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An interval version of separation by semispaces in max–min convexity[☆]

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ABSTRACT

In this paper, we study separation of a closed box from a max–min convex set by max–min semispaces. This can be regarded as an interval extension of the known separation results. We give a constructive proof of the separation in the case when the box satisfies a certain condition, and we show that the separation is never possible when the condition is not satisfied. We also study the separation of two max–min convex sets by a box and by a box and a semispace.

1. Introduction

Consider the closed interval $\mathcal{B} = [0, 1]$ endowed with the operations $\vee = \max$, $\wedge = \min$. This is a well-known distributive lattice, and like any distributive lattice it can be considered as a semiring

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equipped with addition $\vee$ and multiplication $\wedge$. Importantly, both operations are idempotent, $a \vee a = a$ and $a \wedge a = a$, and closely related to the order:

$$a \vee b = b \Leftrightarrow a \leqslant b \Leftrightarrow a \wedge b = a. \tag{1}$$

For standard literature on lattices and semirings see e.g. [2,11].

We consider $\mathcal{B}^n$, the cartesian product of n copies of $\mathcal{B}$, and equip this cartesian product with the operations of taking componentwise addition $\vee$: $(x \vee y)_i := x_i \vee y_i$ for $x, y \in \mathcal{B}^n$ and $i = 1, \ldots, n$, and scalar $\wedge$-multiplication: $(a \wedge x)_i := a \wedge x_i$ for $a \in \mathcal{B}$, $x \in \mathcal{B}^n$ and $i = 1, \ldots, n$. Thus $\mathcal{B}^n$ is considered as a semimodule over $\mathcal{B}$ [11]. Alternatively, one may think in terms of vector lattices [2].

A subset C of $\mathcal{B}^n$ is said to be *max–min convex* if the relations

$$x, y \in C, \quad \alpha, \beta \in \mathcal{B},\ \alpha \vee \beta = 1$$

imply

$$(\alpha \wedge x) \vee (\beta \wedge y) \in C.$$

The interest in max–min convexity is motivated by the study of tropically convex sets, analogously defined over the semiring $\mathbb{R}_{\max}$, which is the completed set of real numbers $\mathbb{R} \cup \{-\infty\}$ endowed with operations of idempotent addition $a \oplus b := \max(a, b)$ and multiplication $a \otimes b := a + b$. Constructed in [24,25], tropical convexity and its lattice-theoretic generalizations received much attention and rapidly developed over the last decades; see [1,5,6,8,12,16–18] among many others.

The matrix algebra developed over the max–min semiring, see [4,10,21] and references therein, is another related area. Max–min semimodules in $\mathcal{B}^n$, like max–min eigenspaces of matrices, are specific max–min convex sets. However, there is no immediate relation between max–min matrix algebra and the present article.

In this article, we continue the study of max–min convex structures started in [19,20,14,15]. We are interested in separation of max–min convex sets by semispaces.

The set

$$\begin{aligned} [x, y]_M &= \{(\alpha \wedge x) \vee (\beta \wedge y) \in \mathcal{B}^n \mid \alpha, \beta \in \mathcal{B}, \alpha \vee \beta = 1\} \\ &= \{\max(\min(\alpha, x), \min(\beta, y)) \in \mathcal{B}^n \mid \alpha, \beta \in \mathcal{B}, \max(\alpha, \beta) = 1\} \end{aligned} \tag{2}$$

is fundamental for max–min convexity, it is called the *max–min segment* (or briefly, the *segment*) *joining x and y*. As in the ordinary convexity in the real linear space, a set is max–min convex if and only if any two points are contained in it together with the max–min segment joining them. The max–min segments have been described in [19,22].

Other relevant types of convex sets studied in the literature (see [20,14,15]) are max–min semispaces, hemispaces, halfspaces and hyperplanes. We recall below the definitions and the relationships between these notions.

For $z \in \mathcal{B}^n$, we call a subset S of $\mathcal{B}^n$ a *max–min semispace* (or, briefly, a *semispace*) *at z*, if it is a maximal (with respect to set-inclusion) max–min convex set avoiding z. Semispaces come from the abstract convexity, see e.g. [23]. One of their main application is in separation results: the family of semispaces is the smallest intersectional basis for the family of all convex sets. We recall (see [20]) that in $\mathcal{B}^n$ there exist at least one and most $n+1$ semispaces at each point $z \in \mathcal{B}^n$, and exactly $n+1$ at each finite point. Moreover, each convex set avoiding z is contained in at least one of those semispaces [20].

Another object from abstract convexity, which can also be straight forwardly introduced in max–min convexity, is the *hemispace*: this is any (max–min) convex set whose complement is also (max–min) convex. Hemispaces are used in separation results.

More general, one can introduce segments, and consequently define convex sets, in any semimodule X over a semiring with multiplicative unity. Zimmermann [25] showed that under some assumptions, the segments of X satisfy *Pasch-Peano* axiom:

$$\begin{aligned} &\forall x, y_1, y_2, z_1, z_2 \in X, \\ &z_i \in [x, y_i], \quad i = 1, 2 \Rightarrow [y_2, z_1] \cap [y_1, z_2] \neq \emptyset. \end{aligned}$$

When the segments in a semimodule satisfy Pasch-Peano axiom, which is the case for max–min convexity [25], a classical theorem of Kakutani (see [3] for a proof) tells us that for any two non-intersecting convex sets C_1 and C_2 there exists a hemispace H containing C_1 such that the complement of H contains C_2. The proof of Kakutani theorem is non-constructive and uses Zorn's Lemma. A constructive proof of this theorem in the special case of max–min convexity is, to the authors' knowledge, an open problem.

