Linear Algebra and Its Applications 435 (2011) 1736–1757

Contents lists available at ScienceDirect

Linear Algebra and Its Applications

journal homepage: www.elsevier.com/locate/laa

Max-algebraic attraction cones of nonnegative irreducible matrices ☆

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ARTICLE INFO

Article history:
Available online 25 March 2011

Submitted by J.J. Loiseau

AMS classification:
15A80
15A06
15A23
05C38
93B05

Keywords:
Max-plus algebra
Tropical algebra
Diagonal similarity
Cyclicity
Imprimitive matrix

ABSTRACT

It is known that the max-algebraic powers A^r of a nonnegative irreducible matrix are ultimately periodic. This leads to the concept of attraction cone $\mathrm{Attr}(A,t)$, by which we mean the solution set of a two-sided system $\lambda^t(A)A^r \otimes x = A^{r+t} \otimes x$, where r is any integer after the periodicity transient $T(A)$ and $\lambda(A)$ is the maximum cycle geometric mean of A. A question which this paper answers, is how to describe $\mathrm{Attr}(A,t)$ by a concise system of equations without knowing $T(A)$. This study requires knowledge of certain structures and symmetries of periodic max-algebraic powers, which are also described. We also consider extremals of attraction cones in a special case, and address the complexity of computing the coefficients of the system which describes attraction cone.

1. Introduction

The max-algebraic *cyclicity theorem* states that if $A \in \mathbb{R}_+^{n\times n}$ is irreducible and its maximal cycle geometric mean $\lambda(A)$ equals 1, then the sequence of max-algebraic powers A^k becomes periodic after some finite transient time $T(A)$, and that the ultimate period of A^k is equal to the cyclicity of the critical graph. Cohen et al. [17,18] seem to be the first to discover this, see also [4,8,19,32]. Generalizations to reducible case, computational complexity issues and important special cases of the cyclicity theorem have been extensively studied in [8,20,28,29,36,37].

☆ This research was supported by EPSRC Grant RRAH12809 and RFBR Grant 08-01-00601.
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doi:10.1016/j.laa.2011.02.038

The cyclicity theorem naturally leads to the concept of *attraction cone* $\mathrm{Attr}(A, t)$, by which we mean the solution set of a two-sided system $\lambda^t(A)A^r \otimes x = A^{r+t} \otimes x$ for any $r \geqslant T(A)$ (all such systems are equivalent). In particular, $\mathrm{Attr}(A, 1)$ is the set of vectors such that $\{A^k \otimes x,\ k \geqslant 1\}$ converges to a max-algebraic eigenvector of A (i.e., vector x such that $A \otimes x = \lambda(A)x$ holds).

The problem of characterizing attraction cones is related to some control problems in max-linear systems. For instance, given some "periodic regime", it is desirable to describe its attraction domain consisting of vectors which are driven to that regime by means of repeated action of max-linear operator. For Mairesse [34], periodic regime is a given set of vectors such that the sequence $\{A^k x,\ k \geqslant 1\}$, for the vectors x from the corresponding attraction domain, remains within that set after large enough time. He studies attraction domains for 3×3 matrices by examining all possible special cases, in each case providing a nice graphical interpretation of the result.

Braker [6, Section 8.3] studies eigenvector attraction spaces, which are the same as $\mathrm{Attr}(A, 1)$ of the present paper. He gives a sufficient condition when for $x \in \mathrm{Attr}(A, 1)$, the critical part of all vectors in $\{A^k \otimes x,\ k \geqslant 1\}$ is constant and equals the critical part of a particular max-algebraic eigenvector of A. No description of attraction spaces was developed in [6], whether in terms of max-linear systems or bases of max-algebraic spaces. The present paper offers such a description, in terms of a two-sided system of max-linear equations whose coefficients can be computed from the entries of A in a polynomial time. Using the elimination method of Butkovič and Hegedüs [10], improved and implemented recently by Allamigeon et al. [3], a basis of the solution set of any such two-sided system can be also obtained.

Our main tool is a special diagonal similarity scaling $A \mapsto X^{-1}AX$ called *visualization*, which brings A to a form where all the entries of A do not exceed $\lambda(A)$. Such scalings can be traced back to a work of Fiedler and Pták [26], and the present paper studies their role in max algebra continuing the thread of [9,12,13,22–25,39,42,44].

Nodes and edges that belong to the cycles attaining $\lambda(A)$ constitute the *critical graph*, which is very important for max algebra [4,19,31,32].

One of the main benefits of visualization is that it highlights the relation of max-algebraic periodicity to Boolean periodicity, leading from the spectral projector of Cohen et al. [17,18], see also Baccelli et al. [4], to CSR-representations introduced in Sergeev [42], see also Sergeev and Schneider [43] where a more general notion of CSR-expansions is introduced and studied. For the max-algebraic powers of nonnegative irreducible matrices, CSR-representation means that $A^t = \lambda^t(A)C \otimes S^t \otimes R$ for $t \geqslant T(A)$, where C and R are extracted from some Kleene star (see page 1743), and S is diagonally similar to the Boolean adjacency matrix of the critical graph.

A key role in Boolean periodicity is played by cyclic classes. Being inserted with S into the CSR-representation, they determine the whole periodic dynamics of A^r by cyclically permuting critical rows and columns. This naturally leads to circulant symmetries of periodic powers, and means that the dimension can be effectively reduced from n to $\check{c} + \overline{c}$, where $\check{c}$ is the total number of cyclic classes and $\overline{c}$ is the number of non-critical nodes. We show in Theorem 4.3, page 1747, that the equations of the system $\lambda^t(A)A^r \otimes x = A^{r+t} \otimes x$ that correspond to the non-critical nodes are redundant, and the remaining critical subsystem can be written as $S^t \otimes R \otimes x = R \otimes x$. This subsystem conveniently breaks into several chains of equations, which correspond to the strongly connected components of the critical graph.

The chains of equations can be further reduced by means of the chain cancellation, which generalizes the well-known rule $ax \oplus b = cx \oplus d \Leftrightarrow b = cx \oplus d$ if $a < c$. The coefficients of the reduced system come from the *core matrix*, whose entries are maxima in the blocks of A^t determined by strongly connected components of the critical graph together with noncritical nodes. The main result of the paper appears as Theorem 4.5, page 1748, where the system for $\mathrm{Attr}(A, 1)$ is explicitly written in terms of the core matrix and cyclic classes. This leads to further research problems: (1) how to efficiently compute the coefficients of the system, (2) how to describe the extremal solutions (in the sense of [11]).

We briefly outline the contents of the paper. Section 2 is devoted to necessary preliminaries on max algebra, Boolean algebra, cyclic classes, and some results on the properties of max-algebraic powers in the periodic regime. In Section 3, we describe concepts and constructions associated with the periodic regime, which are core matrix, CSR-representation and circulants. In Section 4, we derive the system for attraction cone, and study its extremal solutions and the computation of coefficients in the case of strongly connected critical graph. In Section 5 we give two examples illustrating the results of the paper.

We remark that the periodicity of max algebraic powers of matrices studied here can be regarded as an "exactly solvable model" of max-plus semigroups studied by Merlet [35].

Parts of the present paper and Ref. [42] have been included in the monograph of Peter Butkovič [8, Chapter 8].

2. Preliminaries

2.1. Kleene star and maximum cycle geometric mean

By *max algebra* we understand the analogue of linear algebra developed over the max-times semiring $\mathbb{R}_{\max,\times}$ which is the set of nonnegative numbers $\mathbb{R}_+$ equipped with the operations of "addition" $a \oplus b := \max(a,b)$ and the ordinary multiplication $a \otimes b := a \times b$. Zero and unity of this semiring coincide with the usual 0 and 1. The operations of the semiring are extended to the nonnegative matrices and vectors in the same way as in conventional linear algebra. That is if $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$ are matrices of compatible sizes with entries from $\mathbb{R}_+$, we write $C = A \oplus B$ if $c_{ij} = a_{ij} \oplus b_{ij} = \max(a_{ij}, b_{ij})$ for all i, j and $C = A \otimes B$ if $c_{ij} = \bigoplus_k a_{ik}b_{kj} = \max_k(a_{ik}b_{kj})$ for all i, j. If A is a square matrix over $\mathbb{R}_+$ then the iterated product $A \otimes A \otimes \cdots \otimes A$ in which the symbol A appears k times will be denoted by A^k.

The *max-plus semiring* $\mathbb{R}_{\max,+} = (\mathbb{R} \cup \{-\infty\}, \oplus = \max, \otimes = +)$, developed over the set of real numbers $\mathbb{R}$ with adjoined element $-\infty$ and the ordinary addition playing the role of multiplication, is another isomorphic "realization" of max algebra. In particular, $x \mapsto \exp(x)$ yields an isomorphism between $\mathbb{R}_{\max,+}$ and $\mathbb{R}_{\max,\times}$. In the max-plus setting, the zero element is $-\infty$ and the unity is 0. We will use the possibility to switch to this setting in the last section of this paper, which contains examples. The main benefit is that max-plus operations with integers are much easier to perform by hand.

Let $A \in \mathbb{R}_+^{n\times n}$. Consider the formal series

$$A^* = I \oplus A \oplus A^2 \oplus \cdots , \tag{1}$$

where I denotes the identity matrix with entries

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

Series (1) is a max-algebraic analogue of $(I - A)^{-1}$. It is called the *Kleene star* in the case of convergence to a finite matrix. It can be shown that for any $A \in \mathbb{R}_+^{n\times n}$,

$$\begin{aligned} A \text{ is a Kleene star } &\Leftrightarrow\ A^2 = A,\ a_{ii} = 1 \quad \forall i \\ &\Leftrightarrow\ a_{ii} = 1,\ a_{ij}a_{jk} \leqslant a_{ik} \quad \forall i, j, k. \end{aligned} \tag{2}$$

Below a^*_{ij} will denote the (i, j) entry of A^*.

Series (1) converges to a finite matrix if and only if $\lambda(A) \leqslant 1$, where

$$\lambda(A) = \bigoplus_k \bigoplus_{i_1,\ldots,i_k} (a_{i_1i_2}a_{i_2i_3}\ldots a_{i_ki_1})^{1/k} \tag{3}$$

is called the maximum cycle geometric mean (briefly m.c.g.m.) of A. In this case $A^* = I \oplus A \oplus \cdots \oplus A^{n-1}$. See Carré [14], or [32] Lemma 2.2, or [8] Proposition 1.6.10 for reference. The operation of taking m.c.g.m. is homogeneous: $\lambda(\mu A) = \mu\lambda(A)$, and hence any matrix with $\lambda(A) \neq 0$ can be scaled so that $\lambda(\frac{1}{\lambda(A)}A) = 1$. Matrices with m.c.g.m. equal to 1 will be called *definite*.

