|\leq 4M(\min(2m,n))^2$, and it has no more than $4M(\min(2m,n))^6$ slopes when A,B have integer entries.*
3. *When A,B have integer entries, both $\overline{\phi}(\lambda)$ and the spectrum of (A,B) can be reconstructed in pseudopolynomial time.*

See [GS10] for more specific results and treatment of special cases.

4.2. **Examples of spectral functions.** We further describe a couple of special cases when $\phi(\lambda)$ can be identified with the least 1-Lipschitz ($|f(\lambda_1) - f(\lambda_2)| \leq |\lambda_1 - \lambda_2|$) function having a given zero-level set Λ, which we denote by $\phi_\Lambda(\lambda)$.

First we present the following example constructed in [**?**] and the last section of [GS10].

$$\begin{aligned} A &= \begin{pmatrix} \dots & a_i & b_i & c_i & \dots \\ \dots & 2a_i & 2b_i & 2c_i & \dots \end{pmatrix}, \\ B &= \begin{pmatrix} \dots & 0 & 0 & 0 & \dots \\ \dots & a_i & c_i & b_i & \dots \end{pmatrix}, \end{aligned} \tag{34}$$

where $a_i \leq c_i < a_{i+1}$ for $i = 1, \dots, t-1$ and $b_i := \frac{a_i + c_i}{2}$. For this example $\overline{\phi}(\lambda) = \phi_\Lambda$ where Λ is a given sequence of finite intervals and points $[a_1, c_1], \dots, [a_t, c_t]$.

For example, consider

$$\begin{aligned} A &= \begin{pmatrix} 1 & 1.5 & 2 & 2.2 & 2.3 & 2.4 & 3 \\ 2 & 3 & 4 & 4.4 & 4.6 & 4.8 & 6 \end{pmatrix}, \\ B &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1.5 & 2.2 & 2.4 & 2.3 & 3 \end{pmatrix} \end{aligned} \tag{35}$$

The spectral function is shown on Figure 2 (left).

For a different example suppose that B equals the max-plus identity matrix I. Then we have

$$h_\mu = \mu + A^\sharp x \wedge -\mu + Ax, \tag{36}$$

and $\overline{\phi}(\mu) = r(h_\mu)$.

We will show that $\overline{\phi}(\mu) = \phi_\Lambda(\mu)$ where Λ is the spectrum of $Ax = \lambda x$. Since Λ is the zero-level set of $\overline{\phi}(\mu)$ which is 1-Lipschitz, we already have that $\overline{\phi}(\mu) \geq \phi_\Lambda(\mu)$ So we have to show the reverse inequality $\overline{\phi}(\mu) \leq \phi_\Lambda(\mu)$. It is implied by the following result.

Proposition 17. *Let*

$$\mu + A^\sharp x \wedge -\mu + Ax = r + x, \tag{37}$$

where $\operatorname{supp}(x) = M$ *and let* λ *be the maximum cycle mean (i.e. the greatest eigenvalue) of the principal submatrix* A_{MM}. *Then* $r \leq - \mid \lambda - \mu \mid$.

Proof. As only submatrix A_{MM} has nontrivial impact in (37), we can assume w.l.o.g. that $M = \{1, \dots, n\}$. We will consider two cases, in each of them we argue by contradiction.

Case 1: $\mu < \lambda$. Suppose that the claim is false and $r > \mu - \lambda$. Then (37) implies that $\mu + A^\sharp x \geq r + x > \mu - \lambda + x$, hence $A^\sharp x > -\lambda + x$ and $Ax < \lambda + x$. But if λ is the