It is shown in [20] that any max–min semispace is a max–min hemispace.

A *max–min hyperplane* is the set of points in $\mathcal{B}^n$ that satisfies a max–min linear equation:

$$(a_1 \wedge x_1) \vee \cdots \vee (a_n \wedge x_n) \vee a_{n+1} = (b_1 \wedge x_1) \vee \cdots \vee (b_n \wedge x_n) \vee b_{n+1},$$

with $a_i, b_i \in \mathcal{B}$, $i = 1, \ldots, n+1$. Similarly, a *max–min halfspace* is the set of points in $\mathcal{B}^n$ that satisfy the definition above with equality replaced by $\leqslant$. In contrast to the case of the usual linear space, here one needs an affine function on each side of the equality/inequality sign. Indeed, if we regard the operation $a \vee b$ as an addition, it does not have an inverse and one cannot move terms from one side of the equality/inequality sign to the other.

The structure of max–min hyperplanes is described in [14]. In particular, there are examples of hyperplanes that are not halfspaces. It follows from (1) that any max–min halfspace is a max–min hyperplane. The relationship between max–min hyperplanes and max–min semispaces is described in [15]: a semispace in $\mathcal{B}^n$ is a hyperplane if and only if it is a semispace at a point belonging to the main diagonal of $\mathcal{B}^n$. It follows from [14] that, in general, hyperplanes are not hemispaces, and hence not semispaces either. See Fig. 2.3 in [14], showing an example of hyperplane that does not have a connected complement, thus cannot be a hemispace. However, it is easy to show that a max–min halfspace is a hemispace.

In the max–min case, the hyperplanes cannot be used to separate a point from a max–min convex set. An example to this was first published in [14], followed by a simpler one in [15]. This is in contrast with very optimistic results in the tropical convexity and its lattice-theoretic generalizations [5,6,8,9,24], which behave like the ordinary convexity in linear spaces in this respect.

In this paper, we study the following interval version of the semispace separation: given a box B, i.e. a Cartesian product of closed intervals, and a max–min convex set C, decide whether it is possible to construct a semispace which contains C and avoids B. In Section 2 we give our main result, Theorem 1, which shows that such separation is indeed possible when B satisfies a certain condition. This condition holds true in particular when B does not contain points with coordinates equal to 1, or when B reduces to a point. When the condition is not satisfied, we show that the separation by semispaces is never possible. However, separation can be saved if we also allow hemispaces of a certain kind. As a corollary of Theorem 1, we also recover the description of semispaces due to Nitica and Singer [20]. In Section 3 we study the separation of two convex sets by a box and by a box and a semispace. We show that this separation is always possible in $\mathcal{B}^2$, and we provide a counterexample in $\mathcal{B}^3$.

Fig. 1 summarizes the types of separation considered in this paper. The convex sets that need to be separated are colored in black, and the separating boxes or semispaces are colored in gray. The sets C_1, C_2 and C are convex and B is a box.

In view of the recent development of tropical interval linear algebra in [13] and [7, Chapter 6], the present paper may be seen as related to yet undeveloped area of the interval tropical convexity.

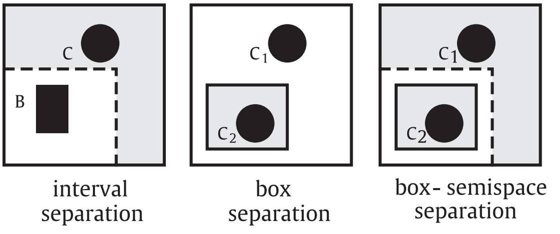

Fig. 1. Separation types, $n = 2$.

2. Separation of boxes from max–min convex sets

For any point $x^0 = (x_1^0, \ldots, x_n^0) \in \mathcal{B}^n$ we define a family of subsets $S_0(x^0), \ldots, S_n(x^0)$ in $\mathcal{B}^n$. The sets are introduced in [20, Proposition 4.1]. Recall that x^0 is called *finite* if it has all coordinates different from zeros and ones. Without loss of generality we may assume that

$$x_1^0 \geqslant \cdots \geqslant x_n^0. \tag{3}$$

The set $\{x_1^0, \ldots, x_n^0\}$ admits a natural subdivision into ordered subsets such that the elements of each subset are either equal to each other or are in strictly decreasing order, say,

$$\begin{aligned} x_1^0 &= \cdots = x_{k_1}^0 > \cdots > x_{k_1+l_1+1}^0 = \cdots = x_{k_1+l_1+k_2}^0 > \ldots \\ &> x_{k_1+l_1+k_2+l_2+1}^0 = \cdots = x_{k_1+l_1+k_2+l_2+k_3}^0 > \ldots \\ &> x_{k_1+l_1+\cdots+k_{p-1}+l_{p-1}+1}^0 = \cdots = x_{k_1+l_1+\cdots+k_{p-1}+l_{p-1}+k_p}^0 \\ &> \cdots > x_{k_1+l_1+\cdots+k_p+l_p}^0 (= x_n^0). \end{aligned} \tag{4}$$

Let us introduce the following notations:

$$L_0 = 0, \quad K_1 = k_1, \quad L_1 = K_1 + l_1 = k_1 + l_1, \tag{5}$$

$$K_j = L_{j-1} + k_j = k_1 + l_1 + \cdots + k_{j-1} + l_{j-1} + k_j \quad (j = 2, \ldots, p), \tag{6}$$

$$L_j = K_j + l_j = k_1 + l_1 + \cdots + k_j + l_j \quad (j = 2, \ldots, p); \tag{7}$$

we observe that $l_j = 0$ if and only if $K_j = L_j$.