To $A = (a_{ij}) \in \mathbb{R}_+^{n\times n}$ we can associate the weighted digraph $\mathcal{D}(A) = (N(A), E(A))$, with the set of nodes $N(A) = \{1, \ldots, n\}$ and the set of edges $E(A) = \{(i, j) \mid a_{ij} \neq 0\}$ with weights $w(i, j) = a_{ij}$.

Suppose that $\pi = (i_1, \ldots, i_p)$ is a path in $\mathcal{D}(A)$, then the *weight* of π is defined to be $w(\pi, A) = a_{i_1 i_2} a_{i_2 i_3} \ldots a_{i_{p-1} i_p}$ if $p > 1$, and 1 if $p = 1$. If $i_1 = i_p$ then π is called a cycle.

It will be important that the entries of A^k, denoted by $a_{ij}^{(k)}$, are equal to the maximal weights of paths of length k connecting i to j, and the non-diagonal entries of A^* (denoted a_{ij}^*) represent just the maximal weights of paths from i to j, with no length restrictions.

When any two nodes of $\mathcal{D}(A)$ can be connected to each other by paths with nonzero weight (equivalently, they belong to a cycle), $\mathcal{D}(A)$ is called *strongly connected* and A is called *irreducible*. Otherwise, A is called *reducible*.

A cycle $\pi = (i_1, \ldots, i_k)$ in $\mathcal{D}(A)$ is called *critical*, if $\lambda(A) = (w(\pi, A))^{1/k}$. Every node and edge that belongs to a critical cycle is called *critical*. The set of critical nodes is denoted by $N_c(A)$, the set of critical edges is denoted by $E_c(A)$. The *critical digraph* of A, further denoted by $\mathcal{C}(A) = (N_c(A), E_c(A))$, is the digraph which consists of all critical nodes and critical edges of $\mathcal{D}(A)$. For definite $A \in \mathbb{R}_+^{n\times n}$, it follows that A^* is well defined, $AA^* \leqslant A^*$, and in particular $a_{ii}^* = 1$ and $a_{ij}a_{ji}^* \leqslant 1$ for all i, j. Moreover it can be shown that

$$(i, j) \in E_c(A) \Leftrightarrow a_{ij}a_{ji}^* = 1, \tag{4}$$

since any $(i, j) \in E_c(A)$ lies on a critical cycle and the nondiagonal entries of A^* represent weights of certain paths. Further we assume without loss of generality that the critical graph occupies first c nodes, i.e., that $N_c(A) = \{1, \ldots, c\}$.

For any $A \in \mathbb{R}_+^{n\times n}$, $\lambda(A)$ plays the role of the largest eigenvalue, with respect to the max-algebraic eigenproblem $A \otimes x = \lambda x$. If A is definite then any column of $A_{\cdot i}^*$ with $i \in N_c(A)$ is a max-algebraic eigenvector of A. Moreover if A is irreducible then A^* is finite and any such vector is finite. The structure of max-algebraic eigenspace (or as we would say, eigencone) was established by Gondran and Minoux [30] and Cuninghame-Green [19, Chapter 15]. See also [4, Theorem 3.23], [31, Section 6.4], [32, Lemmas 2.7 and 2.8] or [8, Theorem 4.4.8] for more specific reference.

There are certain transformations that do not change max-algebraic structures associated with nonnegative matrices. Consider a positive $x \in \mathbb{R}_+^n$ and define

$$X = \operatorname{diag}(x) := \begin{pmatrix} x_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & x_n \end{pmatrix}. \tag{5}$$

The transformation $A \mapsto X^{-1}AX$ is called a *diagonal similarity scaling* of A. Such transformations do not change $\lambda(A)$ and $\mathcal{C}(A)$ [24]. They commute with max-algebraic multiplication of matrices and hence with the operation of taking the Kleene star. Geometrically, they correspond to automorphisms of $\mathbb{R}_+^n$, both in the case of max algebra and in the case of nonnegative linear algebra. Further we define a particularly convenient form of matrices in max algebra.

A matrix $A \in \mathbb{R}_+^{n\times n}$ is called *visualized*, if

$$a_{ij} \leqslant \lambda(A), \ \forall i, j = 1, \ldots, n \tag{6}$$

$$a_{ij} = \lambda(A), \ \forall (i, j) \in E_c(A) \tag{7}$$

Note that (7) can be deduced from (6) and the definition of $\lambda(A)$ (3).

Visualization scalings were known already to Afriat [1] and Fiedler and Pták [26], and motivated extensive study of matrix scalings in nonnegative linear algebra, see e.g. [24,25,38,39]. We remark that some constructions and facts related to application of visualization scaling in max algebra and beyond have been observed in connection with max algebraic power method [22,23], behavior of matrix powers [9], max-balancing [38,39] and tropical methods in eigenvalue perturbation theory [2].

As observed by Fiedler and Pták [26], a matrix $A \in \mathbb{R}_+^{n\times n}$ with $\lambda(A) = 1$ (i.e., as we call it, definite) can be scaled to a visualized form by any column of A^*. A more detailed description of visualization scalings is given in [44] in terms of the convex cone $\{x \mid A \otimes x \leqslant x\}$ and its relative interior.

The present paper, along with [42,43], can be seen as a follow-up of [12,13,21,44] which established basic techniques and terminology of visualization used here.

2.2. Max algebra and Boolean matrices

Max algebra is related to the algebra of Boolean matrices. The latter algebra is defined over the Boolean semiring $\mathcal{S}$ which is the set $\{0, 1\}$ equipped with logical operations "OR" $a \oplus b := a \vee b$ and "AND" $a \otimes b := a \wedge b$. Clearly, Boolean matrices can be treated as objects of max algebra, as a very special but crucial case.

For a strongly connected graph, its cyclicity is defined as the g.c.d. of the lengths of all cycles (or equivalently, all simple cycles). If the cyclicity is 1 then the graph is called *primitive*, otherwise it is called *imprimitive*. We will not distinguish between cyclicity (or primitivity) of a Boolean matrix A and the associated digraph $\mathcal{D}(A)$. Further we recall an important result of Boolean matrix theory.

Proposition 2.1 (see Brualdi and Ryser [7]). *Let $A \in \mathcal{S}^{n\times n}$ be irreducible, and let γ_A be the cyclicity of $\mathcal{D}(A)$ (which is strongly connected). Then for each $k \geqslant 1$, there exists a permutation matrix P such that $P^{-1}A^kP$ has r irreducible diagonal blocks, where $r = \gcd(k, \gamma_A)$, and all elements outside these blocks are zero. The cyclicity of all these blocks is γ_A/r.*

In max algebra, let $A \in \mathbb{R}_+^{n\times n}$. Define the Boolean *critical matrix* $A^{[C]} = (a_{ij}^{[C]})$ by

$$a_{ij}^{[C]} = \begin{cases} 1, & (i,j) \in E_c(A) \\ 0, & (i,j) \notin E_c(A). \end{cases} \tag{8}$$

Let $A, B \in \mathbb{R}_+^{n\times n}$. Assume that $\mathcal{C}(A)$ has n_c strongly connected components (briefly, s.c.c.) $\mathcal{C}_\mu$ for $\mu = 1, \ldots, n_c$, with cyclicities γ_μ. Denote by N_μ the set of nodes in $\mathcal{C}_\mu$. Denote by $B_{\mu\nu}$ the block of B extracted from the rows with indices in N_μ and columns with indices in N_ν.

The following proposition was obtained in [13], as a corollary of Proposition 2.1. See [42, Proposition 3.3] for a simple proof, which uses visualisation.

Proposition 2.2. *Let $A \in \mathbb{R}_+^{n\times n}$ and $\lambda(A) \neq 0$.*

1. $\lambda(A^k) = \lambda^k(A)$.
2. $(A^k)^{[C]} = (A^{[C]})^k$.
3. *For each $k \geqslant 1$, there exists a permutation matrix P such that $(P^{-1}A^kP)_{\mu\mu}^{[C]}$, for each $\mu = 1, \ldots, n_c$, has $r_\mu := \gcd(k, \gamma_\mu)$ irreducible blocks and all elements outside these blocks are zero. The cyclicity of all blocks in $(P^{-1}A^kP)_{\mu\mu}^{[C]}$ is equal to γ_μ/r_μ.*

For a path π in a digraph $G = (N, E)$, where $N = \{1, \ldots, n\}$, denote by $l(\pi)$ the length of π, i.e., the number of edges traversed by π.

Proposition 2.3 (see Brualdi-Ryser [7]). *Let $G = (N, E)$ be a strongly connected digraph with cyclicity γ_G. Then the lengths of any two paths connecting $i \in N$ to $j \in N$ (with i, j fixed) are congruent modulo γ_G.*

Proposition 2.3 implies that the following *equivalence relation* can be defined: $i \sim j$ if there exists a path π from i to j such that $l(\pi) \equiv 0 (\mathrm{mod}\, \gamma_G)$. The equivalence classes of G with respect to this relation are called *cyclic classes* [5,40,41]. The cyclic class of i will be denoted by $[i]$.

Consider the following *access relations* between cyclic classes: $[i] \rightarrow_t [j]$ if there exists a path π from a node in $[i]$ to a node in $[j]$ such that $l(\pi) \equiv t (\mathrm{mod}\, \gamma_G)$. In this case, a path π with $l(\pi) \equiv t (\mathrm{mod}\, \gamma_G)$ exists between any node in $[i]$ and any node in $[j]$. Further, by Proposition 2.3 the length of any path between a node in $[i]$ and a node in $[j]$ is congruent to t, so the relation $[i] \rightarrow_t [j]$ is well-defined. Class $[j]$ is called *adjacent* to $[i]$ if $[i] \rightarrow_1 [j]$.

Cyclic classes can be computed in $O(|E|)$ time by digraph condensation methods [5,7].

The notion of cyclic classes and access relations can be generalized to the case when G has n_c disjoint strongly connected components G_μ with cyclicities γ_μ, for $\mu = 1, \ldots, n_c$ (this is how the critical graph in max algebra looks like). In this case we write $i \sim j$ if i, j belong to the same component G_μ and there exists a path π from i to j such that $l(\pi) \equiv 0 (\text{mod}\, \gamma_\mu)$. If $l(\pi) \equiv t (\text{mod}\, \gamma_\mu)$, then we write $[i] \to_t [j]$. In this case the *cyclicity* of G is $\gamma := \text{l.c.m.}(\gamma_\mu)$, $\mu = 1, \ldots, n_c$.