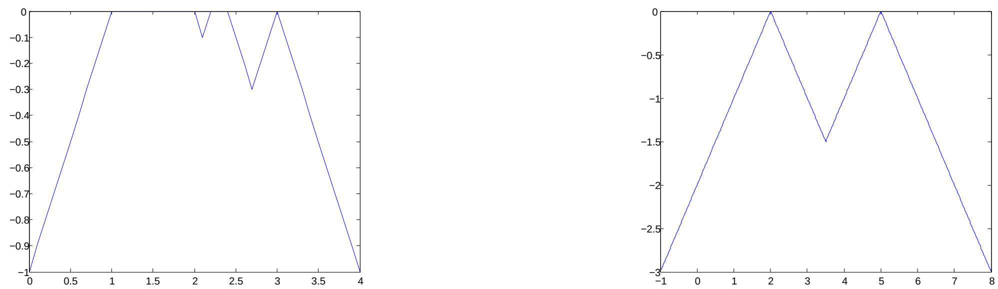

FIGURE 2. The spectral function of (A, B) (35) (left) and (A, I) (38) (right)

maximum cycle mean of A, for any critical index i it must hold that $(Ax)_i = \lambda + x_i$, for any x which satisfies $Ax \leq \lambda + x$. Hence $r \leq \mu - \lambda$.

Case 2: $\lambda > \mu$. Suppose that the claim is false and $r > \lambda - \mu$. Then (37) implies that $-\mu + Ax \geq r + x > \lambda - \mu + x$ hence $Ax > \lambda + x$. But as λ is the greatest eigenvalue of A, for each positive x there exists i such that $(Ax)_i \leq \lambda + x_i$. The contradiction shows that $r \leq \lambda - \mu$. $\square$

For example consider [But10] Example 4.5.9:

$$A = \begin{pmatrix} 0 & 3 & -\infty & -\infty & -\infty & -\infty \\ 1 & 1 & -\infty & -\infty & -\infty & -\infty \\ -\infty & -\infty & 4 & -\infty & -\infty & -\infty \\ -\infty & -\infty & 0 & 3 & 1 & -\infty \\ -\infty & -\infty & -\infty & -1 & 2 & -\infty \\ -\infty & -\infty & -\infty & 1 & -\infty & 5 \end{pmatrix} \tag{38}$$

The spectral function of the corresponding eigenproblem is shown in Figure 2 (right).

4.3. **Symmetric case.** One obtains the following simple bound on the spectrum, as an immediate application of min-max functions.

Theorem 18 (Gaubert and Sergeev[GS10])**.** *Suppose A and B do not have $-\infty$ columns. The spectrum of $Ax = \lambda + Bx$ lies within $[-r(A^\sharp B), r(B^\sharp A)]$.*

Proof. Notice that $Ax = \lambda Bx$ is equivalent to $Ax \leq \lambda + Bx$ and $Ax \geq \lambda + Bx$. By Proposition 6, these imply that $r(\lambda + A^\sharp B) \geq 0$ and $r(-\lambda + B^\sharp A) \geq 0$. Using +-homogeneity we get the claim. $\square$

There are other bounds described in [But10], Section 9.1, and [GS10] which do not require mean-payoff game computations, but they are not so tight in practice. Note that when $-r(A^\sharp B) = r(B^\sharp A)$ the above bound gives an immediate criterion for existence of generalized eigenvalues, as well as an explicit formula and mean-payoff game techniques for the computation of the unique possible eigenvalue and its verification. To this end, we now consider the case of symmetric $A, B \in (\mathbb{R} \cup \{-\infty\})^{n\times n}$.

Lemma 19. *Let $A, B \in (\mathbb{R} \cup \{-\infty\})^{n\times n}$ be symmetric and $x \in \mathbb{R}^n$. Then $A^\sharp Bx = -AB^\sharp(-x)$.*

Proof. We verify using the symmetry of A and B that

$$\begin{aligned}\min_k -a_{ki} + (\max_j b_{kj} + x_j) &= -\max_k a_{ik} - (\max_j b_{jk} + x_j) = \\ &= -\max_k a_{ik} + \min_j -b_{jk} - x_j.\end{aligned} \tag{39}$$

□

Using (13) and Lemma 19 we obtain that $\chi_j(B^\sharp A) = -\chi_j(BA^\sharp)$ and $-\chi_i(A^\sharp B) = \chi_i(AB^\sharp)$ for each $i, j \in [n]$. Combining this with Proposition 4 we obtain the following.