We are ready to define the sets $S_i(x^0)$. We need to distinguish the cases when some of the coordinates of the point x^0 are zeros or ones, since some of the sets $S_i(x^0)$ become empty in that case.

Definition 1. (a) If x^0 is finite, then:

$$S_0(x^0) = \{x \in \mathcal{B}^n | x_i > x_i^0 \text{ for some } i \text{ in } 1 \leqslant i \leqslant n\}, \tag{8}$$

$$\begin{aligned} &S_{K_j+q}(x^0) = \{x \in \mathcal{B}^n | x_{K_j+q} < x_{K_j+q}^0, \text{ or } x_i > x_i^0 \text{ for some } i \text{ in } K_j + q + 1 \leqslant i \leqslant n\} \\ &(q = 1, \ldots, l_j; j = 1, \ldots, p) \text{ if } l_j \neq 0, \end{aligned} \tag{9}$$

$$\begin{aligned} &S_{L_{j-1}+q}(x^0) = \{x \in \mathcal{B}^n | x_{L_{j-1}+q} < x_{L_{j-1}+q}^0, \text{ or } x_i > x_i^0 \text{ for some } i \text{ in } K_j + 1 \leqslant i \leqslant n\} \\ &(q = 1, \ldots, k_j; j = 1, \ldots, p \text{ if } k_1 \neq 0, \text{ or } j = 2, \ldots, p \text{ if } k_1 = 0). \end{aligned} \tag{10}$$

(b) If there exists an index $i \in \{1, \ldots, n\}$ such that $x_i^0 = 1$, but no index j such that $x_j^0 = 0$, then the sets are $S_1(x^0), \ldots, S_n(x^0)$ of part (a).

(c) If there exists an index $j \in \{1, \ldots, n\}$ such that $x_j^0 = 0$, but no index i such that $x_i^0 = 1$, then the sets are $S_0(x^0), S_1(x^0), \ldots, S_{\beta-1}(x^0)$ of part *(a)*, where

$$\beta := \min\{1 \leqslant j \leqslant n | \ x_j^0 = 0\}. \tag{11}$$

(d) If there exist an index $i \in \{1, \ldots, n\}$ such that $x_i^0 = 1$, and an index j such that $x_j^0 = 0$, then the sets are $S_1(x^0), \ldots, S_{\beta-1}(x^0)$ of part *(a)*, where β is given by *(11)*.

For future reference, we call the sets $S_i(x^0)$, $0 \leq i \leq n$, $x^0 \in \mathcal{B}^n$, admissible.

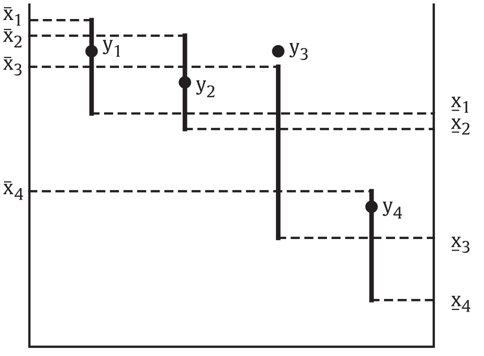

Fig. 2. Illustration of condition (12). The point y should not be in C.

Proposition 1 [20]. *For any $x^0 \in \mathcal{B}^n$ the sets $S_i(x^0)$, $0 \le i \le n$, are max–min convex.*

In the following $[a, c]$ denotes the ordinary interval on the real line $\{b\colon\ a \leqslant b \leqslant c\}$, provided $a \leqslant c$ (and possibly $a = c$).

We investigate the separation of a box $B = [\underline{x}_1, \overline{x}_1] \times \cdots \times [\underline{x}_n, \overline{x}_n]$ from a max–min convex set $C \subseteq \mathcal{B}^n$, by which we mean that there exists a set S described in Definition 1, which contains C and avoids B.

Assume that $\overline{x}_1 \geqslant \cdots \geqslant \overline{x}_n$ and suppose that $t(B)$ is the greatest integer such that $\overline{x}_{t(B)} \geqslant \underline{x}_i$ for all $1 \leqslant i \leqslant t(B)$. We will need the following condition:

$$\text{If } (\overline{x}_1 = 1)\ \&\ (y_l \geqslant \underline{x}_l,\ 1 \le l \le n)\ \&\ (\overline{x}_l < y_l \text{ for some } l \leqslant t(B)),\ \text{ then } y \notin C. \tag{12}$$

Note that if the box is reduced to a point and if $\overline{x}_1 = 1$, then $t(B) = 1$. Hence $\overline{x}_l = 1$ for all $l \leqslant t(B)$ and $\overline{x}_l < y_l$ is impossible. It follows that (12) always holds true in the case of a point.

Fig. 2 shows an illustration of condition (12). One has $t(B) = 3$, $\overline{x}_1 = 1$, and the point $y = (y_1, y_2, y_3, y_4)$ satisfies $y_l \geqslant \underline{x}_l$, $1 \le l \le 4$ and $\overline{x}_3 < y_3$ for $3 \leqslant t(B) = 3$, hence $y \in C$ is not allowed.

The formulation of our main result will also use an *oracle* answering the question, whether or not a given max–min convex set $C \subseteq \mathcal{B}^n$ lies in an admissible set S (see Definition 1). As in the conventional convex geometry or tropical convex geometry, this question can be answered in $O(mn)$ time if C is a convex hull of m points. Indeed it suffices to answer whether any of the inequalities defining S is satisfied for each of the m points generating C.