We will be interested in the cyclic classes of critical graphs, and below we also give an interpretation of these, in terms of the Boolean matrix $A^{[C]}$. Let $A \in \mathbb{R}_+^{n\times n}$. Following Brualdi and Ryser [7] we can find an ordering of the indices such that any submatrix $A_{\mu\mu}^{[C]}$, which corresponds to an imprimitive component $\mathcal{C}_\mu$ of $\mathcal{C}(A)$, will be of the form

$$\begin{pmatrix} \mathbf{0} & A_{s_1s_2}^{[C]} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & A_{s_2s_3}^{[C]} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & A_{s_{k-1}s_k}^{[C]} \\ A_{s_ks_1}^{[C]} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{pmatrix}, \tag{9}$$

where $k = \gamma_\mu$. Indices s_i, for $i = 1, \ldots, k$, correspond to cyclic classes. More precisely, the cyclic class corresponding to s_i is adjacent to the cyclic class corresponding to s_{i+1} for $i = 1, \ldots, k-1$, and the cyclic class of s_k is adjacent to the cyclic class of s_1. By Proposition 2.2 part 2, when A is raised to power k, $A^{[C]}$ is also raised to the same power over the Boolean algebra. Any power of $A^{[C]}$ has a block-permutation form similar to (9), with a different pattern of nonzero blocks.

Theorem 5.4.11 of [33] implies that the sequence $(A^k)^{[C]} = (A^{[C]})^k$ becomes periodic after $k \leqslant (n-1)^2 + 1$, with period $\gamma = \text{lcm}(\gamma_\mu)$, $\mu = 1, \ldots, n_c$. In the periodic regime, all entries of nonzero blocks are equal to 1. This fact is one step from the periodicity of matrices in max algebra.

Further we always assume that the cyclic classes are *properly arranged* meaning that any submatrix $A_{\mu\mu}^{[C]}$ is of the form (9).

2.3. Spectral projector and periodicity in max algebra

We assume as before that the critical graph $\mathcal{C}(A)$ occupies the first c nodes. For a definite $A \in \mathbb{R}_+^{n\times n}$, consider the matrix $Q(A)$ with entries

$$q_{ij} = \bigoplus_{k=1}^{c} a_{ik}^* a_{kj}^*, \quad i, j = 1, \ldots, n. \tag{10}$$

The max-linear operator whose matrix is $Q(A)$, is a max-linear *spectral projector* associated with A, in the sense that it projects $\mathbb{R}_+^n$ on the max-algebraic eigencone $\{x \mid A \otimes x = x\}$, see [4] Subsection 3.7.3. We also note that $Q(A)$ is important for the policy iteration algorithm of [16].

We will need the following property of $Q(A)$ which follows directly from (10).

Proposition 2.4. *For a definite $A \in \mathbb{R}_+^{n\times n}$, any column (or row) of $Q(A)$ with index in $1, \ldots, c$ is equal to the corresponding column (or row) of A^*.*

This operator is closely related to the periodicity questions.

Theorem 2.5 (Baccelli et al. [4], Theorem 3.109)**.** *Let $A \in \mathbb{R}_+^{n\times n}$ be irreducible and definite, and let all s.c.c. of $\mathcal{C}(A)$ have cyclicity 1. Then there is an integer $T(A)$ such that $A^r = Q(A)$ for all $r \geqslant T(A)$.*

It can be easily shown that Theorem 2.5 can be generalized to the situation when A is in a blockdiagonal form with irreducible definite diagonal blocks $A_{11}, \ldots, A_{uu}$ so that

$$A = \begin{pmatrix} A_{11} & \dots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & A_{uu} \end{pmatrix}. \tag{11}$$

If A is irreducible and definite, then A^k for all $k \geqslant 1$ can be transformed by a simultaneous permutation of columns and rows to the blockdiagonal form (11), where all blocks $A_{11}, \dots, A_{uu}$ are irreducible and definite.

Indeed, let us show that $A_{11}, \dots, A_{uu}$ are definite (i.e., that the m.c.g.m. of each block is 1). On one hand, it is evident when A is assumed to be visualised, that the m.c.g.m. of each block does not exceed 1. On the other hand, any column $A^*_{\cdot i}$ with $1 \leqslant i \leqslant c$ is a positive max-algebraic eigenvector of A and hence of A^k. This shows that each block of A^k has an eigenvector with eigenvalue $\lambda(A)$, hence the m.c.g.m. of each block, being equal to the largest max-algebraic eigenvalue of that block, cannot be less than 1.

It follows from Proposition 2.2 part 3 that all components of $\mathcal{C}(A^\gamma)$ are primitive, where γ is the cyclicity of $\mathcal{C}(A)$.

These arguments lead us to the following extension of Theorem 2.5.

Theorem 2.6. *Let $A \in \mathbb{R}_+^{n\times n}$ be irreducible and definite, and let γ be the cyclicity of $\mathcal{C}(A)$. There exists $T(A)$ such that*

1. *$A^{t\gamma} = Q(A^\gamma)$ for all $t\gamma \geqslant T(A)$;*
2. *$A^{r+\gamma} = A^r$ for all $r \geqslant T(A)$.*

$T(A)$ will be called the *transient*, and the powers A^r for $r \geqslant T(A)$ will be called *periodic powers*.

The following properties of periodic powers were established in [42], Propositions 4.4 and 4.5. Both properties are deduced there from Theorem 2.6 above (though not explicitly stated in that paper). Here and in the sequel $a_{ij}^{(k)}$ will denote the (i,j) entry of A^k, with a few explicitly stated exceptions from this rule.

Proposition 2.7. *Let $A \in \mathbb{R}_+^{n\times n}$ be a definite and irreducible matrix, and let $t \geqslant 0$ be such that $t\gamma \geqslant T(A)$. Then for every integer $l \geqslant 0$*

$$A_{k\cdot}^{t\gamma+l} = \bigoplus_{i=1}^{c} a_{ki}^{(t\gamma)} A_{i\cdot}^{t\gamma+l}, \; A_{\cdot k}^{t\gamma+l} = \bigoplus_{i=1}^{c} a_{ik}^{(t\gamma)} A_{\cdot i}^{t\gamma+l}, \quad 1 \leqslant k \leqslant n. \tag{12}$$

In the following $T_c(A)$ denotes the number after which the critical rows and columns of A^r become periodic. It is shown in [42] that $T_c(A) \leqslant n^2$. It can be conjectured that $T_c(A) \leqslant (n-1)^2 + 1$, the same bound as for the Boolean periodicity.

Proposition 2.8. *Let $A \in \mathbb{R}_+^{n\times n}$ be a definite and irreducible matrix, and let $i, j \in \{1, \dots, c\}$ be such that $[i] \to_l [j]$, for some $0 \leqslant l < \gamma$.*

1. *For any $r \geqslant T_c(A)$, there exists $t \geqslant 0$ such that*
$$a_{ij}^{(t\gamma+l)} A_{\cdot i}^r = A_{\cdot j}^{r+l}, \; a_{ij}^{(t\gamma+l)} A_{j\cdot}^r = A_{i\cdot}^{r+l}. \tag{13}$$
2. *If A is visualized, then for all $r \geqslant T_c(A)$*
$$A_{\cdot i}^r = A_{\cdot j}^{r+l}, \; A_{j\cdot}^r = A_{i\cdot}^{r+l}. \tag{14}$$

Let us also note the following corollaries of Proposition 2.8.

Corollary 2.9. *Let $A \in \mathbb{R}_+^{n\times n}$ be irreducible and visualized. For $r \geqslant T_c(A)$, all rows (and all columns) of A^r with indices in the same cyclic class are identical.*

Corollary 2.10. *Let $A \in \mathbb{R}_+^{n\times n}$ be irreducible. For $r \geqslant T_c(A)$ and $i \in N_\mu$, $A^r_{\cdot i} = A^{r+\gamma_\mu}_{\cdot i}$ and $A^r_{i\cdot} = A^{r+\gamma_\mu}_{i\cdot}$.*

Proof. If A is visualized then this follows from Proposition 2.8, part 2. Now observe that the equalities $A^r_{\cdot i} = A^{r+\gamma_\mu}_{\cdot i}$ and $A^r_{i\cdot} = A^{r+\gamma_\mu}_{i\cdot}$ are invariant under diagonal similarity scaling. □

3. Properties of periodic powers

3.1. Core matrix

In the sequel we always assume that $A \in \mathbb{R}_+^{n\times n}$ is irreducible. Let $\mathcal{C}(A)$ consist of n_c components $\mathcal{C}_\mu$ with cyclicities γ_μ, for $\mu = 1, \ldots, c$. Let $\gamma = \text{l.c.m.}\ (\gamma_\mu)$ and $\bar{c}$ be the number of non-critical nodes. Further it will be convenient (though artificial) to consider, together with these components, also "non-critical components" $\mathcal{C}_\mu$ for $\mu = n_c+1, \ldots, n_c+\bar{c}$, whose node sets N_μ consist of just one non-critical node, and whose set of edges is empty.

Consider the block decomposition of A^r for $r \geqslant 1$, induced by the subsets N_μ for $\mu = 1, \ldots, n_c+\bar{c}$. The submatrix of A^r extracted from the rows in N_μ and columns in N_ν will be denoted by $A^{(r)}_{\mu\nu}$. If A is visualized and definite, we define the corresponding *core matrix* $A^{Core} = (\alpha_{\mu\nu}),\ \mu, \nu = 1, \ldots, n_c + \bar{c}$ by

$$\alpha_{\mu\nu} = \max\{a_{ij} \mid i \in N_\mu,\ j \in N_\nu\}. \tag{15}$$

The entries of $(A^{Core})^*$ will be denoted by $\alpha^*_{\mu\nu}$. Their role is investigated in the next theorem.

