

Multiorder, Kleene Stars and Cyclic Projectors in the Geometry of Max Cones

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ABSTRACT. Max cones are the subsets of the nonnegative orthant \mathbb{R}_+^n of the n -dimensional real space \mathbb{R}^n closed under scalar multiplication and component-wise maximisation. Their study is motivated by some practical applications which arise in discrete event systems, optimal scheduling and modelling of synchronization problems in multiprocessor interactive systems. We investigate the geometry of max cones, concerning the role of the multiorder principle, the Kleene stars, and the cyclic projectors.

The multiorder principle is closely related to the set covering conditions in max algebra, and gives rise to important analogues of some theorems of convex geometry. We show that, in particular, this principle leads to a convenient representation of certain nonlinear projectors onto max cones.

The Kleene stars are fundamental in max algebra since they accumulate weights of optimal paths and yield generators for max-algebraic eigenspaces of matrices. We examine the role of their column spans called Kleene cones, as building blocks in the Develin-Sturmfels cellular decomposition. Further we show that the cellular decomposition gives rise to new max-algebraic objects which we call row and column Kleene stars. We relate these objects to the max-algebraic pseudoinverses of matrices and to tropical versions of the colourful Carathéodory theorem.

The cyclic projectors are specific nonlinear operators which lead to the so-called alternating method for finding a solution to homogeneous two-sided systems of max-linear equations. We generalize the alternating method to the case of homogeneous multi-sided systems, and we give a proof, which uses the cellular decomposition idea, that the alternating method converges in a finite number of iterations to a positive solution of a multi-sided system if a positive solution exists. We also present new bounds on the number of iterations of the alternating method, expressed in terms of the Hilbert projective distance between max cones.

2000 *Mathematics Subject Classification.* Primary: 15A48, 52B11; Secondary: 52A20.

Key words and phrases. Max-plus algebra, tropical algebra, tropical convexity, Kleene star, cyclic projection, projective distance.

This work is supported by the EPSRC grant RRAH12809, the RFBR grant 08-01-00601 and the joint RFBR/CNRS grant 05-01-02807.

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1. Introduction

The nonnegative orthant \mathbb{R}_+^n of the n -dimensional real space \mathbb{R}^n can be viewed as an n -dimensional free semimodule over the max-times semiring, which is the set of nonnegative numbers \mathbb{R}_+ equipped with the operations of 'addition' $a \oplus b := \max(a, b)$ and the ordinary multiplication $a \otimes b := a \times b$. The max-times semiring is denoted by $\mathbb{R}_{\max, \times} = (\mathbb{R}_+, \oplus = \max, \otimes = \times)$. Zero and unity of the semiring coincide with the usual 0 and 1. For instance, in this semiring $2 \otimes 3 = 6$ and $2 \oplus 3 = 3$. Subsemimodules of $\mathbb{R}_+^n = \mathbb{R}_{\max, \times}^n$ are the subsets of \mathbb{R}_+^n closed under the componentwise maximization \oplus , and the usual multiplication by nonnegative scalars. These subsemimodules will be called *max cones*, due to their obvious analogy with convex cones. In a very important special case, max cones can indeed be convex cones, but in general they are not convex, i.e., not stable under the usual componentwise addition.

By max algebra we understand linear algebra over the semiring $\mathbb{R}_{\max, \times}$, extending the \max, \times arithmetic to nonnegative matrices and vectors in the usual way. For instance, if $A = (a_{ij})$ and $B = (b_{ij})$ are two matrices of appropriate sizes, then $(A \oplus B)_{ij} = a_{ij} \oplus b_{ij}$, or $(A \otimes B)_{ij} = \bigoplus_k a_{ik} b_{kj}$. The iterated product $A \otimes A \otimes \dots \otimes A$ in which the symbol A appears k times will be denoted by A^k . We assume that $A^0 := I$, the unit matrix. The sets like $\{1, \dots, m\}$ or $\{1, \dots, n\}$ will be denoted by $[m]$ or $[n]$ respectively, and for a set of indices M , the number of elements in M will be denoted by $|M|$.

The idempotency of addition $a \oplus a = a$ and the lack of subtraction are important features of max algebra that make it different from the nonnegative linear algebra.

Max algebra has been known for some time, and we mention here the pioneering works of Cuninghame-Green [18, 19], Yoeli [47], Vorobyev [45], Carré [12], Gondran and Minoux [27], K. Zimmermann [48], and U. Zimmermann [50], among many others. Max algebra is often presented in the settings which seem to be different from $\mathbb{R}_{\max, \times}$, namely, over semirings $\mathbb{R}_{\max, +} = (\mathbb{R} \cup \{-\infty\}, \oplus = \max, \otimes = +)$ (max-plus semiring), $\mathbb{R}_{\min, +} = (\mathbb{R} \cup \{+\infty\}, \oplus = \min, \otimes = +)$ (tropical or min-plus semiring), or most exotically $\mathbb{R}_{\min, \times} = (\mathbb{R}_+ \cup \{+\infty\}, \oplus = \min, \otimes = \times)$ (min-times semiring). All these semirings are isomorphic to each other and to $\mathbb{R}_{\max, \times}$. Max algebra has important practical applications which arise in discrete event systems and scheduling problems [2, 19, 23], and in modelling of synchronization problems in multiprocessor interactive systems [10].

More generally, max algebra can be seen as a branch of tropical mathematics, which is a rapidly developing field with applications in mathematical physics, optimal control, algebraic geometry and other research areas. See [36] for a recent survey, and [34, 35] for recent collections of papers.

The similarity between max cones and convex cones was understood in the very beginning by Vorobyev [45], who used the name 'extremally convex cones' (instead of semimodules or spaces). K. Zimmermann [49] defined extremally convex sets, or tropically/max-plus convex sets as it would be called now, and proved a separation theorem of a point from a closed convex set. This theorem was generalized and more transparent proofs were given by Samborskiĭ and Shpiz [42], Litvinov et al. [37], Cohen et al. [15, 16], and also Develin and Sturmfels [22], Joswig [32]. We note that the separation theorem of a point from a closed max cone, given below as Theorem 2.6, is essentially the same result. In the ordinary convex geometry, separation of a point from a convex set easily leads to the separation of two convex

sets from each other. However, analogous statements for max cones arise differently and are related to the investigation of certain nonlinear projectors onto max cones, and their compositions called cyclic projectors, see Gaubert and Sergeev [26] and Theorems 4.2 and 4.3 below. Remarkably, these cyclic projectors also appear in the study of two-sided max-linear systems of equations, see Cuninghame-Green and Butkovič [20], and lead to a pseudopolynomial method for finding solutions to such systems. This will be discussed in the last section of the paper. We also note here that cyclic projectors are special case of the multiplicative version of the min-max functions studied in [13, 14, 41].

The geometry of max cones can be thought of as a special case of the multiorder convexity, a concept introduced by Martínez-Legaz and Singer [38]. Although this idea was made explicit only recently in a work by Nițică and Singer [39], it is closely related to the set-covering conditions for $A \otimes x = b$ systems in max algebra [2, 19, 45]. The multiorder principle, see Propositions 2.1 and 2.3 below, leads to easy proofs of many statements concerning generators, extremals and bases of max cones, see Butkovič et al. [11], including the tropical Carathéodory theorem, and Minkowski's theorem about extremals of closed cones (also Gaubert and Katz [24]). The multiorder principle is also important for the tropical convexity approach, meaning works of Develin, Sturmfels, Joswig, Yu et al. [6, 22, 32], since it describes max cones as intersections of staircases, and their extremals as elements of bases of monomial ideals.

Yet another approach to the geometry of max cones, though strongly related to the previous one, is to represent max cones as cellular complexes, or, roughly speaking, as unions of ordinary convex cones. This approach was put forward by Develin and Sturmfels [22], and called cellular decomposition. The atoms of this decomposition are well-known to specialists in convex geometry and combinatorics, see Joswig and Kulas [33] for more details. As it was noticed in [43], these atoms are column spans of uniquely defined Kleene stars, a fundamental concept in max algebra.

The aim of the present paper is to bring together some geometric and algebraic ideas discussed above. Section 2 discusses the multiorder principle and related results. In particular, we show that this principle leads to a convenient new representation of the nonlinear projectors mentioned above. In Section 3 we recall the concept of Kleene stars and examine the role of their column spans called Kleene cones as building blocks in the Develin-Sturmfels cellular decomposition. Further we show that, in turn, the cellular decomposition gives rise to new max-algebraic objects which we call row and column Kleene stars. We relate these new concepts to the max-algebraic pseudoinverses of matrices and to tropical versions of the colourful Carathéodory theorem. In Section 4 we generalize the alternating method of Cuninghame-Green and Butkovič [20] to the case of multisided systems $A^{(1)} \otimes x^1 = \dots = A^{(k)} \otimes x^k$. We give a proof, based on the cellular decomposition idea, that if the system has a positive solution, then the method converges to a positive solution in a finite number of steps. We also present new bounds for the number of iterations in the max-plus integer case, and in the general case when there are no solutions, in terms of the Hilbert projective distance between max cones.