Theorem 1. *Let $B = [\underline{x}_1, \overline{x}_1] \times \cdots \times [\underline{x}_n, \overline{x}_n]$, and let $C \subseteq \mathcal{B}^n$ be a max–min convex set avoiding B. Suppose that B and C satisfy (12). Then there is a set S described by Definition 1, which contains C and avoids B. This set is constructed in at most $n + 1$ calls to the oracle.*

Proof. If $\overline{x}_i < 1$ for all i, then we try to separate B from C by $S_0(\overline{x}_1, \ldots, \overline{x}_n)$ given by (8). Suppose we fail. Then there exists $y \in C$ such that $y_i \le \overline{x}_i$ for all i.

Otherwise, $1 = \overline{x}_1 \geqslant \cdots \geqslant \overline{x}_n$. Let $\underline{x}_l = \max_{k \leqslant t(B)} \underline{x}_k$, and define $u \in \mathcal{B}^n$ by

$$u_i = \begin{cases} \underline{x}_l, & \text{if } i \leqslant t(B), \\ \overline{x}_i, & \text{if } i > t(B). \end{cases} \tag{13}$$

It follows from the definition of $t(B)$ that $\underline{x}_l = \max_{1 \le i \le n} u_i$.

We try to separate B from C by $S_l(u)$, which is given by (9) or (10). If we fail then there exists $y \in C$ such that $y_i \leqslant \overline{x}_i$ for all $i > t(B)$ (and trivially $y_i \leqslant \overline{x}_i$ for $\overline{x}_i = 1$).

Thus we either separate C from B, or there is a point $y \in C$ such that $y_i \leqslant \overline{x}_i$ for all $i > t(B)$. In the latter case, condition (12) and $B \cap C = \emptyset$ assure that there is at least one i such that $y_i < \underline{x}_i$. Indeed,

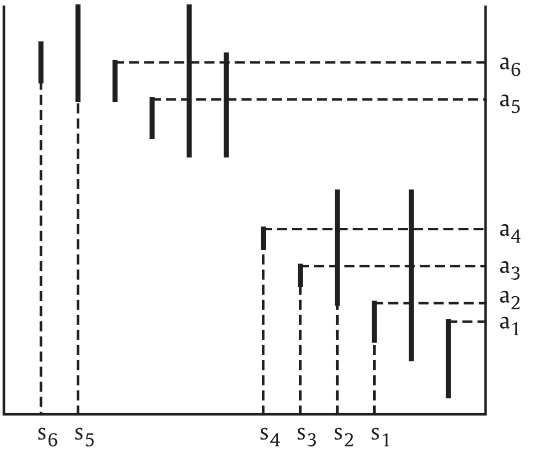

Fig. 3. Construction of a_i and s_i in the proof of Theorem 1.

otherwise if $\underline{x}_i \leq y_i$ for all i and $y_i \leqslant \overline{x}_i$ for all $i \leq t(B)$ then $y \in B$; and if $\underline{x}_i \leq y_i$ for all i and $\overline{x}_i < y_i$ for some $i \leq t(B)$ then $y \notin C$ by condition (12).

Now assume without loss of generality that $\underline{x}_1 \geqslant \cdots \geqslant \underline{x}_n$ (the order of $\overline{x}_i$ is now arbitrary).

The set $\{1, \ldots, n\}$ is naturally partitioned by the following procedure. See Fig. 3 for an illustration. The segments $[\underline{x}_i, \overline{x}_i]$ are drawn vertically and counted from left to right.

Let s_1 be the smallest number such that $\underline{x}_{s_1} \leqslant \overline{x}_i$ for all $i = s_1, \ldots, n$.

If $s_1 \neq 1$ then there exists $t_1 \in \{s_1, \ldots, n\}$ such that $\underline{x}_{s_1-1} > \overline{x}_{t_1}$. In this case let T_1 be the set of such t_1. Otherwise if $s_1 = 1$ we take $T_1 = \{1, \ldots, n\}$. In Fig. 3 one has $T_1 = \{12\}$.

We define

$$a_1 = \min\{\overline{x}_i : i \in T_1\}. \tag{14}$$

We have

$$\underline{x}_i \leqslant a_1 \leqslant \overline{x}_i \quad \forall\, i = s_1, \ldots, n. \tag{15}$$

Thus a_1 is a common level in all intervals $[\underline{x}_{s_1}, \overline{x}_{s_1}], \ldots, [\underline{x}_n, \overline{x}_n]$, but not $[\underline{x}_{s_1-1}, \overline{x}_{s_1-1}]$.

If $s_1 = 1$ then we stop. Otherwise we proceed by induction. Let s_k be the smallest number such that $\underline{x}_{s_k} \leqslant \overline{x}_i$ for all $i \in \{s_k, \ldots, n\} \backslash T_1 \cup \ldots \cup T_{k-1}$. Note that $s_k < s_{k-1}$. If $s_k \neq 1$ then there exists $t_k \in \{s_k, \ldots, n\}$ such that $\underline{x}_{s_k-1} > \overline{x}_{t_k}$. In this case let T_k be the set of such t_k. Otherwise if $s_k = 1$, then define $T_k := \{1, \ldots, n\} \backslash T_1 \cup \ldots \cup T_{k-1}$.

We take

$$a_k = \min\{\overline{x}_i : i \in T_k\}. \tag{16}$$

We have

$$\underline{x}_i \leqslant a_k \leqslant \overline{x}_i \quad \forall\, i = \{s_k, \ldots n\} \backslash T_1 \cup \ldots \cup T_{k-1}. \tag{17}$$

Thus a_k is a common level in all intervals $[\underline{x}_{s_k}, \overline{x}_{s_k}], \ldots, [\underline{x}_n, \overline{x}_n]$ excluding the intervals with indices in $T_1 \cup \ldots \cup T_{k-1}$ which are below that level. The interval $[\underline{x}_{s_k-1}, \overline{x}_{s_k-1}]$ is above a_k (and possibly several other such levels going into T_k).