Theorem 3.1. *Let $A \in \mathbb{R}_+^{n\times n}$ be a definite visualized matrix and $r \geqslant T_c(A)$. Let $\mu, \nu = 1, \ldots, n_c + \bar{c}$ be such that at least one of these indices is critical. Then the maximal entry of the block $A^{(r)}_{\mu\nu}$ is equal to $\alpha^*_{\mu\nu}$.*

Proof. The entry $\alpha^*_{\mu\nu}$ is the maximal weight over paths from μ to ν, with respect to the matrix A^{Core}. We take such a path $(\mu_1, \ldots, \mu_l)$ with maximal weight, where $\mu_1 := \mu$ and $\mu_l = \nu$. With this path we can associate a path π in $\mathcal{D}(A)$ defined by $\pi = \tau_1 \circ \sigma_1 \circ \tau_2 \circ \ldots \circ \sigma_{l-1} \circ \tau_l$, where τ_i are critical paths which entirely belong to the components $\mathcal{C}_{\mu_i}$, and σ_i are edges with maximal weight connecting $\mathcal{C}_{\mu_i}$ to $\mathcal{C}_{\mu_{i+1}}$. Such a path exists since any two nodes in the same component $\mathcal{C}_\mu$ can be connected to each other by critical paths if μ is critical, and if μ is non-critical then $\mathcal{C}_\mu$ consists just of one node. The weights of τ_i are equal to 1, hence the weight of π is equal to $\alpha^*_{\mu\nu}$. It follows from the definition of $\alpha_{\mu\nu}$ and $\alpha^*_{\mu\nu}$ that $\alpha^*_{\mu\nu}$ is the greatest weight over all paths which connect nodes in $\mathcal{C}_\mu$ to nodes in $\mathcal{C}_\nu$. As at least one of the indices μ, ν is critical, there is freedom in the choice of the paths τ_1 or τ_l which can be of arbitrary length. Assume w.l.o.g. that μ is critical. Then for any r exceeding the length of $\sigma_1 \circ \tau_2 \circ \ldots \circ \sigma_{l-1} \circ \tau_l$ which we denote by $l_{\mu\nu}$, the block $A^{(r)}_{\mu\nu}$ contains an entry equal to $\alpha^*_{\mu\nu}$ which is the greatest entry of the block. Taking the maximum $T'(A)$ of $l_{\mu\nu}$ over all ordered pairs (μ, ν) with μ or ν critical, we obtain the claim for $r \geqslant T'(A)$. Since at $r \geqslant T_c(A)$ the critical rows and columns of A^r are periodic, the claim also holds for $r \geqslant T_c(A)$. □

3.2. CSR-representation

For a definite visualized matrix $A \in \mathbb{R}_+^{n\times n}$, the statements of Propositions 2.7 and 2.8 can be combined in the following. Let $C \in \mathbb{R}_+^{n\times c}$ and $R \in \mathbb{R}_+^{c\times n}$ be matrices extracted from the first c columns (resp. rows) of $Q(A^\gamma)$ (or equivalently $(A^\gamma)^*$), and let $S := A^{[C]}$, the critical matrix of A defined by (8).

Theorem 3.2. *Let $A \in \mathbb{R}_+^{n\times n}$ be definite and visualized. For $r \geqslant T(A)$ and $r \equiv l (\text{mod}\ \gamma)$, $A^r = C \otimes S^l \otimes R$. In particular, the submatrices of critical columns of A^r, resp. critical rows of A^r, are expressed as $C \otimes S^l$, resp. $S^l \otimes R$.*

Proof. By (10) $Q(A^\gamma)=C\otimes R$, and by Theorem 2.6 $A^{\gamma t}=Q(A^\gamma)=C\otimes R$ for $\gamma t\geqslant T(A)$. Further C (resp. R) can be extracted from the first c columns (resp. rows) of $A^{\gamma t}$. Thus the claim is already proved for $r=\gamma t\geqslant T(A)$.

As $S=A^{[C]}$, it follows that 1) the (i,j) entry of S^l can be 1 only if $[i]\to_l [j]$, 2) for each pair of classes $[i]\to_l [j]$ and each $i_1\in[i]$ there exists $j_1\in[j]$ such that the (i_1,j_1) entry of S^l equals 1.

Using these two observations and equation (13) applied to $A^{\gamma t}$ and $A^{\gamma t+l}$ for $\gamma t\geqslant T(A)$, we obtain that the critical columns of $A^{\gamma t+l}$ are given by $C\otimes S^l$ and the critical rows of $A^{\gamma t+l}$ are given by $S^l\otimes R$.

Combining this with any of the two equations of (12), we obtain that $A^{\gamma t+l}=C\otimes S^l\otimes R$, for any $\gamma t\geqslant T(A)$. $\square$

Further we observe that the dimensions of periodic powers and the *CSR-representation* established in Theorem 3.2 can be reduced.

The rows and columns with indices in the same cyclic class coincide in any power A^r, where $r\geqslant T_c(A)$ and A is definite and visualized. Hence the blocks of A^r, for $\mu,\nu=1,\ldots,n_c+\overline{c}$ and $r\geqslant T_c(A)$ (in particular, $r\geqslant n^2$), are of the form

$$A^{(r)}_{\mu\nu}=\begin{pmatrix}\tilde a^{(r)}_{s_1t_1}E_{s_1t_1} & \ldots & \tilde a^{(r)}_{s_1t_m}E_{s_1t_m}\\ \vdots & \ddots & \vdots\\ \tilde a^{(r)}_{s_kt_1}E_{s_kt_1} & \ldots & \tilde a^{(r)}_{s_kt_m}E_{s_kt_m}\end{pmatrix}, \tag{16}$$

where k (resp. m) are cyclicities of $\mathcal{C}_\mu$ (resp. $\mathcal{C}_\nu$), indices $s_1,\ldots,s_k$ (resp. $t_1,\ldots,t_m$) correspond to properly arranged cyclic classes of $\mathcal{C}_\mu$ (resp. $\mathcal{C}_\nu$), and $E_{s_it_j}$ are matrices with appropriate dimensions with all entries equal to 1. We assume that $\mathcal{C}_\mu$ has just one “cyclic class” if μ is non-critical.

Formula (16) defines the square matrix $\tilde A^{(r)}$ with $\check c+\overline{c}$ rows and columns, where $\check c$ is the total number of cyclic classes, as matrix with entries $\tilde a^{(r)}_{s_it_j}$. Corresponding to (16), this matrix has blocks

$$\tilde A^{(r)}_{\mu\nu}=\begin{pmatrix}\tilde a^{(r)}_{s_1t_1} & \ldots & \tilde a^{(r)}_{s_1t_m}\\ \vdots & \ddots & \vdots\\ \tilde a^{(r)}_{s_kt_1} & \ldots & \tilde a^{(r)}_{s_kt_m}\end{pmatrix}. \tag{17}$$

It follows that $\tilde A^{(r_1+r_2)}=\tilde A^{(r_1)}\otimes\tilde A^{(r_2)}$ for all $r_1,r_2\geqslant T_c(A)$. In words, the multiplication of any two powers $A^{(r_1)}$ and $A^{(r_2)}$ for $r_1,r_2\geqslant T_c(A)$ reduces to the multiplication of $\tilde A^{(r_1)}$ and $\tilde A^{(r_2)}$.

If we take $r=\gamma t+l$ where $\gamma t\geqslant T(A)$ (instead of $T_c(A)$ above) and denote $\tilde A:=\tilde A^{(\gamma t+1)}$, then due to the periodicity we obtain

$$\tilde A^{(\gamma t+l)}=\tilde A^{((\gamma t+1)l)}=(\tilde A^{(\gamma t+1)})^l=\tilde A^l, \tag{18}$$

so that $\tilde A^{(r)}$ can be regarded as the lth power of $\tilde A$ where $r\equiv l(\mathrm{mod}\ \gamma)$, for all $r\geqslant T(A)$.

Matrices C, R, and S^t for $t\geqslant(n-1)^2+1$ have the same block structure as in (16). This shows that the behavior of periodic powers A^r is fully described by

$$\tilde A^r=\tilde C\otimes\tilde S^l\otimes\tilde R,\quad r\equiv l(\mathrm{mod}\ \gamma), \tag{19}$$

where $\tilde S$ is a $\check c\times\check c$ Boolean matrix with blocks

$$\tilde S_{\mu\mu}=\begin{pmatrix}0&1&0&\cdots&0\\0&0&1&\cdots&0\\ \vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&1\\1&0&0&\cdots&0\end{pmatrix},\quad \tilde S_{\mu\nu}=0 \text{ for } \mu\neq\nu, \tag{20}$$

where $\tilde{C}$ and $\tilde{R}$ are extracted from critical columns and rows of $\tilde{A}^{\gamma} = Q(\tilde{A}^{\gamma})$, or equivalently, formed from the scalars in the blocks of C and R. The dimensions of $\tilde{C}$ and $\tilde{R}$ are $(\check{c}+\bar{c})\times\check{c}$ and $\check{c}\times(\check{c}+\bar{c})$, respectively.

3.3. Circulant properties

Matrix $A\in\mathbb{R}_+^{n\times n}$ is called a *circulant* if there exist scalars $\alpha_0,\ldots,\alpha_{n-1}$ such that $a_{ij}=\alpha_d$ whenever $j-i=d(\text{mod } n)$. This looks like

$$A=\begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} \\ \alpha_{n-1} & \alpha_0 & \alpha_1 & \cdots & \alpha_{n-2} \\ \alpha_{n-2} & \alpha_{n-1} & \alpha_0 & \cdots & \alpha_{n-3} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \alpha_1 & \alpha_2 & \ldots & \ldots & \alpha_0 \end{pmatrix} \tag{21}$$

We also consider the following generalizations of this notion.

Matrix $A\in\mathbb{R}_+^{m\times n}$ will be called a *rectangular circulant*, if $a_{ij}=a_{ps}$ whenever $p=i+t(\text{mod } m)$ and $s=j+t(\text{mod } n)$, for all i,j,t. When $m=n$, this is an ordinary circulant given by (21).

Matrix $A\in\mathbb{R}_+^{m\times n}$ will be called a *block $k\times k$ circulant* when there exist scalars $\alpha_1,\ldots,\alpha_k$ and a block decomposition $A=(A_{ij}),\ i,j=1,\ldots,k$ such that $A_{ij}=\alpha_d E_{ij}$ if $j-i=d(\text{mod } k)$, where all entries of blocks E_{ij} are equal to 1.

$A\in\mathbb{R}_+^{m\times n}$ is called *d-periodic* when $a_{ij}=a_{is}$ if $(s-j)$ mod n is a multiple of d, and when $a_{ji}=a_{si}$ if $(s-j)$ mod m is a multiple of d.

We give an example of 6×9 rectangular circulant A:

$$A=\begin{pmatrix} 0&1&2&0&1&2&0&1&2\\ 2&0&1&2&0&1&2&0&1\\ 1&2&0&1&2&0&1&2&0\\ 0&1&2&0&1&2&0&1&2\\ 2&0&1&2&0&1&2&0&1\\ 1&2&0&1&2&0&1&2&0 \end{pmatrix}.$$

This example provides evidence that a rectangular $m\times n$ circulant consists of ordinary $d\times d$ circulant blocks where $d=$g.c.d.(m,n). In particular, it is d-periodic. Also, there exist permutation matrices P and Q such that PAQ is a block circulant:

$$PAQ=\begin{pmatrix} 0&0&0&1&1&1&2&2&2\\ 0&0&0&1&1&1&2&2&2\\ 2&2&2&0&0&0&1&1&1\\ 2&2&2&0&0&0&1&1&1\\ 1&1&1&2&2&2&0&0&0\\ 1&1&1&2&2&2&0&0&0 \end{pmatrix}.$$

We formalize these observations in the following.