2. The role of multiorder

2.1. Generators, bases and extremals of max cones. Let $S \subseteq \mathbb{R}_+^n$. A vector $u \in \mathbb{R}_+^n$ is called a *max combination* of S if

$$(2.1) \quad u = \bigoplus_{v \in S} \lambda_v v, \quad \lambda_v \in \mathbb{R}_+,$$

where only a finite number of λ_v are nonzero. The set of all max combinations (2.1) of S will be denoted by $\text{span}(S)$. Evidently, $\text{span}(S)$ is a max cone. If $\text{span}(S) = V$, then we call S a set of *generators* for V and say that V is *generated*, or *spanned*, by S . In particular, the set of all max combinations of columns of a matrix A will be denoted by $\text{span}(A)$ and called the *column span* of A . If none of the elements of a generating set S of a max cone V can be expressed as a max combination of other elements, then S is called a (*weak*) *basis* of V .

A vector $v \in V$ is called an *extremal* of V , if

$$v = u \oplus w, \quad u, w \in V \Rightarrow v = u \text{ or } v = w.$$

Extremals are analogous to extremal rays of convex cones. If v is an extremal of V and $\lambda > 0$, then λv is also an extremal.

For all $i = 1, \dots, n$ define the following preorder relation.

$$u \leq_j v \Leftrightarrow uu_j^{-1} \leq vv_j^{-1}, \quad u_j \neq 0, \quad v_j \neq 0.$$

The classes of proportional elements (i.e. rays) are the equivalence classes of these preorder relations. The importance of these relations for the geometry of max cones is expressed by the following principle. Denote $\text{supp}(y) := \{i \mid y_i \neq 0\}$.

PROPOSITION 2.1. *Let $V = \text{span}(S)$, $S \subseteq \mathbb{R}_+^n$. Then the following are equivalent.*

1. $y \in V$.
2. For all $j \in \text{supp}(y)$ there exists $v \in S$ such that $v \leq_j y$.

This principle appeared as a set covering condition, see Proposition 3.12 below, already in the works of Vorobyev [45] and Zimmermann [48], and in the above form (or with a subtle difference) it appeared quite recently in the works of Joswig [32], Nițică and Singer [39], and Butkovič et al. [11], see also [9] and [22].

As it was remarked by Nițică and Singer [39], the above proposition means that the geometry of max cones is a special case of the *multiorder convexity* [38]. In the multiorder convexity, one has a set of order relations, and a point y is said to belong to the convex hull of S , if for any order there is a point in S which precedes y with respect to that order.

The following proposition is the Tropical Carathéodory Theorem, see Helbig [28], Develin and Sturmfels [22], and also [11, 25]. Note that it follows from Proposition 2.1.

PROPOSITION 2.2. *Let $S \subseteq \mathbb{R}_+^n$. Then $y \in \text{span}(S)$ if and only if there exist k vectors $v^1, \dots, v^k \in S$, where $k = |\text{supp}(y)|$, such that $y \in \text{span}(v^1, \dots, v^k)$.*

The multiorder principle also means the following description of extremals [11].

PROPOSITION 2.3. *Let $V \subseteq \mathbb{R}_+^n$ be a max cone generated by S and let $v \in V$, $v \neq 0$. Then the following are equivalent.*

1. v is an extremal in V .

2. For some $j \in \text{supp}(v)$, v is minimal with respect to \leq_j in V .
3. For some $j \in \text{supp}(v)$, v is minimal with respect to \leq_j in S .

Propositions 2.1 and 2.3 lead to a number of statements about generators, extremals and bases of max cones [11], we mention only the following two of them. An element $u \in \mathbb{R}_+^n$ is called *scaled*, if $\|u\| = 1$, where $\|\cdot\|$ denotes some fixed norm (say, the ordinary norm or the max norm). For the following proposition see Butkovič et al. [11], and also [22, 46] for closely related statements.

PROPOSITION 2.4. *Let E be the set of scaled extremals in a max cone $V \subseteq \mathbb{R}_+^n$ and let $S \subseteq \mathbb{R}_+^n$ consist of scaled elements. Then the following are equivalent.*

1. The set S generates V and none of the elements in S are redundant.
2. $S = E$ and S generates V .
3. The set S is a basis for V .

Proposition 2.4 means that if a scaled basis of a max cone exists, then it is unique and consists of all scaled extremals, i.e., all the elements that are minimal with respect to some preorder relation \leq_i . In particular, a scaled basis of a finitely generated max cone V exists and is unique, and the cardinality of this basis will be called the *max-algebraic dimension* of V .

The following result is analogous to Minkowski's theorem about extremal points of convex sets, and was obtained independently by Gaubert and Katz [24] and Butkovič et al. [11].

PROPOSITION 2.5. *Let $V \subseteq \mathbb{R}_+^n$ be a closed max cone. Then V is generated by its set of extremals, and any vector in V is a max combination of no more than n extremals.*

Note that any finitely generated max cone is closed ([11, 32]). One may also think of colourful extensions of Propositions 2.2 and 2.5 in the sense of Bárány [4], and progress in this direction is due to Gaubert and Meunier [25], see also Theorem 3.22 below.

2.2. Projectors and separation. Given a closed max cone $V \subseteq \mathbb{R}_+^n$, we can define a *nonlinear projector* P_V by

$$(2.2) \quad P_V(y) = \max\{v \in V \mid v \leq y\}.$$

This operator is homogeneous: $P_V(\lambda y) = \lambda P_V(y)$, isotone: $y^1 \leq y^2 \Rightarrow P_V(y^1) \leq P_V(y^2)$, nonincreasing: $P_V(y) \leq y$, and continuous, see [16] for the proof. For any vector y there are coordinates which do not change under the action of the projector: $P_V(y)_i = y_i$. These coordinates will be called *sleepers*. Projectors lead to separation theorems of the following kind, see [16, 22, 26, 32] and introduction for some historical remarks.

THEOREM 2.6. *Let $V \subseteq \mathbb{R}_+^n$ be a closed max cone and let $y \in \mathbb{R}_+^n$ be not in V . Then there exist a positive vector \tilde{y} and a max cone $\tilde{V} \supseteq V$ containing positive vectors such that the set*

$$(2.3) \quad \mathcal{H} = \left\{v \mid \bigoplus_{i=1}^n \tilde{y}_i^{-1} v_i \geq \bigoplus_{i=1}^n (P_{\tilde{V}}(\tilde{y}))_i^{-1} v_i\right\}$$

contains V but not y . If y is positive and V contains positive vectors, then one can take $\tilde{y} = y$ and $\tilde{V} = V$.

The set \mathcal{H} defined in (2.3) is an instance of the max analogue of a halfspace, which is generally a set of the form $\{v \mid \bigoplus_{i=1}^n u_i^1 v_i \geq \bigoplus_{i=1}^n u_i^2 v_i\}$.

By comparing this to (2.3) we see that a separating halfspace has both u^1 and u^2 positive and $u^1 \leq u^2$, so that the inequality in (2.3) can be replaced by equality:

$$(2.4) \quad \mathcal{H} = \{v \mid \bigoplus_{i=1}^n \tilde{y}_i^{-1} v_i = \bigoplus_{i=1}^n (P_{\tilde{V}}(\tilde{y}))_i^{-1} v_i\}.$$

The relation of Theorem 2.6 to the multiorder principle was made explicit by Joswig [32]. Denote, for any positive y , $\Delta_i(y) = \{u \in \mathbb{R}_+^n \mid u \leq_i y\}$. Observe that $\bigcup_{i=1}^n \Delta_i(y) = \mathbb{R}_+^n$, and that the separating halfspace defined by (2.3) or equivalently (2.4) can also be written as

$$(2.5) \quad \mathcal{H} = \bigcup_{i \in \text{sl}(P_{\tilde{V}}, \tilde{y})} \Delta_i(P_{\tilde{V}}(\tilde{y})),$$

where $\text{sl}(P_{\tilde{V}}, \tilde{y})$ is the set of sleepers, i.e., the indices k such that $(P_{\tilde{V}}(\tilde{y}))_k = \tilde{y}_k$. Thus, in terms of the multiorder, the separation theorem says that, given a point y and a closed max cone V , there is a point $P_{\tilde{V}}(\tilde{y})$ such that the union of some sectors $\Delta_i(P_{\tilde{V}}(\tilde{y}))$ contains the whole V while the complement of this union contains y .