In Fig. 3, the sets T_i are $T_1 = \{12\}, T_2 = \{10\}, T_3 = \{8\}, T_4 = \{7, 9, 11\}, T_5 = \{4\}, T_6 = \{1, 2, 3, 5, 6\}$.

Next we recall our point $y \in C$. It has $y_i < \underline{x}_i$ for some i. Denote $K = \{l : y_l > \overline{x}_l\}$. Pick the greatest i such that $y_i < \underline{x}_i$ (note that for such i we necessarily have $\underline{x}_i > 0$), and let $s_k \leqslant i < s_{k-1}$, which implies $\underline{x}_j \leqslant y_j \leqslant \overline{x}_j$ for all $j \in \{s_{k-1}, \ldots, n\}\backslash K$. We try to separate B from C by the sets

$$S_i(u^i) = \{x \in \mathcal{B}^n : x_i < \underline{x}_i \text{ or } x_j > \overline{x}_j \text{ for some } j \in T_1 \cup \ldots \cup T_{k-1}\}, \tag{18}$$

where u^i can be defined by

$$u^i_l = \begin{cases} \underline{x}_l, & l < i, \\ \underline{x}_i, & l \geqslant i \text{ and } l \notin T_1 \cup \cdots \cup T_{k-1}, \\ \overline{x}_l, & l \in T_1 \cup \cdots \cup T_{k-1}, \end{cases} \tag{19}$$

for all i with $y_i < \underline{x}_i$ and $s_k \leqslant i < s_{k-1}$. Indeed, (18) is of the form (9) or (10), where u^i is substituted for x^0. (In particular, it can be checked that $u^i_1 \geqslant \cdots \geqslant u^i_n$.)

Suppose the separation always fails. Then it gives us points $x^i \in C$ such that

$$x^i_i \geqslant \underline{x}_i \quad \text{and} \quad x^i_j \leqslant \overline{x}_j \; \forall j \in T_1 \cup \cdots \cup T_{k-1}. \tag{20}$$

Then (17) implies that

$$\underline{x}_i \leqslant a_k \wedge x^i_i \leqslant \overline{x}_i \quad \text{and} \quad a_k \wedge x^i_j \leqslant \overline{x}_j \; \forall j = s_k, \ldots, n, \tag{21}$$

since $a_k \in [\underline{x}_j, \overline{x}_j]$ for $j \in \{s_k, \ldots, n\}\backslash T_1 \cup \cdots \cup T_{k-1}$ by (17), and we use (20) for $j \in T_1 \cup \cdots \cup T_{k-1}$. The point

$$z = \bigvee_i (a_k \wedge x^i) \vee y \in C \tag{22}$$

will be in some sense better than y. Indeed, (21) implies that $\underline{x}_i \leqslant z_i \leqslant \overline{x}_i$ for all $i \in \{s_k, \ldots, n\}\backslash K$, versus $\underline{x}_i \leqslant y_i \leqslant \overline{x}_i$ for all $i = \{s_{k-1}, \ldots, n\}\backslash K$. As $z \geqslant y$ we have $z_i > \overline{x}_i$ for all $i \in K$.

Proceeding with this improvement we obtain a point z in C which satisfies $\underline{x}_i \leqslant z_i$ for all i and $z_i \leqslant \overline{x}_i$ for all $i \in \{1, \ldots, n\}\backslash K$. This contradicts either $B \cap C = \emptyset$, or condition (12). This contradiction shows that we should succeed with separation at some stage. Clearly, the number of calls to the oracle does not exceed $n + 1$. $\square$

We note that Theorem 1 also yields a method which verifies condition (12) in no more than $n + 1$ calls to the oracle.

The box B can be a point and in this case condition (12) always holds true. Therefore, some known results on max–min semispaces [20] can be deduced from Theorem 1. The following statement is an immediate corollary of Theorem 1 and Proposition 1.

Corollary 1 [20]. *Let $x \in \mathcal{B}^n$ and $C \subseteq \mathcal{B}^n$ be a max–min convex set avoiding x. Then C is contained in one of the admissible $S_i(x)$, $1 \leq i \leq n$, introduced in Definition 1. Consequently admissible sets are the family of semispaces at x.*

Proof. The proof of Theorem 1 applied to $B = \{x\}$ shows that any max–min convex set avoiding x is contained in one of the sets $S_i(x)$. Proposition 1 implies that these sets are max–min convex and do not contain x. Obviously, they are not included in each other. If $S_i(x)$ is not maximal, let S be a max–min convex set strictly containing $S_i(x)$. Then Theorem 1 implies that there exists other $S_j(x)$, $i \neq j$, such that $S \subset S_j(x)$. But this implies $S_i(x) \subset S_j(x)$, a contradiction. Hence $S_i(x)$ are all maximal and $\{S_i(x)\}_i$ is the family of semispaces at x. $\square$

Thus we recover a result of [20] that Definition 1 actually yields all semispaces at a given point.

We now show that separation by semispaces is impossible when B and C do not satisfy (12).

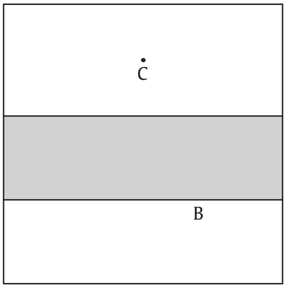

Fig. 4. Forbidden 2-dim interval separation of convex sets.