Theorem 20. *Let $A, B \in (\mathbb{R} \cup \{-\infty\})^{n\times n}$ be symmetric and σ^*, resp. τ^* be optimal strategies of Max, resp. Min. If $i = \tau^*(j)$ or $j = \sigma^*(i)$ where $i, j \in [n]$, then*

$$\chi_j(B^\sharp A) = -\chi_j(BA^\sharp) = -\chi_i(A^\sharp B) = \chi_i(AB^\sharp) \tag{40}$$

Corollary 21. *Let $A, B \in (\mathbb{R}\cup\{-\infty\})^{n\times n}$ be symmetric, then the components of $\chi(A^\sharp B)$ are inverted entries of $\chi(B^\sharp A)$, and analogously for $\chi(AB^\sharp)$ and $\chi(BA^\sharp)$.*

For a min-max function f, denote $\underline{\chi}(f) := \min_i \chi_i(f)$.

Theorem 22. *Let $A, B \in (\mathbb{R}\cup\{-\infty\})^{n\times n}$ be symmetric. Then every eigenvalue of (A, B) belongs to $[\underline{\chi}(B^\sharp A), \overline{\chi}(B^\sharp A)]$. When $B^\sharp A$ (equivalently $A^\sharp B$) has finite eigenvector, $r(B^\sharp A)$ is the only possible eigenvalue of (A, B).*

Proof. The first part of the claim follows from Theorem 18 and Corollary 21. Further, to have a finite eigenvector is the same as to have an invariant halfline (χ, v) with all components of χ equal to each other. Hence if the min-max function $B^\sharp A$ has a finite eigenvector, the spectral bounds reduce to one point $r(B^\sharp A)$. □

We also present the following variation of second part of Theorem 22, which improves an observation of Binding and Volkmer [BV07a], see also [But10] Theorem 9.1.6.

Theorem 23. *Let $A, B \in (\mathbb{R} \cup \{-\infty\})^{n\times n}$ and let $x \in (\mathbb{R}\cup\{-\infty\})^n$ satisfy $Ax = \lambda + Bx$ so that Ax (and $\lambda + Bx$) is finite. Then $\lambda = r(B^\sharp A)$ is the unique generalized eigenvalue of (A, B).*

Proof. We start with inequalities $\lambda + A^\sharp Bx \geq x$ and $-\lambda + B^\sharp Ax \geq x$. Applying $\lambda + B$ and A to these inequalities we obtain for $y = Ax = \lambda + Bx$ that

$$\lambda + BA^\sharp y \geq y, \quad -\lambda + AB^\sharp y \geq y. \tag{41}$$

This shows that both $\chi(\lambda + BA^\sharp) \geq 0$ and $\chi(A(\lambda + B)^\sharp) \geq 0$. But this is only possible when $\chi(\lambda + BA^\sharp) = 0$ and $\chi(A(\lambda + B)^\sharp) = 0$ in which case $\lambda = r(B^\sharp A) = -r(A^\sharp B)$. □

Future research: Using this theorem it is possible to describe all eigenvalues of such symmetric pairs A and B where both A and B are such that for any subset $N_1 \subset [n]$, for each $i \in N_1$ there exists $j \in N_1$ with a_{ij} and b_{ij} are finite. If this condition holds, then each eigenvector leads to a pair of symmetric (priincipal) submatrices of A and B from which the corresponding eigenvalue can be computed. In p;articular, it follows that the spectrum of such pairs (A, B) consists of isolated points.

If no assumptions on symmetric matrices are made, then the eigenproblem is of the same complexity as in the general case, due to the following linear-algebraic arguments (similar to [But10] Theorem 9.1.6).

Proposition 24. *Let $A, B \in (\mathbb{R}\cup\{-\infty\})^{m\times n}$. Suppose that $Ax = \lambda + Bx$ has eigenvalues such that Ax is finite. Then either both $Ax = \lambda + Bx$ and $A^T y = \mu + B^T y$ have unique eigenvalue $\mu = \lambda$ with Ax finite, or $Ax = \lambda + Bx$ has several eigenvalues such that Ax is finite and the transposed problem $A^T y = \lambda + B^T y$ does not have eigenvalues at all.*