If a max cone is generated by the columns of a matrix $A \subseteq \mathbb{R}_+^{n \times m}$, then, denoting $P_A := P_{\text{span}(A)}$, we deduce from (2.2) that

$$(2.6) \quad P_A(y) = A \otimes (\bar{A} \otimes' y),$$

where \bar{A} is the *Cuninghame-Green inverse* of A defined by $\bar{a}_{ij} = a_{ji}^{-1}$, and \otimes' denotes the min-times matrix product. When calculating (2.6), we put by convention that $0^{-1} = \infty$ and $0 \otimes +\infty = 0$. In this form (2.6), the nonlinear projectors were studied by Cuninghame-Green [19]. We also note that formula (2.6) represents a projector as a min-max function in the sense of [13, 14, 41], with addition being replaced by multiplication.

When V is an arbitrary closed max cone, P_V can be expanded in infinite sum of 'elementary' projectors using the following 'scalar product', or an instance of residuation [15, 16]:

$$y/v := \min_{i \in \text{supp}(v)} y_i v_i^{-1} = \max\{\lambda \mid \lambda v \leq y\}.$$

Namely,

$$(2.7) \quad P_V(y) = \bigoplus_{v \in V} y/v \ v.$$

Formula (2.6) is a special case of (2.7), when V is finitely generated. Using the multiorder, we can obtain the following refinement of (2.7). Denote by \wedge the componentwise minimum of vectors in \mathbb{R}_+^n .

THEOREM 2.7. *Suppose that $V \subseteq \mathbb{R}_+^n$ is a closed max cone. Then for any $y \in \mathbb{R}_+^n$, the components $(P_V(y))_i$, for $i \in \text{supp}(y)$, are equal to*

$$(2.8) \quad (P_V(y))_i = \bigoplus_{v \in E_i} y/v \ v_i,$$

where E_i is the set of scaled points of V , minimal with respect to \leq_i . The projector P_V is linear with respect to the componentwise minimum \wedge if and only if every set E_i is a singleton.

PROOF. Writing (2.7) componentwise, we have that

$$(P_V(y))_i = \max_{v \in V: v_i \neq 0} (v_i \min_{k: v_k \neq 0} y_k v_k^{-1}) = \max_{v \in V: v_i \neq 0} \min_{k: v_k \neq 0} y_k (v_k v_i^{-1})^{-1}.$$

By Proposition 2.5, any closed max cone has a scaled basis E . Denote by E_i the set of scaled vectors minimal with respect to \leq_i , then for all $v \in V$ and any $i \in \text{supp}(v)$ there is $v^i \in E_i$ such that $v^i \leq_i v$ and hence $(v_k^i (v_i^i)^{-1})^{-1} \geq (v_k v_i^{-1})^{-1}$ for all k . This proves (2.8), and (2.8) implies that if all the sets E_i consist of one element, then the projector is expressed by a min-times matrix. Now suppose that there is an i such that E_i has at least two elements, say, u and v . Then $P_V(u) = u$ and $P_V(v) = v$. If the projector is linear with respect to the componentwise minimum \wedge , then $P_V(uu_i^{-1} \wedge vv_i^{-1}) = uu_i^{-1} \wedge vv_i^{-1}$, hence $w = uu_i^{-1} \wedge vv_i^{-1} \in V$. As $w_i = 1$, we have that $w \leq_i v$ and $w \leq_i u$. As u and v are both minimal with respect to \leq_i , w is not equal to either of them, which leads to a contradiction with the minimality of u and v . The proof is complete. \square

3. The role of Kleene stars

3.1. Kleene stars and Kleene cones. We start this section with some necessary definitions. Let $A = (a_{ij}) \in \mathbb{R}_+^{n \times n}$. The weighted digraph $D_A = (N(A), E(A))$, whose nodes are $N(A) = [n]$ and whose edges $E(A) = N(A) \times N(A)$ have weights $w(i, j) = a_{ij}$, is called the *digraph associated* with A . Suppose that $\pi = (i_1, \dots, i_p)$ is a path in D_A , then the *weight* of π is defined to be $w(\pi, A) = a_{i_1 i_2} a_{i_2 i_3} \dots a_{i_{p-1} i_p}$ if $p > 1$, and 0 if $p = 1$. A path which begins at i and ends at j will be called an $i \rightarrow j$ *path*. If the starting node of a path coincides with the end node then the path is called a *cycle*.

A path π is called *positive* if $w(\pi, A) > 0$. If for all $i, j \in [n]$ there exists a positive $i \rightarrow j$ path, then A is called *irreducible*.

The *maximum cycle geometric mean* of A , further denoted by $\lambda(A)$, is defined by the formula

$$\lambda(A) = \max_{\sigma} \mu(\sigma, A),$$

where the maximisation is taken over all cycles in the digraph and

$$\mu(\sigma, A) = w(\sigma, A)^{1/k}$$

denotes the *geometric mean* of the cycle $\sigma = (i_1, \dots, i_k, i_1)$.

The following fact was proved by Carré [12], see also [2, 19].

PROPOSITION 3.1. *Let $A \in \mathbb{R}_+^{n \times n}$. The series*

$$(3.1) \quad A^* = I \oplus A \oplus A^2 \oplus \dots$$

converges to a finite limit and is equal to $I \oplus A \oplus \dots \oplus A^{n-1}$ if and only if $\lambda(A) \leq 1$. In this case also $\lambda(A^) \leq 1$.*

The matrix series A^* defined by (3.1) is called the *Kleene star* of A , which comes from the theory of automata, see Conway [17]. Kleene stars enjoy the property $(A^*)^2 = A^*$, i.e., they are *multiplicatively idempotent*. Their diagonal entries are all equal to 1, i.e., the Kleene stars are *increasing*. Actually these two properties are also sufficient for a matrix to be a Kleene star, and further by a Kleene star we will also mean any matrix with these two properties. We also note that $(A^*)^2 = A^*$ implies that $(A^*)^* = A^*$.

A max cone will be called a *Kleene cone* if it can be represented as max-algebraic column span of a Kleene star.

In terms of the multiorder, we can say that a matrix A is a Kleene star if and only if $a_{ii} = 1$ for all $i \in [n]$ and $A_{\cdot i} \leq_i A_{\cdot k}$ for all i, k such that $a_{ik} \neq 0$. That is, A is a Kleene star if and only if $a_{ii} = 1$ and $A_{\cdot i}$ is the unique minimum of $\text{span}(A^*)$ with respect to \leq_i for all $i \in [n]$, so that all the sets E_i defined in Theorem 2.7 are singletons. The last sentence of Theorem 2.7 can be now formulated as follows.

PROPOSITION 3.2. *P_V is a min-times linear operator if and only if V is a Kleene cone. If $V = \text{span}(A)$, where A is a Kleene star, then $P_V(y) = \overline{A} \otimes' y$ for all y .*

Kleene stars play crucial role in the description of max-algebraic eigenvectors and subeigenvectors of nonnegative matrices. If for some x and λ we have that $A \otimes x = \lambda x$, then λ is a *max-algebraic eigenvalue* of A , and x is a *max-algebraic eigenvector* associated with this eigenvalue. Analogously, x is called a *max-algebraic subeigenvector* associated with λ , if $A \otimes x \leq \lambda x$.

The well-known Perron-Frobenius theorem has a max-algebraic analogue [2, 3, 19, 45].

THEOREM 3.3. *Let $A \in \mathbb{R}_+^{n \times n}$.*

1. *A has a max-algebraic eigenvalue, and the number of such eigenvalues is less than or equal to n .*
2. *$\lambda(A)$ is the largest eigenvalue of A .*
3. *If A is irreducible, then $\lambda(A)$ is the unique max-algebraic eigenvalue of A and all eigenvectors associated with $\lambda(A)$ are positive.*

The set of eigenvectors associated with a fixed eigenvalue λ is a max cone, and analogously the set of subeigenvectors associated with a fixed λ is a max cone, so they will be called the *eigencone* and the *subeigencone* associated with λ . For a nonnegative square matrix $A \in \mathbb{R}_+^{n \times n}$ the eigencone associated with 1 will be denoted by $V(A)$, and the subeigencone associated with 1 will be denoted by $V^*(A)$. A matrix $A \in \mathbb{R}_+^{n \times n}$ is called *definite*, if $\lambda(A) = 1$. We do not lose much generality when considering definite matrices, as for any matrix A with $\lambda(A) \neq 0$, the matrix $A/\lambda(A)$ is definite and has the same eigenvectors and subeigenvectors as A .

Any subeigencone is a Kleene cone, and the other way around.

PROPOSITION 3.4. *Let $A \in \mathbb{R}_+^{n \times n}$ be definite, then $V^*(A) = V(A^*) = \text{span}(A^*)$.*

PROOF. First note that by Proposition 3.1, if $\lambda(A) = 1$ then A^* exists and $\lambda(A^*) = 1$.

We show that $V^*(A) = V(A^*)$. Suppose that $A^* \otimes x = x$, then $A \otimes x \leq x$, because $A \leq A^*$. If $A \otimes x \leq x$, then $(I \oplus A) \otimes x = x$ and also $A^* \otimes x = x$, since $A^m \otimes x \leq x$ for any m (due to the isotonicity of matrix multiplication).