Theorem 2. *Suppose that $B = [\underline{x}_1, \overline{x}_1] \times \cdots \times [\underline{x}_n, \overline{x}_n]$ and a max–min convex set $C \subseteq \mathcal{B}^n$ are such that $B \cap C = \emptyset$ but condition (12) does not hold. Then there is no semispace that contains C and avoids B.*

Proof. We assume that $\overline{x}_1 \geqslant \cdots \geqslant \overline{x}_n$. Since (12) does not hold, we have $\overline{x}_1 = 1$. Also there exists $z \in C$, such that for some indices $k \leq t(B)$ we have

$$\max_i \{\underline{x}_i : i \leqslant k\} \leqslant \overline{x}_k < z_k, \tag{23}$$

but

$$\underline{x}_i \leqslant z_i \leqslant \overline{x}_i \tag{24}$$

for all the others. Any semispace of the type S_0 given by (8) intersects B since $\overline{x}_1 = 1$. If we assume by contradiction that a separating semispace exists, then it must be of the type $S_i(x^0)$ given by (9) or (10). Further, we claim that this semispace must contain the set $\{x: x_k > x_k^0\}$ for some k such that $z_k > x_k^0 \geqslant \overline{x}_k$, otherwise either it does not contain z or it intersects B. Indeed $z \in S_i(x^0)$ implies

$$z_i < x_i^0, \quad \text{or } z_j > x_j^0 \text{ for } j \text{ such that } x_i^0 > x_j^0. \tag{25}$$

If $z_j > x_j^0$ is true for some j such that $x_j^0 < \overline{x}_j$, then $y \in S_i(x^0)$ for each $y \in B$ with $y_j = \overline{x}_j$, hence $B \cap S_i(x^0) \neq \emptyset$, a contradiction. If $z_i < x_i^0$ is true, then $x_i^0 > \underline{x}_i$ implying $y \in S_i(x^0)$ for each $y \in B$ with $y_i = \underline{x}_i$, hence again $B \cap S_i(x^0) \neq \emptyset$, a contradiction. Thus $z_k > x_k^0$ must hold for at least one k, and necessarily with $x_k^0 \geqslant \overline{x}_k$.

This also implies that $i \neq k$, for the type of the semispace above. Then we must have $x_i^0 > x_k^0 \geqslant \overline{x}_k$ and $\{x | x_i < x_i^0\} \subseteq S_i(x^0)$. If $i < k$ then $x_i^0 > \overline{x}_k \geqslant \underline{x}_i$ due to (23), and if $i > k$ then also $x_i^0 > \overline{x}_k \geqslant \overline{x}_i \geqslant \underline{x}_i$ by the ordering of $\overline{x}_i$. Hence the set $\{x: x_i < x_i^0\}$, which is contained in $S_i(x^0)$, intersects B and the separation is impossible. $\square$

Remark 1. A simple example of interval non-separation is shown in Fig. 4. The box is $B = [0, 1] \times [a, b]$ where $0 \leq a \leqslant b < 1$ and the convex set is $C = \{z\}$ where $z = (z_1, z_2)$ with $z_2 > b$. Note that B and C do not satisfy condition (12).

Remark 2. Theorem 1 can be easily modified to allow any case, if in addition to the admissible sets from Definition 1 we also allow the sets

$$T_0^M(x^0) = \{x: x_i > x_i^0 \text{ for some } i \in M\}, \tag{26}$$

where M is a proper subset of $\{1, 2, \ldots, n\}$.

By Corollary 1, these sets cannot be semispaces. Nevertheless, they are *hemispaces*. Indeed, a point $x \in T_0^M$ is characterized by $x_i > x_i^0$ for some $i \in M$. So if $x, y \in T_0^M(x^0)$, then one has $x_i > x_i^0$ and

$y_j > x_j^0$ for some $i, j \in M$. It follows now from (2) that any point $z \in [x, y]$ has either $z_i > x_i^0$ or $z_j > x_j^0$. The complement of T_0^M is max–min convex as the intersection of the max-min convex sets $\{x_i : x_i \leqslant x_i^0\}$, for i in M.

The condition $C \subseteq T_0^M(x^0)$ can be verified by the same type of oracle as in Theorem 1.

3. Separation of two max–min convex sets

Throughout this section, we assume that $\mathcal{B}^n = [0, 1]^n$ is endowed with the induced topology coming from the standard Euclidean topology of $\mathbb{R}^n$. All topological notions used in the sequel: boundary, closure etc. refer to this topology.

We will investigate the separation of two disjoint closed max–min convex sets by a box, property called in the introduction *box separation*, and by a box and a semispace, property called in the introduction *box-semispace separation*. Both properties are illustrated in Fig. 1.

We recall the structure of 2-dimensional max–min segments as presented in [19]. Pictures of all types of max–min segments are shown in Fig. 5, taken from [19].