Proposition 3.3. *Let $A \in \mathbb{R}_+^{m\times n}$ be a rectangular circulant, and let d =g.c.d.(m, n).*

1. *A is d-periodic.*
2. *There exist permutation matrices P and Q such that PAQ is a block $d \times d$ circulant.*

Proof. 1. There are integers t_1 and t_2 such that $d = t_1 m + t_2 n$. Using the definition of rectangular circulant we obtain $a_{ij} = a_{is}$, if $s = j + t_1 m (\text{mod } n)$, and hence if $s = j + d (\text{mod } n)$. Analogously for rows, we obtain that $a_{ji} = a_{si}$, if $s = j + t_2 n (\text{mod } m)$, and hence if $s = j + d (\text{mod } m)$.
2. As A is d-periodic, all rows such that $i + d = j (\text{mod } m)$ are equal, so that $\{1, \ldots, m\}$ can be divided in d groups each with m/d indices, in such a way that $A_{i\cdot} = A_{j\cdot}$ if i and j belong to the same group. We can find a permutation matrix P such that $A' = PA$ will have rows $A'_{1\cdot} = \ldots = A'_{d\cdot} = A_{1\cdot}$, $A'_{d+1\cdot} = \ldots = A'_{2d\cdot} = A_{2\cdot}$, and so on. Analogously we can find a permutation matrix Q such that $A'' = PAQ$ will have columns $A''_{\cdot 1} = \ldots = A''_{\cdot d} = A'_{\cdot 1}, A''_{\cdot d+1} = \ldots = A''_{\cdot 2d} = A'_{\cdot 2}$, and so on. Then A'' has blocks (A''_{ij}) for $i, j = 1, \ldots, d$. of dimension $n/d \times m/d$, where $A''_{ij} = a_{ij} E_{ij}$, and E_{ij} has all entries 1. As A is d-periodic, it can be shown that its submatrix extracted from the first d rows and columns, is a circulant. Hence A'' is a block circulant. □

When g.c.d.$(m, n) = 1$, the rectangular circulant is 1-periodic, i.e., a constant matrix.

Proposition 3.4. *Let $A \in \mathbb{R}_+^{n\times n}$ be a definite visualized matrix which admits block decomposition (16), and $r \geqslant T(A)$. Let $\mathcal{C}_\mu, \mathcal{C}_\nu$ be two (possibly equal) components of $\mathcal{C}(A)$, and $d = g.c.d.(\gamma_\mu, \gamma_\nu)$.*

1. *$\tilde{A}^{(r)}_{\mu\nu}$ is a rectangular circulant (which is a circulant if $\mu = \nu$).*
2. *For any critical μ and ν, there is a permutation P such that $(P^T \tilde{A} P)^{(r)}_{\mu\nu}$ is a block $d \times d$ circulant matrix.*
3. *If r is a multiple of γ_μ, then $\tilde{A}^{(r)}_{\mu\mu}$ are circulant Kleene stars, where all off-diagonal entries are strictly less than* 1.

Proof. 1. Using Eq. (14) we see that for all (i, j) and (k, l) such that $k = i + t (\text{mod}\, \gamma_\mu)$ and $l = j + t (\text{mod}\, \gamma_\nu)$,

$$\tilde{a}^{(r)}_{s_k t_l} = \tilde{a}^{(r+t)}_{s_i t_l} = \tilde{a}^{(r)}_{s_i t_j}.$$

2. If $\mu = \nu$ then $P = I$, and if $\mu \neq \nu$ then P is any permutation matrix such that its "subpermutations" for N_μ and N_ν are given by P and Q of Proposition 3.3.

3. Part 1 shows that $\tilde{A}^{(r)}_{\mu\mu}$ are circulants for any $r \geqslant T(A)$ and critical μ. If r is a multiple of γ_μ then, using the periodicity of critical rows and columns (Corollary 2.10) we obtain that $\tilde{A}^{(r)}_{\mu\mu}$ are submatrices of $\tilde{A}^\gamma = Q(\tilde{A}^\gamma)$ and hence of $(\tilde{A}^\gamma)^*$. This implies, using (2), that they are Kleene stars. As the μth component of $\mathcal{C}(\tilde{A})$ is just a cycle of length γ_μ, the corresponding component of $\mathcal{C}(\tilde{A}^{\gamma_\mu})$ consists of γ_μ loops, showing that the off-diagonal entries of $\tilde{A}^{(r)}_{\mu\mu}$ are strictly less than 1. □

4. Attraction cones

4.1. A system for attraction cone

Let $A \in \mathbb{R}_+^{n\times n}$ and $\lambda(A) = 1$. The *attraction cone* $\text{Attr}(A, t)$, where $t \geqslant 1$ is an integer, is the set which consists of all vectors x, for which there exists an integer r such that $A^r \otimes x = A^{r+t} \otimes x$, and hence this is also true for all integers greater than or equal to r. Actually we can speak of any $r \geqslant T(A)$, due to the following observation.

Proposition 4.1. *Let A be irreducible and definite. The systems $A^r \otimes x = A^{r+t} \otimes x$ are equivalent for all $r \geqslant T(A)$.*

Proof. Let x satisfy $A^s \otimes x = A^{s+t} \otimes x$ for some $s \geqslant T(A)$, then it also satisfies this system for all greater s. Due to the periodicity, for all k from $T(A) \leqslant k \leqslant s$ there exists $l > s$ such that $A^k = A^l$. Hence $A^k \otimes x = A^{k+t} \otimes x$ also hold for $T(A) \leqslant k \leqslant s$. □

Corollary 4.2. $\operatorname{Attr}(A, t) = \operatorname{Attr}(A^t, 1)$.

Proof. By Proposition 4.1, $\operatorname{Attr}(A, t)$ is solution set to the system $A^r \otimes x = A^{r+t} \otimes x$ for any $r \geqslant T(A)$. It follows that $A^{tl} \otimes x = A^{t(l+1)} \otimes x$ for any $l \geqslant T(A)/t$, and hence for any $l \geqslant T(A^t)$. □

An equation of $A^r \otimes x = A^{r+t} \otimes x$ whose index (meaning the index of the corresponding row of A^r and A^{r+t}) is in $\{1, \ldots, c\}$ will be called *critical*, and the subsystem of equations with indices in $\{1, \ldots, c\}$ will be called the *critical subsystem*.

Theorem 4.3. *Let A be irreducible and definite and let $r \geqslant T(A)$. Then $A^r \otimes x = A^{r+t} \otimes x$ is equivalent to its critical subsystem, which can be written as $S^t \otimes R \otimes x = R \otimes x$.*

Proof. Consider an equation of $A^r_{k\cdot} \otimes x = A^{r+t}_{k\cdot} \otimes x$ whose index is non-critical. Using (12) it can be written as

$$\bigoplus_{i=1}^{c} a_{ki}^{(r)} A^r_{i\cdot} \otimes x = \bigoplus_{i=1}^{c} a_{ki}^{(r)} A^{r+t}_{i\cdot} \otimes x, \tag{22}$$

hence it is a max combination of equations in the critical subsystem.

As the systems $A^{r+t} \otimes x = A^r \otimes x$ are equivalent to each other for all $r \geqslant T(A)$, we can assume w.l.o.g. that r is a multiple of γ. It follows now from Theorem 3.2 that the critical subsystem above can be written as $S^t \otimes R \otimes x = R \otimes x$. □

It is proved in [42], following the idea of [40], that the coefficients of the system of $A^r \otimes x = A^{r+t} \otimes x$ for $r \geqslant T(A)$ can be obtained in $O(n^3 \log n)$ operations, by means of matrix squaring and permutation on cyclic classes.

Next we show how the specific circulant structure of A^r at $r \geqslant T(A)$ can be exploited, to cancel the redundant terms in the system of equations for the attraction cone $\operatorname{Attr}(A, 1)$. Due to Theorem 3.1 the core matrix $A^{Core} = \{\alpha_{\mu\nu} \mid \mu, \nu = 1, \ldots, n_c\}$, and its Kleene star $(A^{Core})^* = \{\alpha^*_{\mu\nu} \mid \mu, \nu = 1, \ldots, n_c\}$ will be of special importance. We introduce the notation

$$\begin{aligned} M_\nu^{(r)}(i) &= \{j \in N_\nu \mid a_{ij}^{(r)} = \alpha^*_{\mu\nu}\},\ i \in N_\mu,\ \forall \nu : \mathcal{C}_\nu \neq \mathcal{C}_\mu, \\ K^{(r)}(i) &= \{t > c \mid a_{it}^{(r)} = \alpha^*_{\mu\nu(t)}\},\ i \in N_\mu, \end{aligned} \tag{23}$$

where $\mathcal{C}_\mu$ and $\mathcal{C}_\nu$ are s.c.c. of $\mathcal{C}(A)$, N_μ and N_ν are their node sets, and $\nu(t)$ in the second definition denotes the index of the non-critical component which consists of the node t. The sets $M_\nu^{(r)}(i)$ are non-empty for any ν, critical i and $r \geqslant T_c(A)$, due to Theorem 3.1. However, $K^{(r)}(i)$ may be empty.

The results of Subsection 3.3 lead to the following properties of $M_\nu^{(r)}(i)$ and $K^{(r)}(i)$.

Proposition 4.4. *Let $r \geqslant T_c(A)$ and $\mu, \nu \in \{1, \ldots, n_c\}$.*

1. *If $[i] \to_t [j]$ then $M_\nu^{(r+t)}(i) = M_\nu^{(r)}(j)$ and $K^{(r+t)}(i) = K^{(r)}(j)$.*
2. *Each $M_\nu^{(r)}(i)$ is the union of some cyclic classes of $\mathcal{C}_\nu$.*
3. *Let $i \in N_\mu$ and $d = g.c.d.(\gamma_\mu, \gamma_\nu)$. Then, if $[p] \subseteq M_\nu^{(r)}(i)$ and $[p] \to_d [s]$ then $[s] \subseteq M_\nu^{(r)}(i)$.*
4. *Let $i, j \in N_\mu$ and $p, s \in N_\nu$. Let $[i] \to_t [j]$ and $[p] \to_t [s]$. Then $[p] \subseteq M_\nu^{(r)}(i)$ if and only if $[s] \subseteq M_\nu^{(r)}(j)$.*

Next we establish the cancellation rules which will enable us to write out a concise system of equations for the attraction cone $\operatorname{Attr}(A, 1)$.

If $a < c$, then

$$\{x\colon ax \oplus b = cx \oplus d\} = \{x\colon b = cx \oplus d\}. \tag{24}$$

Now consider a system of equations over max algebra:

$$\bigoplus_{i=1}^{n} a_{1i}x_i \oplus c_1 = \bigoplus_{i=1}^{n} a_{2i}x_i \oplus c_2 = \cdots = \bigoplus_{i=1}^{n} a_{ni}x_i \oplus c_n. \tag{25}$$

Suppose that $\alpha_1, \ldots, \alpha_n \in \mathbb{R}_+$ are such that $a_{li} \leqslant \alpha_i$ for all l and i, and $S_l = \{i \mid a_{li} = \alpha_i\}$ for $l = 1, \ldots, n$. Let S_l be such that $\bigcup_{l=1}^{n} S_l = \{1, \ldots, n\}$. Repeatedly applying the elementary cancellation law described above, we obtain that (25) is equivalent to

$$\bigoplus_{i\in S_1} \alpha_i x_i \oplus c_1 = \bigoplus_{i\in S_2} \alpha_i x_i \oplus c_2 = \cdots = \bigoplus_{i\in S_n} \alpha_i x_i \oplus c_n. \tag{26}$$

We will refer to the equivalence between (25) and (26), which we acknowledge to [21], as to *chain cancellation*.