We show that $V(A^*) = \text{span}(A^*)$. It is immediate that $V(A^*) \subseteq \text{span}(A^*)$, as $V(A) \subseteq \text{span}(A)$ for any matrix A . If A^* converges, then $A \otimes A^* = A \oplus A^2 \oplus \dots$, so $A \otimes A^* \leq A^*$ meaning that each column of A^* is a subeigenvector of A . Hence $\text{span}(A^*) \subseteq V^*(A)$. \square

The positivity of subeigenvectors is addressed in the following observation.

PROPOSITION 3.5. *Let $A \in \mathbb{R}_+^{n \times n}$ be such that $a_{ii} = 1$ for all $i \in [n]$. Then $V^*(A)$ contains a positive vector if and only if A is definite.*

PROOF. The “if” part: If A is definite, then by Proposition 3.4 $V^*(A) = \text{span}(A^*)$ and we can take, for a positive subeigenvector of A , any max combination of all the columns of A^* with positive coefficients.

The “only if” part: Suppose that there exists a positive x such that $A \otimes x \leq x$, and take a cyclic permutation $\tau = (i_1, \dots, i_k)$ of a subset of $[n]$. Then we have that $a_{i_l i_{l+1}} x_{i_{l+1}} \leq x_{i_l}$ for $l \in [k]$, assuming $i_{k+1} := i_1$. Multiplying all these inequalities and cancelling the coordinates of x we have that $w(\tau, A) \leq 1$. Hence $\lambda(A) \leq 1$. As all diagonal entries are equal to 1, we have that $\lambda(A) = 1$. \square

Proposition 3.4 implies that if A is a Kleene star, then

$$\text{span}(A) = V(A) = V(A^*) = V^*(A) = \{x \mid a_{ij}x_j \leq x_i, i, j \in [n]\},$$

and it is not hard to see the following.

PROPOSITION 3.6. *Let K be a max cone in \mathbb{R}_+^n . Then it is a Kleene cone if and only if for some matrix B it is the solution set of the system of inequalities $b_{ij}x_j \leq x_i$, $i, j \in [n]$, satisfied by at least one positive x .*

PROOF. The “if” part: If the system is satisfied by a positive x , then $b_{ii} \leq 1$ for all $i \in [n]$. Take $\tilde{B} := I \oplus B$, then \tilde{B} has all diagonal entries equal to 1, $K = V^*(\tilde{B})$ and there is a positive $x \in V^*(\tilde{B})$. By Proposition 3.5, \tilde{B} is definite, and by Proposition 3.4, $K = \text{span}((\tilde{B})^*)$.

The “only if” part: If K is a Kleene cone $\text{span}(A^*)$, then by Proposition 3.4 and Proposition 3.5 we can take $B := A^*$. \square

The above observations imply that Kleene cones are convex cones, and that they have many close relatives in the realm of combinatorial geometry, see Joswig and Kulas [33].

One may think of various systems of inequalities describing the same Kleene cone. However, the Kleene star which defines this cone is unique [43].

PROPOSITION 3.7. *Suppose that A and B are two Kleene stars. Then $A = B$ if and only if $\text{span}(A) = \text{span}(B)$.*

We now describe the bases of $V(A)$ and $V^*(A)$, for a definite matrix $A \in \mathbb{R}_+^{n \times n}$. The cycles with the cycle geometric mean equal to 1 are called *critical*, and the nodes and the edges of D_A that belong to critical cycles are called *critical*. The set of critical nodes is denoted by $N_c(A)$, the set of critical edges is denoted by $E_c(A)$, and the *critical digraph* of A , further denoted by $C(A) = (N_c(A), E_c(A))$, is the digraph that consists of all critical nodes and critical edges of D_A . All cycles of $C(A)$ are critical [2]. For two vectors x and y , we write $x \sim y$ if $x = \lambda y$ for $\lambda > 0$. The following theorem follows from well-known results on the max-algebraic spectral theory [2, 19, 23].

THEOREM 3.8. *Let $A \in \mathbb{R}_+^{n \times n}$ be definite, and let $M(A)$ denote a set of indices such that for each strongly connected component of $C(A)$ there is a unique index in $M(A)$ which belongs to that component.*

1. *The following statements are equivalent: $A_{\cdot i}^* \sim A_{\cdot j}^*$, $A_i^* \sim A_j^*$, i and j belong to the same strongly connected component of $C(A)$.*
2. *Any column of A^* is a max extremal of $\text{span}(A^*)$.*

3. The subeigencone of A , which is the eigencone of A^* , is

$$V^*(A) = V(A^*) = \left\{ \bigoplus_{i \in M(A)} \alpha_i A_{.i}^* \oplus \bigoplus_{j \notin C(A)} \alpha_j A_{.j}^*, \alpha_i, \alpha_j \in \mathbb{R}_+ \right\},$$

and none of the columns of A^* in this description are redundant.

4. The eigencone of A is

$$V(A) = \left\{ \bigoplus_{i \in M(A)} \alpha_i A_{.i}, \alpha_i \in \mathbb{R}_+ \right\},$$

and none of the columns of A^* in this description are redundant.

Proposition 2.4 and Theorem 3.8 imply that extremals of $V^*(A)$ are precisely the columns of A^* , so the columns of A^* , after eliminating the proportional ones, constitute the basis of $V^*(A) = \text{span}(A^*)$, and the columns whose indices belong to $C(A)$ constitute the basis of $V(A)$. Denote by $n_c(A)$ the number of strongly connected components in $C(A)$, and denote by $\overline{N}_c(A)$ the set of nodes that are not critical. Theorem 3.8 yields the following corollary.

PROPOSITION 3.9. *For any definite matrix $A \in \mathbb{R}_+^{n \times n}$, the max-algebraic dimension of the subeigencone of A is equal to $n_c(A) + |\overline{N}_c(A)|$. The max-algebraic dimension of the eigencone is equal to $n_c(A)$.*

Kleene cones are both convex cones and max cones. They are inhabitants of two worlds, that of max algebra and tropical convexity, and that of nonnegative linear algebra and ordinary convexity. One might think of an interplay between these worlds. For a definite matrix A , define the linear space

$$(3.2) \quad L(C(A)) = \{x \in \mathbb{R}^n \mid a_{ij}x_j = x_i, (i, j) \in E_c(A)\}.$$

A proof of the following theorem can be found in [44].

THEOREM 3.10. *Let $A \in \mathbb{R}_+^{n \times n}$ be a definite matrix. Then $L(C(A))$ is the linear hull of the convex cone $V^*(A)$. The linear dimension of $V^*(A)$, i.e., the dimension of $L(C(A))$, is equal to the max-algebraic dimension of $V^*(A)$, i.e., to $n_c(A) + |\overline{N}_c(A)|$.*

The intersection of Kleene cones is again a Kleene cone. More precisely, we have the following proposition, see Butkovič [7] for the case $k = 2$. The proof is based on the formula $(A^* \oplus B^*)^* = (A^* \otimes B^*)^*$, which follows from $(A \oplus B)^* = A^* \otimes (B \otimes A^*)^*$ [17], and on the observations above.

PROPOSITION 3.11. *Let $A^{(1)}, \dots, A^{(k)} \in \mathbb{R}_+^{n \times n}$ be Kleene stars. The following are equivalent.*

1. $\bigcap_{i=1}^k \text{span}(A^{(i)})$ contains a positive vector.
2. $\lambda(\bigoplus_{i=1}^k A^{(i)}) = 1$.
3. $\lambda(\bigotimes_{i=1}^k A^{(\pi(i))}) = 1$ for some permutation π of $\{1, \dots, k\}$.
4. $\lambda(\bigotimes_{i=1}^k A^{(\pi(i))}) = 1$ for all permutations π of $\{1, \dots, k\}$.

If any of these equivalent conditions are true, then

$$(3.3) \quad \bigcap_{i=1}^k \text{span}(A^{(i)}) = \text{span}\left(\left(\bigoplus_{i=1}^k A^{(i)}\right)^*\right) = \text{span}\left(\left(\bigotimes_{i=1}^k A^{(\pi(i))}\right)^*\right)$$

for all permutations π .