Proof. Suppose that x^1 and x^2 satisfy $Ax^1 = \lambda^1 + Bx^1$ and $Ax^2 = \lambda^2 + Bx^2$. If we assume that there exists y satisfying $A^T y = \mu + B^T y$, then premultiplying by y^T we obtain that $y^T Ax^1 = \lambda_1 + y^T Bx^1 = \mu + y^T Bx^1$, and $y^T Ax^2 = \lambda_2 + y^T Bx^2 = \mu + y^T Bx^2$. As $y^T Bx^1$ and $y^T Bx^2$ are finite it follows that $\mu = \lambda_1 = \lambda_2$. □

Suppose now that $Ay = \lambda + By$ has eigenvalues such that Ay is finite. We now turn $Ay = \lambda + By$ into a symmetric eigenproblem:

$$\begin{pmatrix} -\infty & A^T \\ A & -\infty \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda + \begin{pmatrix} -\infty & B^T \\ B & -\infty \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{42}$$

This symmetric eigenproblem splits into two non-symmetric ones, one of them being the transpose of the other. Using Proposition 24 we obtain that this symmetric problem has the same spectrum as $Ay = \lambda + By$. It follows in particular [?] that the spectrum of general symmetric two-sided eigenproblem can be any given sequence of finite intervals and points (see an example above).

5. Tropical linear programming

5.1. **Formulations.** In [BA08] Butkovič and Aminu considered the following tropical analogue of the linear programming problem:

$$\begin{aligned} &\text{minimize } px \ \ \text{(resp. maximize } qx) \\ &\text{subject to: } Ax \vee c \leq Bx \vee d\ ,\ x \in (\mathbb{R} \cup \{-\infty\})^n\ , \end{aligned} \tag{43}$$

where $p, q \in (\mathbb{R} \cup \{-\infty\})^n$, $c, d \in (\mathbb{R} \cup \{-\infty\})^m$, and $A, B \in (\mathbb{R} \cup \{-\infty\})^{m\times n}$. Instead of this problem, we will consider its generalization:

$$\begin{aligned} &\text{minimize } \quad (px \vee r) - (qx \vee s) \\ &\text{subject to: } Ax \vee c \leq Bx \vee d\ ,\ x \in (\mathbb{R} \cup \{-\infty\})^n\ , \end{aligned} \tag{44}$$

The new problem formulation has a good geometric insight, since it means optimization over a one-parametric family of general tropical half-spaces (or hyperplanes).

Formulation (44) can be recast in a more compact way as

$$\min\{\lambda \mid Uy \leq V(\lambda)y\ ,\ y_{n+1} \neq -\infty \ \text{ is solvable}\}\ , \tag{45}$$

where

$$U = \begin{pmatrix} A & c \\ p & r \end{pmatrix} \ \text{ and } \ V(\lambda) = \begin{pmatrix} B & d \\ \lambda + q & \lambda + s \end{pmatrix} \tag{46}$$

have dimensions $(m + 1) \times (n + 1)$.

We will also consider the *separated case* when p, r, q, s satisfy the following complementarity condition: $p_i \vee q_i$ and $r \vee s$ are finite but for each i either $p_i = -\infty$ or $q_i = -\infty$, and also either $r = -\infty$ or $s = -\infty$. This arises from better geometric properties of "separated" halfspaces. Homogenizing we obtain the following problem: find minimal parameter λ such that the system

$$\begin{aligned} A^1x \vee A^2y &\leq B^1x \vee B^2y, \\ ux &\leq \lambda + vy, \end{aligned} \tag{47}$$

where u is the vector of finite components of (p, r) and v is the vector of finite components of (q, s).

Theorem 25. *Bisection method and Newton algorithms described in Subsect. 3.2 can be applied to the tropical linear programming formulations* (43) *and* (44). *Moreover the bisection method is pseudopolynomial, and when all the data are integers bounded by* M*,* *the number of iterations in the bisection method can be reduced to* $\log(8M(\min(m+1, n+1)^2)$.