Using notation (23) and Proposition 4.4 we formulate the main result of the paper.

Theorem 4.5. *Let $A \in \mathbb{R}_+^{n\times n}$ be a visualized matrix and $r \geqslant T_c(A)$ be a multiple of γ. Then* $\operatorname{Attr}(A, 1)$ *is the solution set of the system*

$$\begin{aligned} &\bigoplus_{k\in[i]} x_k \oplus \bigoplus_{\mathcal{C}_\nu \neq \mathcal{C}_\mu} \alpha^*_{\mu\nu} \left(\bigoplus_{k\in M_\nu^{(r)}(i)} x_k \right) \oplus \bigoplus_{t\in K^{(r)}(i)} \alpha^*_{\mu\nu(t)} x_t \\ &\quad = \bigoplus_{k\in[j]} x_k \oplus \bigoplus_{\mathcal{C}_\nu \neq \mathcal{C}_\mu} \alpha^*_{\mu\nu} \left(\bigoplus_{k\in M_\nu^{(r)}(j)} x_k \right) \oplus \bigoplus_{t\in K^{(r)}(j)} \alpha^*_{\mu\nu(t)} x_t, \end{aligned} \tag{27}$$

where $[i]$ and $[j]$ range over all pairs of cyclic classes such that $[i] \to_1 [j]$ and $\mathcal{C}_\mu$ is the component of $\mathcal{C}(A)$ which contains both $[i]$ and $[j]$.

Proof. By Theorem 4.3 the attraction cone $\operatorname{Attr}(A, 1)$ is described by the system $S \otimes R \otimes x = R \otimes x$, which can be written as

$$\bigoplus_k a_{ik}^{(r)} x_k = \bigoplus_k a_{jk}^{(r)} x_k, \ \forall i, j\colon \ [i] \to_1 [j], \tag{28}$$

for any $r \geqslant T_c(A)$.

Proposition 3.4, part 2, implies that all principal submatrices of A^r extracted from critical components have a circulant block structure. In this structure, if r is a multiple of γ, all entries of the diagonal blocks are equal to 1, and the entries of all off-diagonal blocks are strictly less than 1. Hence we can apply the chain cancellation (equivalence between (25) and (26)) and obtain the first terms on both sides of (27). By Theorem 3.1 each block $A_{\mu\nu}$ contains an entry equal to $\alpha^*_{\mu\nu}$. For a non-critical $\nu(t)$, this readily implies that the corresponding "subcolumn" $A_{\mu\nu(t)}$ contains an entry $\alpha^*_{\mu\nu(t)}$. Applying the chain cancellation we obtain the last terms on both sides of (27). Due to the block circulant structure of $A_{\mu\nu}$ with both μ and ν critical, see Proposition 3.4 or Proposition 4.4, we see that each column of such block also contains an entry equal to $\alpha^*_{\mu\nu}$. Applying the chain cancellation we obtain the remaining terms in (27). $\square$

Remark 4.6. We can allow *any* $r \geqslant T_c(A)$ in (27), but then $k \in [i]$ and $k \in [j]$ in the first terms on both sides have to be replaced by $k\colon [i] \to_r [k]$ and $k\colon [j] \to_r [k]$ respectively.

Remark 4.7. As $\mathrm{Attr}(A, t) = \mathrm{Attr}(A^t, 1)$, system (27) also describes more general attraction cones, it only amounts to substituting $\mathcal{C}(A^t)$ for $\mathcal{C}(A)$ and the entries of $((A^t)^{Core})^*$ for $\alpha^*_{\mu\nu}$ (the dimension of this matrix will be different from that of A^{Core} in general, see Proposition 2.2 part 3).

We note that the system for $\mathrm{Attr}(A, 1)$ naturally breaks into several chains of equations corresponding to the s.c.c. of $\mathcal{C}(A)$. If we start with (28), it can be equivalently written as $R \otimes x = H \otimes y$, where $H \in \mathbb{R}_+^{c\times n_c}$ is a Boolean matrix with entries

$$h_{i\mu} = \begin{cases} 1, & \text{if } i \in N_\mu, \\ 0, & \text{otherwise.} \end{cases} \tag{29}$$

We can apply cancellation as described in the proof of Theorem 4.5, to get rid of redundant terms on the left-hand side of the two-sided system.

If $\mathcal{C}(A)$ is strongly connected then H is a vector of all ones, and the two-sided system $R \otimes x = H \otimes y$ becomes essentially one-sided. We treat this case in the next subsections.

4.2. Extremals of attraction cones

System (27) in general consists of several chains of equations corresponding to s.c.c. of $\mathcal{C}(A)$. Each chain is of the form

$$\bigoplus_{i\in T_1} a_i x_i = \bigoplus_{i\in T_2} a_i x_i = \cdots = \bigoplus_{i\in T_m} a_i x_i, \tag{30}$$

where $T_1 \cup \ldots \cup T_m = \{1, \ldots, n\}$ and a_i come from the entries of $(A^{Core})^*$.

Here, we consider only the case of strongly connected $\mathcal{C}(A)$, i.e., only one chain. By scaling $y_i = a_i x_i$ we obtain

$$\bigoplus_{i\in T_1} y_i = \bigoplus_{i\in T_2} y_i = \cdots = \bigoplus_{i\in T_m} y_i, \tag{31}$$

Note that when the critical graph is not strongly connected, we have several chains of equations and the coefficients of (30) in general cannot be scaled to get (31) for each chain at the same time.

By e^i we denote the vector which has the ith coordinate equal to 1 and all the rest equal to 0. Vector $y \in \mathbb{R}^n_+$ will be called *scaled* if $\bigoplus_{i=1}^n y_i = 1$, and set S is called scaled if it consists of scaled vectors. We say that $V \subseteq \mathbb{R}^n_+$ is *generated* by $S \subseteq \mathbb{R}^n_+$ (also, S is a generating set of V) if $S \subseteq V$ and for any $x \in V$ there exist $y^1, \ldots, y^l \in S$ and $\alpha_1, \ldots, \alpha_l \in \mathbb{R}_+$ such that $x = \bigoplus_{i=1}^l \alpha_i y^i$.

We investigate extremal solutions of (31): a solution x is called *extremal* if $x = y \oplus z$ for two other solutions y, z implies that $x = y$ or $x = z$ [11,27]. The following can be deduced from the results of [10,11,27].

Proposition 4.8. *The solution set of any finite system of max-linear equations has a finite generating set. In particular, it is generated by extremal solutions, and any set of scaled generators for the solution set contains all scaled extremal solutions.*

In the next proposition we show that extremal solutions of (31) can have only 0 and 1 components.

Proposition 4.9. *Let y be a scaled solution of (31) and let $0 < y_i < 1$ for some i. Then y is not an extremal.*

Proof. Let $K^< := \{i \mid 0 < y_i < 1\}$ and $K^1 := \{i \mid y_i = 1\}$, and define vectors v^0 and $v^1(k)$ for each $k \in K^<$ by

$$v_i^0 = \begin{cases} 1, & \text{if } i \in K^1 \\ 0, & \text{otherwise} \end{cases}, \quad v_i^1(k) = \begin{cases} 1, & \text{if } i \in K^1 \cup \{k\} \\ 0, & \text{otherwise} \end{cases}. \tag{32}$$

Observe that both v^0 and $v^1(k)$ for any $k \in K^<$, are solutions to (31), different from y. We have:

$$y = v^0 \oplus \bigoplus_{k \in K^<} y_k \cdot v^1(k), \tag{33}$$

hence y is not an extremal. $\square$

Let $T = (t_{ij})$ be the $m \times n$ Boolean matrix defined by

$$t_{ij} = \begin{cases} 1, & \text{if } j \in T_i, \\ 0, & \text{otherwise}, \end{cases} \tag{34}$$

where T_i are from (31).

A subset $K \subseteq \{1, \ldots, n\}$ is called a *covering* of T if each T_i contains an index from K. The following is immediate.

Proposition 4.10. *A scaled vector y is a solution of (31) if and only if $K^1 := \{i \mid y_i = 1\}$ is a covering of T.*

A covering K is called *minimal* if it does not contain any proper subset which is also a covering.

A covering K will be called *nearly minimal* if it contains no more than one proper subcovering K'. Observe that then the complement $K \backslash K'$ consists of just one index. Hence, a covering is nearly minimal if and only if there may exist no more than one $i \in K$ such that $K \backslash \{i\}$ is a covering.

Proposition 4.11. *Extremal solutions of (31) are precisely $v^K := \bigoplus_{i \in K} e_i$, where K is a nearly minimal covering of T.*

Proof. If a covering K is not nearly minimal, then there exist i and j such that $K[i] := K \backslash \{i\}$ and $K[j] := K \backslash \{j\}$ are both coverings of T. Then $v^{K[i]}$ and $v^{K[j]}$ are both solutions and $v = v^{K[i]} \oplus v^{K[j]}$ hence v^K is not extremal.

Conversely, if v^K is not extremal, then there exist $y \neq v^K$ and $z \neq v^K$ such that $v^K = y \oplus z$. Hence $y \leqslant v^K$ and $z \leqslant v^K$, and also y and z are not proportional with each other. By Proposition 4.9, see in particular (33), we can represent y and z as combinations of Boolean solutions of (31). These solutions correspond to coverings, which must be proper subsets of K. At least two of these coverings must be different from each other (since y and z are not proportional), hence K is not nearly minimal. $\square$

Thus, the problem of finding all nearly minimal coverings of a Boolean matrix is equivalent to the problem of finding all extremal solutions of (31).

The following case applies if the critical graph is strongly connected and occupies all nodes.

Corollary 4.12. *Let $T_1, \ldots, T_m$ be pairwise disjoint, then the scaled extremals are precisely all vectors $v^S = \bigoplus_{i \in S} e^i$, where S is an index set which contains exactly one index from each set T_i.*

Proof. Any such set S forms a minimal covering of T, and it can be shown that the solution set of (31) is generated by v^S, so there are no more scaled extremals (or nearly minimal coverings). $\square$

4.3. An algorithm for finding the coefficients of an attraction system

Coefficients of the system of equations which defines attraction cone are determined by the entries of $(A^{Core})^*$ which can be found in $O((n_c + \bar{c})^3)$ operations, where n_c is the number of s.c.c. of $\mathcal{C}(A)$ and $\bar{c}$ is the number of non-critical nodes. However, it remains to find the places where these coefficients appear, i.e., the sets $M_\nu^{(r)}(i)$ and $K^{(r)}(i)$ for $i = 1, \ldots, c$. Solving this problem, we get another polynomial method for computing the coefficients of (27).