PROOF. Complete $\mathbb{R}_{\max, \times}$ with $+\infty$ and assume $a \times +\infty = +\infty$ for any positive a and $0 \times +\infty = 0$. Matrix algebra over this completed semiring is a regular algebra in the sense of [17]. This means in particular that A^* is always defined, $(A^*)^* = A^*$, $(A \oplus B)^* = A^* \otimes (B \otimes A^*)^*$ and $(A \otimes B)^* = I \oplus (A \otimes (B \otimes A)^*)$. If A and B are two Kleene stars, then

$$(3.4) \quad \begin{aligned} (A \otimes B)^* &= I \oplus (A \otimes (B \otimes A)^*) = A \otimes (B \otimes A)^* = \\ &= (A \oplus B)^* = (B \oplus A)^* = (B \otimes A)^*. \end{aligned}$$

It can be shown by induction that $(A^{(1)} \oplus \dots \oplus A^{(k)})^* = (A^{(\pi(1))} \otimes \dots \otimes A^{(\pi(k))})^*$ for any permutation π of $\{1, \dots, k\}$. Using Proposition 3.1 we obtain that $\lambda(\bigoplus_{i=1}^k A^{(i)}) \leq 1$ is true if and only if $\lambda(\bigotimes_{i=1}^k A^{\pi(i)}) \leq 1$ is true for some π , and hence if and only if the same is true for all π . The inequalities here can be replaced by equalities, since all diagonal entries, and hence all eigenvalues, of any product or entrywise maximum of Kleene stars, are greater than or equal to 1. This yields equivalence of 2., 3., and 4.

We now prove the equivalence between 1. and 2., and (3.3). We have that

$$(3.5) \quad V^*\left(\bigoplus_{i=1}^k A^{(i)}\right) = \bigcap_{i=1}^k V^*(A^{(i)}) = \bigcap_{i=1}^k \text{span}(A^{(i)}),$$

where the first equality is immediate, and the second equality follows from Proposition 3.4. Note that all diagonal entries of $\bigoplus_{i=1}^k A^{(i)}$ are 1, and by Proposition 3.5, $V^*(\bigoplus_{i=1}^k A^{(i)})$ contains a positive vector if and only if $\lambda(\bigoplus_{i=1}^k A^{(i)}) = 1$. This, together with (3.5), implies the equivalence between assertions 1. and 2. By Proposition 3.4, $V^*(\bigoplus_{i=1}^k A^{(i)}) = \text{span}((\bigoplus_{i=1}^k A^{(i)})^*)$ since $\lambda(\bigoplus_{i=1}^k A^{(i)}) = 1$, which yields (3.3). \square

3.2. Cellular decomposition. We have described some properties of Kleene cones. Though such cones are very special, they can be viewed as building blocks, or atoms, of any finitely generated max cone. This can be seen as the main idea of the cellular decomposition, an ingenious concept of Develin and Sturmfels [22], which we adjust below to the setting of max cones.

Let $A \subseteq \mathbb{R}_+^{n \times m}$ be a nonnegative matrix with m nonzero columns and n nonzero rows. The *column type* of y with respect to A is defined to be the m -tuple of subsets T_1, \dots, T_m of $[n]$, where every T_j , for $j \in [m]$ is defined by

$$T_j = \{i \in [n] \mid a_{ij}y_i^{-1} \geq a_{kj}y_k^{-1}, k \in [n]\} = \{i \in [n] \mid y \geq_i A_{\cdot j}\}.$$

The *row type* of y with respect to A is an n -tuple of subsets S_1, \dots, S_n of $[m]$, where every S_i , for $i \in [n]$, is defined by

$$\begin{aligned} S_i &= \{j \in [m] \mid a_{ij}y_i^{-1} \geq a_{kj}y_k^{-1}, k \in [n]\} = \{j \in [m] \mid y \geq_i A_{\cdot j}\} = \\ &= \{j \in [m] \mid i \in T_j\}. \end{aligned}$$

The theory of $A \otimes x = y$ systems [2, 9, 19, 22, 45, 48] is based on the following *set covering conditions* for y to be in $\text{span}(A)$, see the proposition below. The multi-order principle (Proposition 2.1) can be seen as a reformulation of these conditions, therefore we leave the proposition below without proof.

PROPOSITION 3.12. *Let $A \in \mathbb{R}_+^{n \times m}$ have all rows and columns nonzero and let $y \in \mathbb{R}_+^n$ be a positive vector with the column type $T = (T_1, \dots, T_m)$ and the row type $S = (S_1, \dots, S_n)$. The following are equivalent.*

1. $y \in \text{span}(A)$;
2. $\bigcup_{i=1}^m T_i = [n]$;
3. none of $S_i, i \in [n]$ are empty.

See also Akian et al. [1] for an infinite-dimensional generalisation in the context of Galois connections.

Following Develin and Sturmfels [22], we can see this from a geometric viewpoint. For any row type S , we define its *region* with respect to A by

$$X_S = \{y \text{ positive} \mid y_k y_i^{-1} \geq a_{kj} a_{ij}^{-1}, \forall k, i, \forall j \in S_i\}.$$

Proposition 3.12 means that the part of $\text{span}(A)$ consisting of all positive vectors is the union of the regions X_S such that S do not contain empty sets ([22], Theorem 15). If X_S is not empty, then the closure of X_S is

$$(3.6) \quad \text{cl}(X_S) = \{y \in \mathbb{R}_+^n \mid a_{kj} a_{ij}^{-1} y_i \leq y_k, \forall k, i, \forall j \in S_i\}.$$

It follows from the results of [22] that the relative interiors of regions build up a cellular decomposition of the positive part of \mathbb{R}_+^n . We will need a weaker statement, but without positivity.

PROPOSITION 3.13. *Suppose that $A \in \mathbb{R}_+^{n \times m}$ has all rows and columns nonzero. Then the max cone $\text{span}(A)$ is the union of $\text{cl}(X_S)$ such that X_S are not empty and S do not contain empty sets.*

PROOF. As A has all rows nonzero, the max cone $\text{span}(A)$ contains positive vectors. By Proposition 3.12 if y is positive, then $y \in \text{span}(A)$ if and only if the row type of y does not contain empty sets. Hence the positive part of $\text{span}(A)$ is the union of nonempty X_S such that S do not contain empty sets. Further, $\text{span}(A)$ is the closure of its positive part. Indeed, $\text{span}(A)$ contains positive vectors and for any $u \in \text{span}(A)$ and a positive $v \in \text{span}(A)$ we can take $w = u \oplus \varepsilon v \in \text{span}(A)$, so that $\|w - u\| \leq \varepsilon \|v\|$ (the max norm) and w is positive. Hence $\text{span}(A)$ is the union of closed regions $\text{cl}(X_S)$ such that X_S are not empty and S do not contain empty sets. \square

From the max-algebraic point of view, an important role in the cellular decomposition is played by *strongly definite matrices*, which are definite matrices with all diagonal entries equal to 1. Note that any Kleene star is a strongly definite matrix.

Observe that $\text{cl}(X_S)$ is the subeigencone of the $n \times n$ matrix $A^S = (a_{ij}^S)$ defined by

$$(3.7) \quad a_{ij}^S = \begin{cases} \bigoplus_{k \in S_j} a_{ik} a_{jk}^{-1}, & \text{if } S_j \neq \emptyset, \\ \delta_{ij}, & \text{if } S_j = \emptyset, \end{cases}$$

where δ_{ij} are Kronecker symbols ($\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ij} = 1$ if $i = j$). It is immediate that all diagonal entries of A^S are equal to 1. We have the following proposition which can be used to compute the generators of any closed region, a preliminary version of this proposition appeared in [43].

PROPOSITION 3.14. *The closed region $\text{cl}(X_S)$ contains positive vectors if and only if A^S is a strongly definite matrix, and in this case $\text{cl}(X_S) = V^*(A^S) = \text{span}((A^S)^*)$.*

PROOF. From (3.6) and (3.7) one infers that $\text{cl}(X_S) = V^*(A^S)$. After that, the claim follows from Proposition 3.4 and Proposition 3.5. \square

Propositions 3.13 and 3.14 have the following consequences.

PROPOSITION 3.15. *For any matrix $A \in \mathbb{R}_+^{n \times m}$ with no zero rows there exist Kleene stars $A^{(1)}, \dots, A^{(l)} \in \mathbb{R}_+^{n \times n}$ such that $\text{span}(A) = \bigcup_{i=1}^l \text{span}(A^{(i)})$.*

PROPOSITION 3.16. *For any matrix $A \in \mathbb{R}_+^{n \times m}$ with no zero rows there exist Kleene stars $A^{(1)}, \dots, A^{(l)} \in \mathbb{R}_+^{n \times n}$ such that for any $y \in \mathbb{R}_+^n$ we have that $P_{Ay} = \overline{A^{(k)}} \otimes' y$ for some k .*

To express the dimension of a region, Develin and Sturmfels [22] introduce the undirected graph G_S : The set of nodes of this graph is $[n]$, it contains all loops (i, i) , and for $i \neq j$ an edge (i, j) belongs to G_S if and only if there exists $k \in S_i \cap S_j$. The following observation relates this notion to max algebra.

PROPOSITION 3.17. *Let $A \in \mathbb{R}_+^{n \times m}$ be a matrix with no zero rows and columns, let $y \in \mathbb{R}_+^n$ be a positive vector and S be the row type of y with respect to A . Then $G_S = C(A^S)$.*

PROOF. Note that as all entries of A^S are equal to 1, the graph $C(A^S)$ contains all loops.