Proof. Recall that (43) is an instance of (44), which is equivalent to problem (45) with U and $V(\lambda)$ as in (46). As (45) is a special case of Problem 1, all methods described in Subsection 3.2 apply. Moreover, we have integrality of the objective function when the entries of A, B are integer, similarly to [But10] Corollary 10.2.6 (also [BA08]). Following the approach of [But10] Subsect. 10.2.3, the middle point in the bisection algorithm can be replaced by its rounding, and the algorithm stops when $U(k) - L(k) = 1$. This shows that the number of iterations can be reduced to $\log(8M(\min(m+1, n+1))^2)$. $\square$

5.2. **Tropical linear programming and mean-payoff games.** The diagrams of mean-payoff games for tropical linear programming are given below.

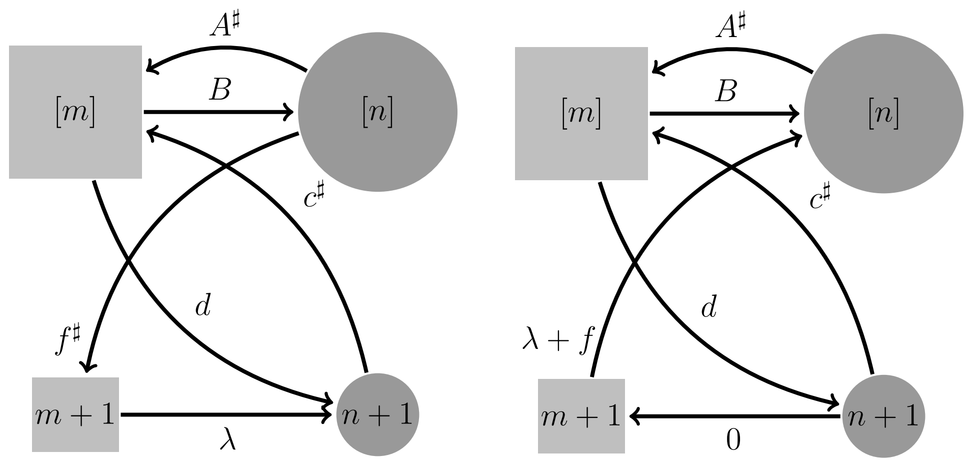

FIGURE 3. Mean-payoff games for tropical linear programming (43): $\min fx$ (left) and $\max fx$ (right)

In all these cases we play the game starting at the node number $n+1$ of Min. Analysing such diagrams, Gaubert et al. [GKS10] conclude that in the tropical linear programming (44) there are important certificates of optimality and unboundedness, which show that the role of Lagrangian multipliers is played here by certain strategies. Below, the subscript λ^* of $\mathcal{G}$ indicates that the weights of certain edges are modified by adding λ (as in the diagrams), and the subscripts σ or τ indicate at all nodes i of Max, resp. j of Min, all edges except for $(i, \sigma(i))$, resp. $(j, \tau(j))$, are removed.

Theorem 26 (Gaubert et al. [GKS10]). *The tropical linear programming problem* (45) *has the optimal value* $\lambda^* \in \mathbb{R}$ *if, and only if,* $\phi(\lambda^*) \geq 0$ *and there exists a strategy* τ *of Min such that the digraph* $\mathcal{G}^{\tau}_{\lambda^*}$ *satisfies the following conditions:*