Here we restrict our attention to the case when $\mathcal{C}(A)$ is strongly connected. In this case there are no second terms on both sides of (27) and we need only $K^{(r)}(i)$. The digraph $\mathcal{D}(A^{Core})$ associated with the matrix A^{Core} consists of one critical node which corresponds to the whole $\mathcal{C}(A)$ and will be denoted by μ, and $\bar{c}$ non-critical nodes $\nu(t)$, for $t > c$. The core matrix is of the form

$$A^{Core} = \begin{pmatrix} 1 & h \\ g & B \end{pmatrix}, \tag{35}$$

the entries $\alpha_{\mu\nu}$ and the entries of h, g and $B = (b_{\nu(s),\nu(t)})$ are given by

$$\begin{aligned} &\alpha_{\mu\mu} = 1, \\ &h_{\nu(t)} = \alpha_{\mu\nu(t)} = \max_{k=1}^{c} a_{kt}, \quad g_{\nu(t)} = \alpha_{\nu(t)\mu} = \max_{k=1}^{c} a_{tk}, \quad t > c, \\ &b_{\nu(s)\nu(t)} = \alpha_{\nu(s)\nu(t)} = a_{st}, \quad s > c, \quad t > c. \end{aligned} \tag{36}$$

Denote by $[\to_m i]$ the cyclic class $[j]$ such that $[j] \to_m [i]$. For each $t > c$, we initialize Boolean c-vectors P_t by

$$P_t(i) = \begin{cases} 1, & \text{if } [\to_1 i] \cap \arg\max_{k=1}^{c} a_{kt} \neq \emptyset \\ 0, & \text{otherwise.} \end{cases} \tag{37}$$

P_t encode the Boolean information associated with h (shifted cyclically by 1).

Further we compute the Kleene star of the non-critical submatrix $B := A_{MM}$, where M denotes the set of non-critical nodes, and store the information on the lengths of paths with maximal weight and length not exceeding $\bar{c}$ in sets U_{st} associated to each entry of B. To compute these sets we use the formula

$$B^* = I \oplus B \oplus \cdots \oplus B^{\bar{c}-1}, \tag{38}$$

where for each entry of this matrix series we find the arguments of maxima.

Note that $b_{ij}^{(t)} < b_{ij}^*$ for all $t \geqslant \bar{c}$ and all i, j, since every path of length exceeding $\bar{c} - 1$ contains a cycle which can be deleted from the path while *strictly increasing* the weight. Therefore, paths with lengths exceeding $\bar{c} - 1$ do not contribute to U_{st}.

To combine the information associated with h and B^*, we recall the max-algebraic version of bordering method due to Carré [14], which computes

$$(A^{Core})^* = \begin{pmatrix} 1 & h^T \\ g & B \end{pmatrix}^* = \begin{pmatrix} 1 & h^T \otimes B^* \\ B^* \otimes g & B^* \oplus B^* \otimes g \otimes h^T \otimes B^* \end{pmatrix}, \tag{39}$$

where $h, g \in \mathbb{R}_+^{\bar{c}}$. Note that all information that we need for system (27), is in the entries of $h^T \otimes B^*$ and in the indices of equations of the system where the entries of $h^T \otimes B^*$ appear. Computing $(h^T \otimes B^*)_i$ means in particular obtaining the "winning" indices

$$W_t = \arg\max_{s>c} h_{\nu(s)} b^*_{\nu(s)\nu(t)}. \tag{40}$$

After that, the idea is to combine P_s with U_{st} for all $s \in W_t$ and unite the obtained indices. More precisely, for each number m stored in U_{st} we define the shifted Boolean vector $P_s^{\to m}$ by

$$P_s^{\to m}(i) = \begin{cases} 1, & \text{if } [\to_{m+1} i] \cap \arg\max_{k=1}^{c} a_{ks} \neq \emptyset \\ 0, & \text{otherwise.} \end{cases} \tag{41}$$

Equivalently,

$$P_s^{\to m}(i) = 1 \Leftrightarrow P_s(j) = 1 \text{ and } [j] \to_m [i]. \tag{42}$$

After that, we define

$$G_t := \bigvee_{s \in W_t} \bigvee_{m \in U_{st}} P_s^{\to m}. \tag{43}$$

Proposition 4.13. *Let $t > c$, and let $r \geqslant T(A)$ be a multiple of γ. Then for all $i \leqslant c$, $t \in K^{(r)}(i)$ if and only if $G_t(i) = 1$.*

Proof. By (42) and (43), $G_t(i) = 1$ if and only if there exist $s \in W_t$ and $m \in U_{st}$ such that $P_s^{\to m}(i) = 1$. Then there exists a path $\pi_1 \circ \tau \circ \pi_2$ where π_1 starts in i, belongs to $\mathcal{C}(A)$ and has length $-m-1 \pmod{\gamma}$, τ is an edge which attains $h_{\nu(s)} = \max_{l=1}^{c} a_{ls}$, and π_2 is entirely non-critical, has length m, weight $b^*_{\nu(s)\nu(t)}$, and connects s to t. The weight of $\pi_1 \circ \tau \circ \pi_2$ is equal to $1 \cdot h_s \cdot b^*_{st} = \alpha^*_{\mu\nu(t)}$, and the length is a multiple of γ, meaning that $a_{it}^{(r)} = \alpha^*_{\mu\nu(t)}$ and $t \in K^{(r)}(i)$.

The other way around, let $t \in K^{(r)}(i)$. Then there exists a path π of length r that connects i to t and has weight $\alpha^*_{\mu\nu(t)} = \bigoplus_s h_{\nu(s)} b^*_{\nu(s)\nu(t)}$. We can decompose $\pi = \pi_1' \circ \tau' \circ \pi_2'$, where τ' is an edge connecting a critical node to a non-critical node $s' > c$, and π_2' has only non-critical nodes. Obviously $w(\pi) \leqslant w(\tau' \circ \pi_2') \leqslant \bigoplus_s h_{\nu(s)} b^*_{\nu(s)\nu(t)} = \alpha^*_{\mu\nu(t)}$. But $w(\pi) = \alpha^*_{\mu\nu(t)}$, hence $w(\tau' \circ \pi_2') = \alpha^*_{\mu\nu(t)}$ and $w(\pi_1') = 1$ so that π_1' entirely belongs to $\mathcal{C}(A)$. As $w(\tau') \leqslant h_{\nu(s')}$ and $w(\pi_2') \leqslant b^*_{\nu(s')\nu(t)}$ but $w(\tau' \circ \pi_2') \leqslant \bigoplus_s h_{\nu(s)} b^*_{\nu(s)\nu(t)}$, we deduce that $w(\tau') = h_{\nu(s')}$ and $w(\pi_2') = b^*_{\nu(s')\nu(t)}$. In particular, the length of π_2' does not exceed $\overline{c} - 1$. Using (42) and (43) we conclude that $G_t(i) = 1$. $\square$

Summarizing above said, we have the following algorithm for computing the coefficients of (27) in the case when $\mathcal{C}(A)$ is strongly connected. Recall that in this case there is no second term on both sides of (27). The computation of coefficients of the third term includes the computation of $h \otimes B^*$ and the sets $K^{(r)}(i)$ for $i \leqslant c$ (in fact we can improve the algorithm since only γ of them are different).

Algorithm 1. *Compute the coefficients of (27) if $\mathcal{C}(A)$ is strongly connected.*

Input. Visualized matrix A, critical graph $\mathcal{C}(A)$ which is strongly connected and the cyclic classes of $\mathcal{C}(A)$.

1. Compute h (takes $c\overline{c}$ operations) and initialize P_t for $t > c$ (takes $O(c^2\overline{c})$ operations).
2. Compute B^* and sets U_{st} for all $s, t > c$. It takes $O(\overline{c}^4)$ operations (see (38)).
3. Compute hB^* and G_t for $t > c$, by (40), (42) and (43). Computation of hB^* and W_t by (40) requires $O(\overline{c}^2)$ operations, computation of shifted Boolean vectors is $O(c\overline{c}^2)$, and the conjunction (43) takes $O(c\overline{c}^3)$ operations.
4. Compute $K^{(r)}(i)$ using Proposition 4.13. This requires $c\overline{c}$ operations.

As the overall complexity does not exceed $O(\overline{c}^4) + O(c^2\overline{c}) + O(c\overline{c}^3) = O(n\overline{c}^3) + O(c^2\overline{c})$ operations, we conclude the following.

Proposition 4.14. *Let $A \in \mathbb{R}_+^{n\times n}$ be visualized, $\mathcal{C}(A)$ be strongly connected, $\overline{c}$ be the number of non-critical nodes, and suppose we know $\mathcal{C}(A)$ and all cyclic classes. Then Algorithm 1 computes the coefficients of the attraction system in no more than $O(\overline{c}^3 n) + O(c^2\overline{c})$ operations.*

It is also important that the eigenvalue and an eigenvector of irreducible matrix can be computed by the policy iteration algorithm of [15], which is very fast in practice. After that, $\mathcal{C}(A)$ and the cyclic classes can be computed in $O(n^2)$ time. Thus we are led to an efficient method of computing the coefficients in the case when A is irreducible and $\mathcal{C}(A)$ is strongly connected, especially in the case when the number of non-critical nodes is small. Note that the case of irreducible A and strongly connected $\mathcal{C}(A)$ is generic when matrices A are real and generated at random. Also, in this generic case it almost never happens that maxima in blocks or among the weights of paths are achieved twice, which means that we do not need to assign Boolean vectors or sets to each entry. In this case the total number of operations in the algorithm is reduced to $O(\overline{c}^3) + O(\overline{c}c)$, which comes from computing h and B^*.

When $\mathcal{C}(A)$ is not strongly connected, the bordering method (39) can be used to obtain an algorithm which operates only with the entries of A^{Core}. However, the complexity of operations with indices and Boolean numbers significantly increases in that general case.