Theorem 3. *Let $C_1, C_2 \in \mathcal{B}^2$, $C_1 \cap C_2 = \emptyset$, be two closed max–min convex sets. Then there exist a permutation $i : \{1, 2\} \to \{1, 2\}$ and a box $B \subset \mathcal{B}^2$ such that $C_{i(1)} \subset B$ and $B \cap C_{i(2)} = \emptyset$.*

Proof. Let

$$\begin{aligned} x_c &:= \max\{x|(x, y) \in C_1 \text{ for some } y\}, \\ y_c &:= \max\{y|(x, y) \in C_1 \text{ for some } x\}. \end{aligned} \tag{27}$$

As C_1 is compact, there exist $(x_c, y), (x, y_c) \in C_1$. Moreover, the convexity of C_1 implies that

$$c := (x_c, y_c) = (x_c, y) \vee (x, y_c) \in C_1. \tag{28}$$

Let

$$\begin{aligned} x_a &:= \min\{x|(x, y) \in C_1 \text{ for some } y\}, \\ y_b &:= \min\{y|(x, y) \in C_1 \text{ for some } x\}. \end{aligned} \tag{29}$$

Consider the points in C_1, guaranteed again by compactness:

$$\begin{aligned} a &:= (x_a, y_a), \\ b &:= (x_b, y_b). \end{aligned} \tag{30}$$

The values y_a and x_b are chosen arbitrarily.

The smallest box in $\mathcal{B}^2$ containing the convex set C_1 is $B_0 := [x_a, x_c] \times [y_b, y_c]$. The point c is the upper right corner of B_0.

We need the following Lemma, which can be proved by drawing all possible special cases and using the structure of max–min segments shown in Fig. 5. This proof is routine and will be omitted.

Lemma 1. *The box B_0 can be partitioned as $B_0 = T_0 \cup T_1 \cup T_2 \cup T_3$, where*

$$\begin{aligned} T_0 &= \{(\alpha \wedge a) \vee (\beta \wedge b) \vee (\gamma \wedge c) : \alpha \vee \beta \vee \gamma = 1\}, \\ T_1 &= B_0 \cap \{(x, y) : x < x_b,\ y < y_a\}, \\ T_2 &= B_0 \cap \{(x, y) : y > y_a,\ x < x_c,\ y > x\}, \\ T_3 &= B_0 \cap \{(x, y) : x > x_b,\ y < y_c,\ y < x\}. \end{aligned}$$

All regions T_0, T_1, T_2, T_3 are max–min convex (or possibly empty).

The regions T_0, T_1, T_2, T_3 are shown in Fig. 6.

Evidently $T_0 \subseteq C_1$ (note that T_0 is the max–min convex hull of a, b, c). In particular, the max–min segments $[a, b]_M$, $[a, c]_M$, $[b, c]_M$ are included in C_1 and any point from C_2 stays away from them. The other regions may contain points from both C_1 and C_2.

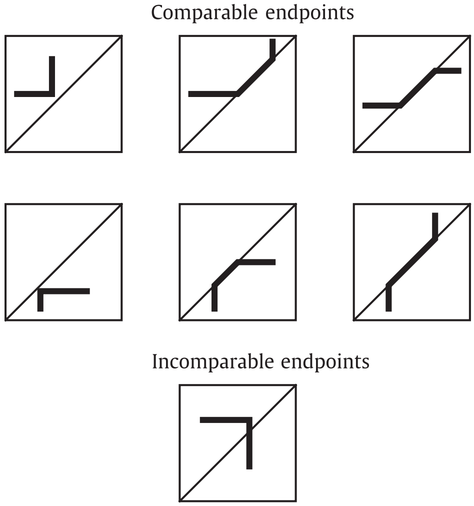

Fig. 5. 2-dim max–min segments.

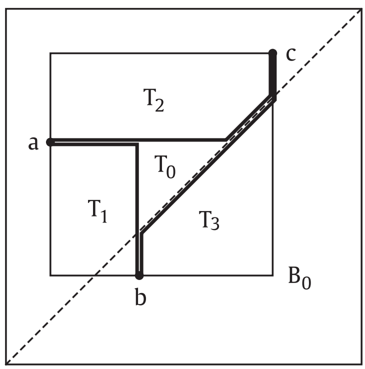

Fig. 6. 2-dim box separation.

We show that if the convex set C_2 intersects one of the regions T_1 and T_2, then there is a box $B_1 \subset \mathcal{B}^2$ such that $C_2 \subset B_1$ and $B_1 \cap C_1 = \emptyset$. Due to the symmetry about the main diagonal and the definitions of T_2 and T_3, there is no need to consider the case where C_2 intersects T_3.

Case 1. Assume C_2 intersects the region T_1.

The intersection C_2' of C_2 with the region T_1 is max–min convex (as the intersection of two max–min convex sets). Thus there exists a point $(x_M, y_M) \in C_2'$, away from the boundary of C_1, and consequently away from the segment $[a, b]_M$, that has the maximum x-coordinate and maximum y-coordinate for C_2'.

We show that C_2 is included in the box $B_1 = [0, x_M] \times [0, y_M]$.

Assume by contradiction that there exists $(x', y') \in C_2$ such that $x' > x_M$ or $y' > y_M$, then $(x'', y'') := (x_M, y_M) \vee (x', y')$ has either $x'' = x_M$ and $y'' = y' > y_M$, or $x'' = x' > x_M$ and $y'' = y_M$, or $(x'', y'') = (x', y')$. If $x'' < x_b$ and $y'' < y_a$ then $(x'', y'') \in C_2'$, which contradicts the maximality of x_M and y_M. Otherwise, the segment $[(x_M, y_M), (x'', y'')]_M$ intersects $[a, b]_M$ and hence $C_1 \cap C_2 \neq \emptyset$, a contradiction.

We show that the box B_1 does not intersect with C_1. Assume that there exists $(x, y) \in B_1 \cap C_1$. Then $x \leqslant x_M$ and $y \leqslant y_M$. There exist $(x', y_M) \in [a, b]_M$ and $(x_M, y') \in [a, b]_M$ such that $x' > x_M$ and

$y' > y_M$. Using these points, we obtain that

$$\begin{aligned} (x_M, y_M) &= (x, y) \vee (x_M \wedge (x', y_M)), \quad \text{if } x_M \geqslant y_M, \\ (x_M, y_M) &= (x, y) \vee (y_M \wedge (x_M, y')), \quad \text{if } x_M \leqslant y_M. \end{aligned} \tag{31}$$

In both cases $(x_M, y_M) \in C_1$ and hence $C_1 \cap C_2 \neq \emptyset$, a contradiction.