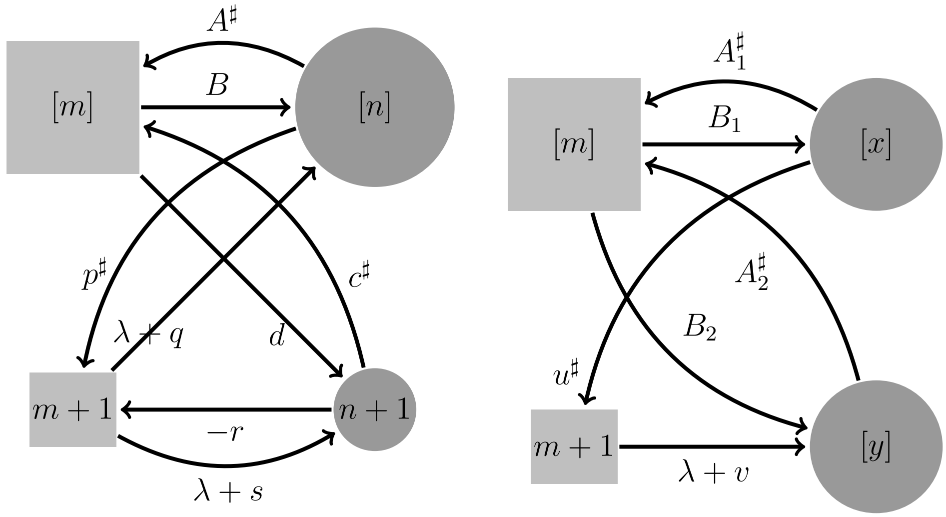

FIGURE 4. Mean-payoff game for the general tropical linear programming (left) and the separated case (right)

(i) *all circuits accessible from node $n+1$ of Min have nonpositive weight,*
(ii) *there is a circuit accessible from node $n+1$ of Min with zero weight,*
(iii) *each circuit of zero weight accessible from node $n+1$ of Min passes through node $m+1$ of Max.*

Moreover, these conditions are always satisfied when τ is left-optimal at point λ^*.

Theorem 27. *The tropical linear programming problem* (45) *is unbounded if, and only if, there exists a strategy σ of Max such that all circuits in the digraph $\mathcal{G}_0^\sigma$ accessed from node $n+1$ of Min do not contain node $m+1$ of Max and have nonnegative weight.*

If the strategies σ or τ and the scalar λ^* are fixed (considered as inputs) the conditions of Theorems 26 and 27, i.e., the validity of the certificates, can be checked in polynomial time.

5.3. **Separated case.** In the maximization case of (43) when A, B, c, d have finite entries, Butkovič and Aminu [BA08] obtained a particular version of Theorem 27 where this certificate of unboundedness reduces to solvability of $Ax \leq Bx$ in (44). An analogous more general result can be obtained for separated problem (47) (Figure 4).

Proposition 28. *Let A, B, c, d have finite entries. Problem* (47) *is unbounded if and only if $A^2 y \leq B^2 y$ has nontrivial solution.*

The quality of bisection method crucially depends on the quality of initial bounds, and the bound $4M(\min(m+1, n+1)^2$ seems to be quite rough. To this end, Butkovič and Aminu obtained nicer bounds for their special formulation, which do not depend on the dimension of the problem, in the case when the entries of A and B are finite. Below we obtain a generalization of their bounds to (44) for the case of separated tropical linear programming (47), when A, B, c, d have finite entries. It can be seen that (43), the formulation of [BA08], corresponds to the following special cases of this formulation: 1) $\min fx$ corresponds to the case when q is one-component (scalar), 2) $\max fx$ corresponds to the case when p is one-component.

We are going to obtain the following lower bound:

$$L = \min_i(\min_j(p_j - b^1_{ij}) + \min_k((\max(a^2_{ik}, b^2_{ik}) - q_k)) \tag{48}$$

Theorem 29. *Suppose that all entries of A_1, B_1, A_2, B_2 are finite, p, q satisfy the complementarity condition, and that $A^2y \leq B^2y$ does not have nontrivial solutions. Then $px - qy \geq L$ for all solutions (x, y) of $A_1x \vee A_2y \leq B_1x \vee B_2y$.*

Proof. We transform the system of constraints to the following form:

$$(A^1 \vee B^1)x \vee (A^2 \vee B^2)y = B^1x \vee B^2y. \tag{49}$$

For the solutions of this system it must hold that

$$\exists i: \ (B^1x)_i \geq ((A^2 \vee B^2)y)_i, \tag{50}$$

otherwise we would have to satisfy the system without B^1x, given that even $(A^2 \vee B^2)y = B^2y$ cannot be satisfied.

Now suppose that for each i we can find such α_i and β_i that

$$\alpha_i + px \geq (B^1x)_i, \quad ((A^2 \vee B^2)y)_i \geq \beta_i + qy, \tag{51}$$

for any solution (x, y) and any i. Then we obtain that a solution (x, y) satisfies

$$\alpha_i + px \geq (B^1x)_i \geq ((A^2 \vee B^2)y)_i \geq \beta_i + qy, \tag{52}$$

for some i and hence $px \geq (\beta_i - \alpha_i) + qy$ for some i. Hence any solution should satisfy $px \geq \min(\beta_i - \alpha_i) + qx$, and $\min(\beta_i - \alpha_i)$ is a lower bound for (47). It remains to find suitable α_i and β_i.