5. Examples

5.1. Circulants

Here we consider a 9×9 example of definite and visualized matrix in *max-plus algebra*

$$A = \begin{pmatrix} -8 & 0 & -1 & -8 & -8 & -9 & -4 & -5 & -1 \\ -4 & -5 & 0 & -2 & -6 & 0 & -7 & -3 & -9 \\ -7 & -9 & -8 & 0 & -8 & -4 & -6 & -9 & -10 \\ -8 & -8 & -10 & -7 & 0 & -4 & -6 & -10 & -1 \\ -2 & -8 & -7 & -4 & -8 & 0 & -3 & -1 & -10 \\ 0 & -1 & -2 & -7 & -10 & -6 & -3 & -6 & -1 \\ -10 & -7 & -7 & -7 & -6 & -1 & -5 & 0 & -9 \\ -8 & -3 & -6 & -8 & -6 & -8 & -5 & -10 & 0 \\ -4 & -3 & -5 & -6 & -6 & -10 & 0 & -6 & -9 \end{pmatrix}. \tag{44}$$

The critical graph of this matrix consists of two s.c.c. comprising 6 and 3 nodes respectively. They are shown in Figures 1 and 2, together with their cyclic classes.

The components of $C(A)$ induce block decomposition

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \tag{45}$$

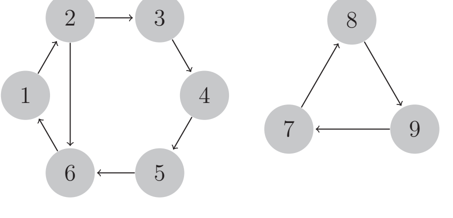

Fig. 1. Critical graph of (44).

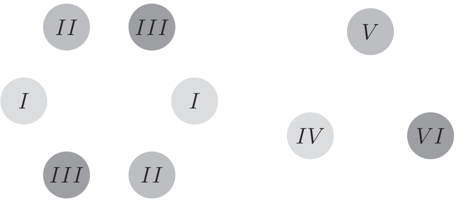

Fig. 2. Cyclic classes of the critical graph.

where

$$A_{11}=\begin{pmatrix}-8&0&-1&-8&-8&-9\\-4&-5&0&-2&-6&0\\-7&-9&-8&0&-8&-4\\-8&-8&-10&-7&0&-4\\-2&-8&-7&-4&-8&0\\0&-1&-2&-7&-10&-6\end{pmatrix},\quad A_{22}=\begin{pmatrix}-5&0&-9\\-5&-10&0\\0&-6&-9\end{pmatrix}\tag{46}$$

The core matrix and its Kleene star are equal to

$$A^{Core}=(A^{Core})^*=\begin{pmatrix}0&-1\\-1&0\end{pmatrix}.\tag{47}$$

By calculating $A, A^2, \ldots$ we obtain that the powers of A become periodic after $T(A)=6$. In the block decomposition of A^6 analogous to (45), we have the following circulants:

$$\begin{aligned}A_{11}^{(6)}&=\begin{pmatrix}0&-1&-2&0&-1&-2\\-2&0&-1&-2&0&-1\\-1&-2&0&-1&-2&0\\0&-1&-2&0&-1&-2\\-2&0&-1&-2&0&-1\\-1&-2&0&-1&-2&0\end{pmatrix},\quad A_{12}^{(6)}=\begin{pmatrix}-2&-1&-1\\-1&-2&-1\\-1&-1&-2\\-2&-1&-1\\-1&-2&-1\\-1&-1&-2\end{pmatrix},\\ A_{21}^{(6)}&=\begin{pmatrix}-3&-1&-2&-3&-1&-2\\-2&-3&-1&-2&-3&-1\\-1&-2&-3&-1&-2&-3\end{pmatrix},\quad A_{22}^{(6)}=\begin{pmatrix}0&-3&-2\\-2&0&-3\\-3&-2&0\end{pmatrix}.\end{aligned}\tag{48}$$

The corresponding blocks of "reduced" power $\tilde{A}^{(6)}$ are

$$\begin{aligned}\tilde{A}_{11}^{(6)}&=\begin{pmatrix}0&-1&-2\\-2&0&-1\\-1&-2&0\end{pmatrix},\quad \tilde{A}_{12}^{(6)}=\begin{pmatrix}-2&-1&-1\\-1&-2&-1\\-1&-1&-2\end{pmatrix},\\ \tilde{A}_{21}^{(6)}&=\begin{pmatrix}-3&-1&-2\\-2&-3&-1\\-1&-2&-3\end{pmatrix},\quad \tilde{A}_{22}^{(6)}=\begin{pmatrix}0&-3&-2\\-2&0&-3\\-3&-2&0\end{pmatrix}.\end{aligned}\tag{49}$$

Note that $\tilde{A}_{11}^{(6)}$ and $\tilde{A}_{22}^{(6)}$ are Kleene stars, with all off-diagonal entries negative.

Using (48), we specialize system (27) to our case, we see that this system of equations for the attraction cone $\mathrm{Attr}(A,1)$ consists of two chains of equations, namely

$$\begin{aligned}x_1\oplus x_4\oplus(x_8-1)\oplus(x_9-1)&=x_2\oplus x_5\oplus(x_7-1)\oplus(x_9-1)\\&=x_3\oplus x_6\oplus(x_7-1)\oplus(x_8-1),\\(x_2-1)\oplus(x_5-1)\oplus x_7&=(x_3-1)\oplus(x_6-1)\oplus x_8\\&=(x_1-1)\oplus(x_4-1)\oplus x_9.\end{aligned}\tag{50}$$

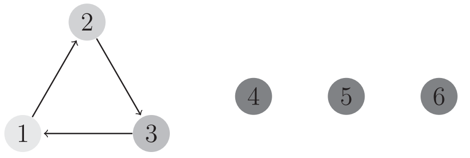

Fig. 3. Critical graph and non-critical nodes of (51).

Note that only 0 and -1, the coefficients of $(A^{Core})^*$ (which is equal to A^{Core} in our example), appear in this system.

5.2. *Algorithm for the strongly connected case*

Here, we consider a 6×6 max-plus example

$$A = \begin{pmatrix} -3 & 0 & -1 & -19 & -7 & -3 \\ -3 & -4 & 0 & -10 & -19 & -16 \\ 0 & -3 & -1 & -10 & -8 & -8 \\ -1 & -4 & -4 & -1 & -1 & -3 \\ -1 & -1 & -4 & -2 & -4 & -1 \\ -4 & -2 & -4 & -1 & -4 & -1 \end{pmatrix}, \tag{51}$$

and apply to it the algorithm described in Subsection 4.3. The critical graph of this matrix consists just of one cycle of length 3, and there are 3 non-critical nodes.

The core matrix in this case is equal to

$$A^{Core} = \begin{pmatrix} 0 & -10 & -7 & -3 \\ -1 & -1 & -1 & -3 \\ -1 & -2 & -4 & -1 \\ -2 & -1 & -4 & -1 \end{pmatrix}$$

Vector $h = (-10\ \ -7\ \ -3)^T$, whose components are computed by

$$h_i = \bigoplus_{k=1}^{3} a_{ki}, \ \text{for } i = 4, 5, 6, \tag{52}$$

comprises 2, 3, 4-components of the first row of A^{Core}. The arguments of maxima in (52) give, after the cyclic shift by one position, the Boolean vectors

$$P_4 = (1\ 0\ 1),\ P_5 = (0\ 1\ 0),\ P_6 = (0\ 1\ 0). \tag{53}$$

These vectors encode, for the corresponding non-critical nodes $t = 4, 5, 6$, the starting cyclic classes (here, just critical nodes!) of paths which go from $\mathcal{C}(A)$ directly to t and whose length is 3.

The non-critical principal submatrix of A and its Kleene star are equal to

$$B = \begin{pmatrix} -1 & -1 & -3 \\ -2 & -4 & -1 \\ -1 & -4 & -1 \end{pmatrix}, \quad B^* = \begin{pmatrix} 0 & -1 & -2 \\ -2 & 0 & -1 \\ -1 & -2 & 0 \end{pmatrix}$$

The lengths of optimal non-critical paths (whose weights are entries of B^*) can be written in the matrix of sets

$$U = \begin{pmatrix} \{0\} & \{1\} & \{2\} \\ \{1,2\} & \{0\} & \{1\} \\ \{1\} & \{2\} & \{0\} \end{pmatrix} \tag{54}$$

Further we compute

$$h^T \otimes B^* = (-10\ \ -7\ \ -3) \otimes \begin{pmatrix} 0 & -1 & -2 \\ -2 & 0 & -1 \\ -1 & -2 & 0 \end{pmatrix} = (-4\ \ -5\ \ -3)$$

The maxima in $\bigoplus_t h_t b^*_{ti}$ for all i are achieved only at $t = 6$, so $W_4 = W_5 = W_6 = \{6\}$. Hence G_4, G_5 and G_6 are shifted P_6 and the shift is determined by the components in the last row of U which is $(\{1\}\ \{2\}\ \{0\})$. From $P_6 = (0\ 1\ 0)$ we conclude that

$$G_4 = (0\ 0\ 1),\ G_5 = (1\ 0\ 0),\ G_6 = (0\ 1\ 0).$$

By Proposition 4.13 we have that $G_i(t) = 1$ if and only if $t \in K^{(r)}(i)$ (where $r \geqslant T(A)$ is a multiple of $\gamma = 3$). Using this rule we obtain that $K^{(r)}(1) = \{5\}$, $K^{(r)}(2) = \{6\}$, $K^{(r)}(3) = \{4\}$, and using the vector of coefficients $h^T \otimes B^* = (-4\ \ -5\ \ -3)$, we can write out the system for attraction cone

$$x_1 \oplus (x_5 - 5) = x_2 \oplus (x_6 - 3) = x_3 \oplus (x_4 - 4). \tag{55}$$

To verify this result, we observe that in our case $T(A) = 8$ and

$$A^8 = \begin{pmatrix} -1 & -1 & 0 & -4 & -6 & -4 \\ 0 & -1 & -1 & -5 & -5 & -4 \\ -1 & 0 & -1 & -5 & -6 & -3 \\ -2 & -1 & -2 & -6 & -1 & -4 \\ -2 & -1 & -1 & -5 & -7 & -4 \\ -2 & -3 & -2 & -6 & -7 & -6 \end{pmatrix}$$

$$A^9 = \begin{pmatrix} 0 & -1 & -1 & -5 & -5 & -4 \\ -1 & 0 & -1 & -5 & -6 & -3 \\ -1 & -1 & 0 & -4 & -6 & -4 \\ -2 & -2 & -1 & -5 & -7 & -5 \\ -1 & -2 & -1 & -5 & -6 & -5 \\ -2 & -2 & -3 & -7 & -7 & -5 \end{pmatrix}$$

Applying cancellation to the critical subsystem of $A^8 \otimes x = A^9 \otimes x$, we obtain (55).

Acknowledgement

The author is grateful to anonymous referees for their careful reading and numerous comments which improved the quality of the paper. He wishes to thank Peter Butkovič for valuable discussions and comments concerning this work. He wishes to thank Hans Schneider for sharing his original point of view on nonnegative matrix scaling and max algebra and suggesting the concept of CSR-representations of Theorem 3.2. The author also acknowledges the work of Trivikram Dokka [21], supervised by Peter Butkovič, which was at the origin of this research.

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