Case 2. Assume now that C_2 intersects T_2, and let $C_2' := C_2 \cap T_2$. Let x_M be the largest x coordinate of a point in C_2' and y_M the smallest y-coordinate of a point in C_2'. Let $(x_0, y_M), (x_M, y_0) \in C_2'$. From the definition of T_2 we have $x_0 \leq y_M$ and $x_M \leq y_0$. Let $[t_1, t_2] := [x_a, x_c] \cap [y_a, y_c]$ (where all segments are ordinary on the real line).

If $y_M \leqslant x_M$, then due to convexity

$$\begin{aligned} (x_0, x_0) &= (x_0, y_M) \vee (x_0 \wedge (x_M, y_0)) \in C_2', \\ (y_M, y_M) &= (x_0, y_M) \vee (y_M \wedge (x_M, y_0)) \in C_2', \end{aligned} \tag{32}$$

and hence the whole diagonal (and max–min) segment $[(x_0, x_0), (y_M, y_M)]_M$ is included in C_2'. It can be observed that any point in the closure of T_2 that belongs to the main diagonal lies in $[a, c]_M$, which is in C_1. Thus $C_1 \cap C_2 \neq \emptyset$, a contradiction, hence we must have $y_M > x_M$.

When $y_M > x_M$, due to convexity we have

$$(x_M, y_M) = (x_0, y_M) \vee (y_M \wedge (x_M, y_0)) \in C_2'. \tag{33}$$

In this case we claim that C_2 is contained in the box $B_1 := [0, x_M] \times [y_M, 1]$, which avoids C_1.

Assume by contradiction that there exists $(x', y') \in C_2$ which does not lie in B_1. This implies that $x' > x_M$ or $y' < y_M$. We also have $y_M > x'$ and $y' > x_M$, otherwise the segment $[(x', y'), (x_M, y_M)]_M$ has points on the main diagonal, in which case it intersects $[a, c]_M$. Consider the convex combinations

$$\begin{aligned} (x', y_M) &= (y_M \wedge (x', y')) \vee (x_M, y_M), \quad \text{if } x' > x_M, \\ (x_M, y') &= (x', y') \vee (y' \wedge (x_M, y_M)), \quad \text{if } y' < y_M \text{ and } x' \leqslant x_M. \end{aligned} \tag{34}$$

Thus we obtain either $(x', y_M) \in C_2$ with $x' > x_M$, or $(x_M, y') \in C_2$ with $y' < y_M$ and $x' \leqslant x_M$, leading to a contradiction with the maximality of x_M or the minimality of y_M.

To prove that B_1 avoids C_1, assume by contradiction that there exists $(x, y) \in C_1$ where $x \leqslant x_M$ and $y \geqslant y_M$. We observe that there is a point $(x_M, y') \in [a, c]_M$, where $y' \leqslant y_M$. Using this point we obtain

$$(x_M, y_M) = (y_M \wedge (x, y)) \vee (x_M, y'), \tag{35}$$

which implies $(x_M, y_M) \in C_1$, hence $C_1 \cap C_2 \neq \emptyset$, a contradiction. □

Theorem 4. *Let $C_1, C_2 \in \mathcal{B}^2$, $C_1 \cap C_2 = \emptyset$, be two closed max–min convex sets that do not intersect the boundary of $\mathcal{B}^2$. Then there exist a permutation $i : \{1, 2\} \to \{1, 2\}$, a box $B \subset \mathcal{B}^2$ and a semispace $S \subset \mathcal{B}^2$ such that $C_{i(1)} \subset B$, $C_{i(2)} \subset S$ and $B \cap S = \emptyset$.*

Proof. The statement follows from Theorem 1 and Theorem 3. Indeed, Theorem 3 implies that either the minimal containing box of C_1 does not intersect with C_2, or the minimal containing box of C_2 does not intersect with C_1. The condition (12) is satisfied due to the fact that the convex sets do not intersect the boundary of $\mathcal{B}^n$ and hence so are the minimal containing boxes. Applying Theorem 1 we obtain the statement. □

Remark 3. We observe that Theorem 3 and Theorem 4 are not valid in dimension 3 or higher.

Let C_1 be the max–min segment $[(a, a, a), (b, b, b)]_M$ and C_2 be the max–min segment $[(b, a, a), (a, b, a)]_M$, where $0 \leqslant a < b \leqslant 1$. It follows from [19] that C_1 is part of the main diagonal, and that C_2 is the concatenation of two pieces with parametrizations $\{(t, b, a) | a \leq t \leq b\}$ and $\{(b, t, a) | a \leq t \leq b\}$. It follows from Fig. 7 that the smallest box containing C_1 is $[a, b]^3$ and the smallest box containing

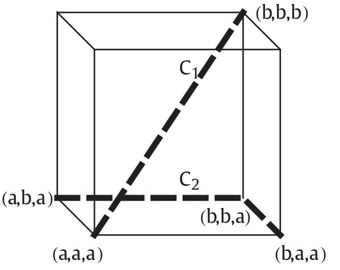

Fig. 7. Forbidden 3-dim box-semispace separation.

C_2 is $[a, b]^2 \times \{a\}$. Since one box is completely included in the other, neither box separation nor box-semispace separation of C_1 and C_2 is not possible.

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