Let $i \neq j$ and $(i, j) \in G_S$, then there exists $k \in S_i \cap S_j$. It follows that $a_{ik}a_{jk}^{-1} = y_i y_j^{-1} \geq a_{il}a_{jl}^{-1}$ for all $l \in S_j$, and therefore $a_{ij}^S = a_{ik}a_{jk}^{-1}$. Analogously, $a_{ji}^S = a_{jk}a_{ik}^{-1}$, and therefore $a_{ij}^S a_{ji}^S = 1$ so that $(i, j) \in C(A^S)$.

Let $(i, j) \in C(A^S)$, then observe that $a_{ij}^S y_j < y_i$ is impossible, because the multiplication with other inequalities over the critical cycle would lead to $1 < 1$. So $a_{ij}^S y_j = y_i$, and hence there exists $k \in S_j$ such that $a_{ik}a_{jk}^{-1} y_j = y_i$. But then also $k \in S_i$ and $(i, j) \in G_S$. \square

The equality $G_S = C(A^S)$ means that $C(A^S)$ is symmetrical and $(i, j) \in G_S$ if and only if (i, j) or equivalently (j, i) belong to $C(A^S)$. Theorem 3.10 and Proposition 3.17 yield the following result, see also Develin and Sturmfels [22], Proposition 17.

THEOREM 3.18. *Let $A \in \mathbb{R}_+^{n \times m}$ be a matrix with no zero rows and columns, let y be a positive vector and S be the row type of y with respect to A , then both max-algebraic and linear dimensions of $\text{cl}(X_S)$ are equal to the number of connected components in G_S .*

3.3. Row and column Kleene stars. For a matrix $A = (a_{ij}) \in \mathbb{R}_+^{n \times n}$ and any permutation $\sigma \in S_n$ (where S_n denotes the group of all permutations of $[n]$) define the *weight* of σ to be $w(\sigma) := \prod_{i=1}^n a_{i\sigma(i)}$. The *max-algebraic permanent* of A is defined as

$$(3.8) \quad \text{per}(A) = \bigoplus_{\sigma \in S_n} w(\sigma),$$

and a permutation, at which the maximum in (3.8) is attained, is called a *maximal permutation*. For any permutation σ , define the diagonal matrix $D^\sigma = (d_{ij}^\sigma)$ by

$$d_{ij}^\sigma = \begin{cases} a_{ij}, & \text{if } j = \sigma(i); \\ 0, & \text{otherwise.} \end{cases}$$

Observe that $A(D^\sigma)^{-1}$ is an instance of A^S , for the type $S = \{\{\sigma(1)\}, \dots, \{\sigma(n)\}\}$. The subeigencone $V^*(A(D^\sigma)^{-1})$ is precisely the closed region $\text{cl}(X^S)$. It contains positive vectors if and only if $A(D^\sigma)^{-1}$ is strongly definite, and this is true if and only if the permutation σ is maximal [9]. This is also equivalent to $(D^\sigma)^{-1}A$ being strongly definite. Further $A(D^\sigma)^{-1}$ will be denoted by $A^{c\sigma}$ and $(D^\sigma)^{-1}A$ will be denoted by $A^{r\sigma}$. The entries of $A^{c\sigma}$ and $A^{r\sigma}$ are

$$(3.9) \quad a_{ij}^{c\sigma} = a_{i\sigma(j)}a_{j\sigma(j)}^{-1}, \quad a_{ij}^{r\sigma} = a_{\sigma^{-1}(i)i}^{-1}a_{\sigma^{-1}(i)j}.$$

The Kleene stars of $A^{c\sigma}$ and $A^{r\sigma}$ will be denoted by $A^{c\sigma*}$ and $A^{r\sigma*}$ and called *column Kleene stars* and *row Kleene stars*, respectively.

The results of Yoeli [47], see also Cuninghame-Green [19], Theorem 27-11, and Izhakian [29, 31] suggest that row and column Kleene stars are related to the max-algebraic pseudoinverses of matrices. The *pseudoinverse* of A is defined, see [19] and [47], as $A^\nabla = (\text{per}(A))^{-1}A^{\text{adj}}$. Here A^{adj} is the *pseudoadjugate* of A defined by $a_{ij}^{\text{adj}} = \text{per}(A_{ji})$, where A_{ji} is the complementary minor to a_{ij} . The following proposition collects some facts about strongly definite matrices, which are due to Yoeli and Cuninghame-Green.

PROPOSITION 3.19. *Let $A \in \mathbb{R}_+^{n \times n}$ be strongly definite.*

1. $I \leq A \leq A^2 \leq \dots \leq A^{n-1} = A^n = \dots$
2. $A^* = A^{n-1}$.
3. $A^* = A^{\text{adj}} = A^\nabla$.

Izhakian [29, 31] studies the products $A \otimes A^\nabla$ and $A^\nabla \otimes A$ over extended tropical semiring, with the main emphasis on the questions of regularity and rank. In this context, he proves [31] that the products $A \otimes A^\nabla$ and $A^\nabla \otimes A$ are Kleene stars. Below we give an elementary proof that over max algebra, these products are equal to column and row Kleene stars, respectively.

THEOREM 3.20. *Let $A \in \mathbb{R}_+^{n \times n}$ have nonzero permanent. For any permutation σ with maximal weight we have that $A^{c\sigma*} = D^\sigma A^\nabla = A \otimes A^\nabla$ and $A^{r\sigma*} = A^\nabla D^\sigma = A^\nabla \otimes A$.*

PROOF. Using (3.9) and the definition of A^{adj} , we write:

$$\begin{aligned} a_{ij}^{\text{adj}} &= \bigoplus_{\pi: \pi(j)=i} \prod_{k \neq j} a_{k\pi(k)} = \bigoplus_{\pi: \pi(j)=i} \prod_{k \neq j} a_{\sigma^{-1}\pi(k), \pi(k)} a_{k, \sigma^{-1}\pi(k)}^{c\sigma} = \\ &= \prod_{k \neq i} a_{\sigma^{-1}(k)k} \cdot \bigoplus_{\pi: \pi(j)=i} \prod_{k \neq j} a_{k, \sigma^{-1}\pi(k)}^{c\sigma} = \\ &= \text{per}(A) \cdot a_{\sigma^{-1}(i)i}^{-1} \cdot \bigoplus_{\pi: \pi(j)=\sigma^{-1}(i)} \prod_{k \neq j} a_{k\pi(k)}^{c\sigma} = \text{per}(A) \cdot a_{\sigma^{-1}(i)i}^{-1} (a^{c\sigma})_{\sigma^{-1}(i)j}^{\text{adj}}. \end{aligned}$$

By Proposition 3.19, $(A^{c\sigma})^{\text{adj}} = A^{c\sigma*}$, so we have obtained that $A^{\text{adj}} = \text{per}(A)(D^\sigma)^{-1}A^{c\sigma*}$, and hence $A^\nabla = (D^\sigma)^{-1}(A^{c\sigma})^*$ and $D^\sigma A^\nabla = (A^{c\sigma})^*$. We now infer that

$$(A \otimes A^\nabla)_{ij} = \bigoplus_k a_{ik} a_{kj}^\nabla = \bigoplus_k a_{ik} a_{\sigma^{-1}(k)k}^{-1} a_{\sigma^{-1}(k)j}^{c\sigma*} = \bigoplus_k a_{i\sigma^{-1}(k)}^{c\sigma} a_{\sigma^{-1}(k)j}^{c\sigma*} = a_{ij}^{c\sigma*}.$$

Thus $A \otimes A^\nabla = A^{c\sigma^*}$. On the other hand, one can similarly obtain that $a_{ij}^{\text{adj}} = \text{per}(A)a_{i\sigma(j)}^{r\sigma^*}a_{j\sigma(j)}^{-1}$ and that $A^\nabla \otimes A = A^\nabla D^\sigma = A^{r\sigma^*}$. \square

Clearly this theorem yields the following corollary the first part of which was obtained in [43]. This corollary means that for any matrix with nonzero permanent, both row Kleene star and column Kleene star are uniquely defined.

PROPOSITION 3.21. *Let $A \in \mathbb{R}_+^{n \times n}$ have nonzero permanent. Then for all permutations σ with maximal weight, the corresponding column Kleene stars $A^{c\sigma^*}$ are equal to each other, and the row Kleene stars $A^{r\sigma^*}$ are also equal to each other.*

The idea of the proof in [43] was to notice that the (sub)eigencones of $A^{c\sigma}$ are the same for all maximal permutations σ , and to use Proposition 3.7 that any Kleene star is uniquely defined by its column span.

For a square matrix A , the span of its column Kleene star is the only region of $\text{span}(A)$ which may have full linear dimension, and the linear dimension of that region determines the *tropical rank* of A , introduced by Develin et al. [21], and also investigated by Izhakian [30]. When the tropical rank is full, the interior of span of the column Kleene star is the *simple image set* of A studied by Butkovič [8]: It is the set of vectors $y \in \mathbb{R}_+^n$ such that $Ax = y$ has a unique solution. In what follows, the span of column Kleene star of A will be called the *essential span* of A .