Notice that (51) can be relaxed by removing x and y:

$$\alpha_i + p \geq B^1_{i\cdot}, \quad (A^2 \vee B^2)_{i\cdot} \geq \beta_i + q, \tag{53}$$

where $B^1_{i\cdot}$ and $A^2 \vee B^2)_{i\cdot}$ denote the ith rows of B^1, resp. $A^2 \vee B^2$. The greatest $-\alpha_i$ and β_i that satisfy (53) are given by

$$-\alpha_i = \min_j (p_j - b^1_{ij}), \quad \beta_i = \min_k ((\max(a^2_{ik}, b^2_{ik}) - q_k). \tag{54}$$

Substituting this into $\min(\beta_i - \alpha_i)$ we obtain L. □

As a special case, let us get a bound for the problem $\max fx$ (given constraints in the form of inequalities and transformed to equalities as above). For this we set $p_{n+1} = 0$ (one-component) , $b^1_{i,n+1} = d_i$, $q = f$, $A^2 \vee B^2 = A$:

$$\min_i \min_k (-d_i + a_{ik} - f_k). \tag{55}$$

Taking the inverse of (55) we obtain an upper bound for the maximization problem (43)(see the same bound with c instead of d in [BA08] or [But10] Chapter 10).

For the problem $\min fx$ we have $p = f$, $B^1 = B$, $q_{n+1} = 0$ (one component), $\max(a^2_{i,n+1}, b^2_{i,n+1}) = c_i$:
$\min_i \min_j (f_j - b_{ij} + c_i)$, which is less than the bound of [BA08] (also [But10] Chapter 10): $\max_{i|c_i>d_i} \min_j (f_j - b_{ij} + c_i)$. Such improvement is possible because in this case A^2 and B^2 are just c and d respectively, and it is easy to see that some of the inequalities in $A^2 y \leq B^2 y$ are always true (where $c_i \leq d_i$, and the rest are always false (where $c_i > d_i$).

Further research: Try to generalize this kind of observations.

5.4. **Numerical experiments.** A set of MATLAB programs was written, in order to implement the methods of solving two-sided eigenproblem $Ax = \lambda + Bx$ as well as the bisection method and Newton iterations for tropical linear programming.

Below we present some graphs showing how bisection method and Newton iterations behave on randomly generated instances of tropical linear programming with entries of matrices and vectors ranging from -500 to 500. The matrices A and B in (43) and (44) are square, with dimensions ranging from 1 to 400.

Figure 5 displays the cases of tropical linear programming (43), where all entries in (43) are finite. Here the certificates of unboundedness reduce to solvability of a tropical two-sided system of inequalities. When a feasible and bounded problem is generated, it is solved by bisection and Newton algorithms. For smaller dimensions the results are compared in order to ensure that the solution is correct.

The bisection method works similarly in both cases, with the number of iterations quickly approaching a constant level of 9 or 10 iterations ($\log 500$ or $\log 1000$). Thin red line represents the run of bisection method (up to 250), and thick red line represents the constant level of 10.