The following theorem, which is a slight generalization of Theorem 8 by Gaubert and Meunier [25], illustrates the role of essential span in the geometry of max cones. It can be thought of as a colourful generalization of Minkowski's theorem for max cones in the sense of Bárány [4].

THEOREM 3.22. *Let $U \subseteq \mathbb{R}_+^n$ be a closed max cone and let $V^1, \dots, V^n \subseteq \mathbb{R}_+^n$ be closed max cones such that the intersection of V^i with U is nontrivial for all $i \in [n]$. Then there exist vectors v^1, \dots, v^n such that v^i is an extremal of V^i , for $i \in [n]$, and $\text{span}(v^1, \dots, v^n)$ has nontrivial intersection with U .*

PROOF. Take any nonzero points $y^1 \in V^1 \cap U, \dots, y^n \in V^n \cap U$ and consider the matrix $A \in \mathbb{R}_+^{n \times n}$ with columns $A_{\cdot i} = y^i$, for $i = 1, \dots, n$. Assume first that A has permutations with nonzero weight. The essential span of A is the closed region $\text{cl}(X_S)$, where $S = \{\{\sigma(1)\}, \dots, \{\sigma(n)\}\}$, for any maximal permutation σ . Take any $u \in \text{cl}(X_S)$, then $u \in U$ and $u \geq_i A_{\cdot \sigma(i)}$ for all i . The column $A_{\cdot \sigma(i)}$ is equal to $y^{\sigma(i)}$ and it belongs to $V^{\sigma(i)}$. Applying Minkowski theorem (Proposition 2.5) and the multiorder principle (Proposition 2.1), we obtain an extremal $v^{\sigma(i)}$ of $V^{\sigma(i)}$ such that $v^{\sigma(i)} \leq_i y^{\sigma(i)} \leq_i u$. Applying Proposition 2.1 again, we see that $u \in \text{span}(v^{\sigma(1)}, \dots, v^{\sigma(n)})$. As $u \in U$, the claim follows.

In the case when A does not have nonzero permutations, an inductive argument using Hall's marriage theorem, see [25], shows that there exist subsets of indices M , N_1 and N_2 such that the submatrix $A_{[N_1, M]}$ is zero, while the submatrix $A_{[N_2, M]}$ is square and has a permutation with nonzero weight. Then the above argument goes with the essential span of that submatrix. \square

4. Cyclic projectors and the alternating method

4.1. Cyclic projectors and separation of several max cones. Let V^1, \dots, V^k be closed max cones in \mathbb{R}_+^n and denote by P_i the projector onto V^i . The composition $P_k \cdots P_1$ will be called the *cyclic projector* associated with V^1, \dots, V^k .

This operator inherits many properties of the sole projector: it is a homogeneous, continuous, isotone and nonincreasing operator. In general, it is not linear with respect to max and min operations. Such operators can be treated by nonlinear Perron-Frobenius theory. In particular, the following theorem of Nussbaum [40] generalizes the well-known Collatz-Wielandt formula for the spectral radius of a nonnegative matrix.

THEOREM 4.1. *Let F be a continuous, homogeneous and isotone operator in \mathbb{R}_+^n . Then the spectral radius of F is equal to*

$$(4.1) \quad r(F) = \inf\{\lambda \mid \exists y \text{ positive: } Fy \leq \lambda y\}.$$

Such operators have no more than one eigenvalue over any set of vectors with the same support, and therefore the total number of their eigenvalues is finite. Formula (4.1) implies that the spectral radius is monotone. Define the *cyclic projective distance* of $y^1, \dots, y^k \in \mathbb{R}_+^n$ by

$$(4.2) \quad \rho_{\text{H}}(y^1, \dots, y^k) = \log \bigoplus_{i_1, \dots, i_k \in M} y_{i_1}^1 (y_{i_1}^2)^{-1} \cdot \dots \cdot y_{i_k}^k (y_{i_k}^1)^{-1},$$

when $\text{supp}(y^1) = \dots = \text{supp}(y^k) = M$, and by $+\infty$ otherwise. In the case $k = 2$ this is the Hilbert projective distance between two points in \mathbb{R}_+^n . An equivalent definition is

$$(4.3) \quad \rho_{\text{H}}(y^1, \dots, y^k) = \log \inf\left\{\prod_{i=1}^k \lambda_i \mid y^i \leq \lambda_i y^{i+1}, i \in [k]\right\},$$

where $y^{k+1} := y^1$. Note that ρ_{H} is stable under multiplication of the arguments by nonzero scalars and under their cyclic permutation. If $\sum_{i=1}^k y_i^i = 1$ for $i \in [k]$, then it follows from (4.3) that $\lambda_i \geq 1$, and $\rho_{\text{H}}(y^1, \dots, y^k) = 0$ if and only if $y^1 = \dots = y^k$. For general $y^1, \dots, y^k \in \mathbb{R}_+^n \setminus \{0\}$, $\rho_{\text{H}}(y^1, \dots, y^k) = 0$ if and only if y^1, \dots, y^k are proportional to each other.

Define the cyclic projective distance between closed max cones V^1, \dots, V^k by

$$(4.4) \quad \rho_{\text{H}}(V^1, \dots, V^k) = \inf_{y^1 \in V^1, \dots, y^k \in V^k} \rho_{\text{H}}(y^1, \dots, y^k).$$

The minimum in (4.4) is attained since ρ_{H} is lower semicontinuous, see Proposition 4.8 below.

The monotonicity of spectral radius is crucial for the following theorem [26].

THEOREM 4.2. *Let V^1, \dots, V^k be closed max cones in \mathbb{R}_+^n . Suppose that y^0 is an eigenvector of $P_k \cdots P_1$ associated with the spectral radius, and consider vectors $y^1 \in V^1, \dots, y^k \in V^k$ defined by $y^1 := P_1 y^0, \dots, y^k := P_k y^{k-1}$. Then*

$$\rho_{\text{H}}(y^1, \dots, y^k) = \rho_{\text{H}}(V^1, \dots, V^k) = -\log r(P_k \cdots P_1).$$

Cyclic projectors also enable to prove a separation theorem for closed max cones [26], with the following ideas in mind. Firstly, formula (4.1) implies the existence of a positive subeigenvector with $\lambda < 1$. Secondly, if we take such a subeigenvector, then its projections onto V^1, \dots, V^k define separating halfspaces, see Theorem 2.6.

THEOREM 4.3. *Let $V^1, \dots, V^k \subseteq \mathbb{R}_+^n$ be closed max cones. If each of V^1, \dots, V^k has a positive vector, then the following are equivalent.*

1. *There exists a positive vector y and $\lambda < 1$: $P_k \cdots P_1 y \leq \lambda y$.*

2. There exist halfspaces $\mathcal{H}^1, \dots, \mathcal{H}^k$ such that $V^1 \subseteq \mathcal{H}^1, \dots, V^k \subseteq \mathcal{H}^k$ and $\bigcap_{i=1}^k \mathcal{H}^i = \{0\}$.
3. $\bigcap_{i=1}^k V^i = \{0\}$.
4. $r(P_k \cdots P_1) < 1$.

The statements 2. and 3. are equivalent even if V^1, \dots, V^k do not have positive vectors.

4.2. The alternating method and its convergence. In what follows we consider the case when $V^1 = \text{span}(A^{(1)}), \dots, V^k = \text{span}(A^{(k)})$, and $A^{(1)}, \dots, A^{(k)}$ are nonnegative matrices with an equal number of nonzero rows. A natural question is to find a positive solution to the system of equations

$$(4.5) \quad A^{(1)} \otimes x^1 = \dots = A^{(k)} \otimes x^k,$$

and the cyclic projectors provide an efficient method for doing this.

ALTERNATING METHOD

Input: Nonnegative matrices $A^{(1)} \in \mathbb{R}_+^{n \times m_1}, \dots, A^{(k)} \in \mathbb{R}_+^{n \times m_k}$ with an equal number n of nonzero rows.

Initialization: Arbitrary positive $y^{(0)} := y^{(1)0}$.

Iteration: Number $l \geq 1$. For all $s = 1, \dots, k$ compute $x^{(l)s} := \overline{A^{(s)}} \otimes' y^{(l)s-1}$ and $y^{(l)s} := A^{(s)} \otimes x^{(l)s}$. Set $x^{(l)} := x^{(l)k}$ and $y^{(l)} := y^{(l)k}$.

Stop: If $y^{(l)} = y^{(l-1)}$, then stop. The vectors $x^{(l)s}$, for $s = 1, \dots, k$, give a solution to system (4.5). Else if $y_i^{(l)} < y_i^{(0)}$ for all $i \in [n]$, then stop. There is no solution.