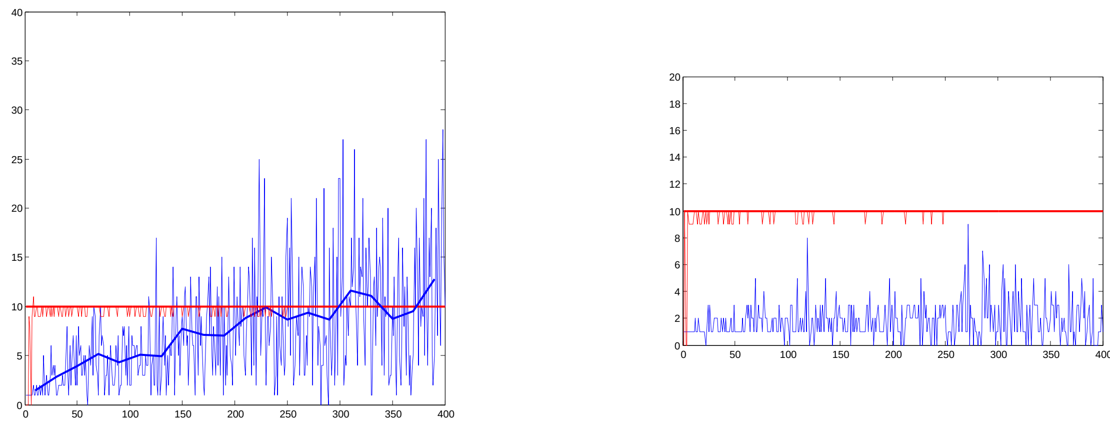

FIGURE 5. The work of bisection method and Newton iteration in the cases of minimization (left) and maximization (right)

Thin blue line represents the run of Newton iterations, and thick blue line represents their average number calculated for each interval of 20 dimensions. In the case of minimization, the average number of Newton iterations slowly grows, being smaller than 10 before $n = 250$, but exceeding 10 at larger dimensions. Naturally the number of iterations for the same dimension may be very different, depending on the configuration and complexity of tropical polytopes (i.e., solution sets of $Ax \vee c \leq Bx \vee d$. In the case of maximization, the number of iterations is usually below 5. We think that this is due to the scarcity of bounded instances of the problem: if a bounded problem is generated, typically after very long sequence of tries where either boundedness or feasibility fail, then it represents a very "thin" polytope whose dimension is much lower than $m = n$ or even reduced to one point. However, it also may be that the maximization is "simpler" than minimization: actually it always suffices to find the greatest point of the solution set (which is straightforward if the generators of the polytope are known). We also remark that there does not seem to be any correlation between the number of iterations of Newton and bisection methods.

Figure 6 displays the cases of general (fractional) tropical linear programming 44 with all entries finite (left) and with the frequency of $-\infty$ entries 0.7. In the case of $-\infty$ entries, we ensure that the set of constraints contains neither $-\infty$ rows on the right-hand side nor $-\infty$ columns on the left-hand side, which is the same as Assumptions 1 and 2 on the mean-payoff game. The case of general tropical linear programming with finite entries

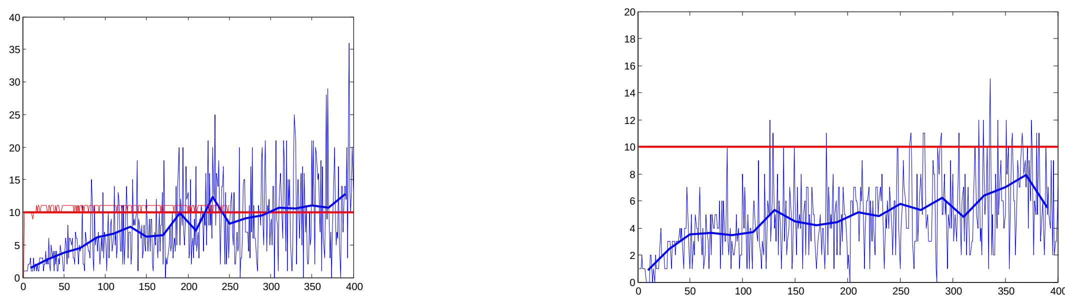

FIGURE 6. The work of bisection method and Newton iteration in the cases of fractional programming with finite entries (left) and the work of Newton iteration if the rate of $-\infty$ entries is 0.7 (right)

shows almost the same picture as in the case of minimization. Here, we would expect that the bisection method which uses more general lower bound (48) is slightly worse, and the Newton iterations are slightly better being closer to the case of maximization. The case when $-\infty$ appear with a regular frequency is even more favourable for Newton iterations, due to the sparsity of $\mathcal{G}$. We preferred to show level 10 as in the other graphs instead of the bound $\log(8Mn^2)$ for bisection method which is always much greater than the number of Newton iterations.

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