Over the semiring $\mathbb{R}_{\max,+} = (\mathbb{R} \cup \{-\infty\}, \oplus = \max, \otimes = +)$ and for $k = 2$, this method was formulated by Cuninghame-Green and Butkovič [20]. The method is essentially a max-algebraic version of the *cyclic projections method* known in optimization theory [5], since $y^{(l)} = P_k \cdots P_1 y^{(l-1)}$.

The first part of the stop condition follows from the fact that P_1, \dots, P_k are nonincreasing projectors onto $\text{span}(A^{(1)}), \dots, \text{span}(A^{(k)})$. Indeed, if $y^{(l-1)} = y^{(l)}$, then the inequalities

$$y^{(l)} \geq P_{k-1} \cdots P_1 y^{(l-1)} \geq \dots \geq P_1 y^{(l-1)} \geq y^{(l-1)}$$

are satisfied with equalities, implying that $y^{(l)s} = P_s \cdots P_1 y^{(l)}$ are equal for all $s \in [k]$ and that $y^{(l)} \in \text{span}(A^{(1)}) \cap \dots \cap \text{span}(A^{(k)})$. As $y^{(l)s} = A^{(s)} \otimes x^{(l)s}$ for $s \in [k]$, we have that $x^{(l)s}$, for $s \in [k]$, give a solution to (4.5).

Also note that the absence of zero rows in the matrices implies that all vectors in the sequence generated by the alternating method are positive and hence any solution, which the alternating method may find, has to be positive.

The following proposition, similar to the results of [20], justifies the second part of the stop condition. It emphasizes the role of *sleepers*, i.e., such indices $i(s) \in [n]$ (for $s = 1, \dots, k$) that $y_{i(s)}^{(1)s} = y_{i(s)}^{(2)s} = \dots$ for the whole sequence $\{y^{(l)s}, l \geq 1\}$, and $j(s) \in [m_s]$ such that $x_{j(s)}^{(1)s} = x_{j(s)}^{(2)s} = \dots$ for the whole sequence $\{x^{(l)s}, l \geq 1\}$. Sleepers will be called *eternal*, if the corresponding coordinates are constant for all

$l \geq 1$, and *temporary*, if the corresponding coordinates are constant up to the last iteration of the alternating method.

PROPOSITION 4.4. *Let $A^{(1)}, \dots, A^{(k)}$, $x^{(l)s}$ and $y^{(l)s}$ be as in the formulation of the alternating method. Then*

1. *temporary sleepers exist for all sequences $\{x^{(l)s}\}$ and $\{y^{(l)s}\}$, $s \in [k]$.*
2. *if (4.5) has a solution, then eternal sleepers exist for all sequences $\{x^{(l)s}\}$ and $\{y^{(l)s}\}$, $s \in [k]$.*
3. *if (4.5) has a positive solution, then $\{x^{(l)s}\}$ and $\{y^{(l)s}\}$, for all $s \in [k]$, are bounded from below by positive vectors.*

PROOF. 1. Assume that for some $s \in [k]$ and $l \geq 1$ we have that all coordinates of $y^{(l)s}$ or $x^{(l)s}$ are strictly less than that of $y^{(1)s}$ or $x^{(1)s}$. Then we have that $y^{(l)s} \leq \mu y^{(1)s}$ or $x^{(l)s} \leq \mu x^{(1)s}$ for some $\mu < 1$. As all matrix multiplications are homogeneous and isotone, we have that $y^{(l)} \leq \mu y^{(1)}$ so that all coordinates of $y^{(l)}$ are strictly less than that of $y^{(0)}$ and the alternating method immediately stops.

2. and 3. Take any $s \in [k]$. If there is a vector y in the intersection of column spans, we can scale it so that $y \leq y^{(1)s}$ and $y_i = y_i^{(1)s}$ for some i . In terms of the multiorder, $y \leq_i y^{(1)s}$ (for this scaling it is essential that $y^{(0)}$ and hence $y^{(1)s}$ are positive). As the projectors are all isotone and y is their fixed point, we have that $y \leq y^{(l)s}$ and $y_i = y_i^{(l)s}$ for the whole sequence. If (4.5) has a positive solution, then the same scaling argument shows that the sequence $\{y^{(1)s}, y^{(2)s}, \dots\}$ is bounded from below by a positive vector. Now note that the same line of argument applies to $\{x^{(l)s}\}$ as well. \square

In what follows we will prove that the alternating method converges to a positive solution if a positive solution exists. We note here that a cyclic projector is a min-max function in the sense of [13, 14, 41], with addition being replaced by multiplication, and the convergence of the alternating method follows from the results of [13, 41] concerning the ultimate periodicity of min-max functions. Below we give a different proof which uses the cellular decomposition idea.

We first investigate the convergence of the alternating method for Kleene stars, which then enables us, using cellular decomposition, to prove the finiteness results for general matrices.

PROPOSITION 4.5. *Suppose that $A^{(1)}, \dots, A^{(k)} \in \mathbb{R}_+^{n \times n}$ are Kleene stars. If $\text{span}(A^{(1)}) \cap \dots \cap \text{span}(A^{(k)})$ contains a positive vector, then the alternating method converges in no more than n iterations.*

PROOF. The alternating method starts with an arbitrary positive initial vector y and repeatedly applies the composition $P_k \cdots P_1$. Due to Proposition 3.2 we have that

$$P_k \cdots P_1 y = \overline{A^{(k)}} \otimes' \dots \otimes' \overline{A^{(1)}} \otimes' y,$$

and hence

$$(P_k \cdots P_1)^m y = (\overline{A^{(k)}} \otimes' \dots \otimes' \overline{A^{(1)}})^m \otimes' y.$$

This means that the stabilization of the alternating method is equivalent to the stabilization of $(A^{(1)} \otimes \dots \otimes A^{(k)})^m \otimes y$ for any positive y . Denote the matrix product $A^{(1)} \otimes \dots \otimes A^{(k)}$ by C . By Proposition 3.11 we have that $\lambda(C) = 1$. We also have that the diagonal entries of C are equal to 1 and hence it is a strongly

definite matrix. By Proposition 3.19 the powers of C stabilize in no more than $n - 1$ steps, and this proves the claim. \square

Now we make use of the cellular decomposition to prove that if there is a positive solution, then the alternating method finds a positive solution in a finite number of steps. First we prove the following technical proposition.

PROPOSITION 4.6. *Suppose that $A^{(1)}, \dots, A^{(k)} \in \mathbb{R}_+^{n \times n}$ have all diagonal entries equal to 1 and suppose that any product D of no more than n of them has $\lambda(D) \leq 1$. Fix a mapping $j : \{1, \dots\} \mapsto \{1, \dots, k\}$. Consider the sequence of products $C^{(m)} = A^{(j(m))} \otimes \dots \otimes A^{(j(1))}$, for $m \geq 1$. Then there exists $m \leq n^k - 1$ such that $C^{(m)} = C^{(m+1)}$.*

PROOF. For the case of just one matrix, this is Proposition 3.19. We argue by induction, assuming the result is true for $k - 1$ matrices and proving it for k . Choose any mapping $\pi : \{1, \dots, n\} \mapsto \{1, \dots, k\}$. Then either for some $m < n^k$ we have that there are no repetitions before that m and

$$C^{(m)} = \bigotimes_{i=1}^n A^{(\pi(i))} \otimes B^{(i)},$$

where each $B^{(i)}$ is a product of less than $n^{k-1} - 1$ matrices, or there is a repetition, and in this case we are done. Hence, for $M = n^k - 1$, either there are repetitions before that M , or the product $C^{(M)} = (c_{ij}^{(M)})$ contains all the above mentioned products. We claim then that

$$(4.6) \quad c_{ij}^{(m)} = \bigoplus_{\pi, i_{n-1}, \dots, i_1} a_{i_{n-1}}^{(\pi(n))} \cdot \dots \cdot a_{i_1 j}^{(\pi(1))}.$$

for all $m \geq M$. Indeed, $c_{ij}^{(m)}$ is greater than or equal to the maximum on the r.h.s. due to the choice of M and since all diagonal entries of all matrices are 1. It is actually equal to this maximum because all products of no more than n matrices have $\lambda \leq 1$, so the weight of any path of length M does not exceed the weight of the simple path obtained after cycle deletion, and the weights of all simple paths are already in (4.6). \square

THEOREM 4.7. *Suppose that $A^{(1)} \in \mathbb{R}_+^{n \times m_1}, \dots, A^{(k)} \in \mathbb{R}_+^{n \times m_k}$ have all rows nonzero and are such that $\text{span}(A^{(1)}) \cap \dots \cap \text{span}(A^{(k)})$ contains a positive vector. Then the alternating method stabilizes in a finite number of steps.*

PROOF. It follows from Proposition 3.15 that for each matrix $A^{(i)}$ we have a Kleene decomposition

$$\text{span}(A^{(i)}) = \bigcup_{l=1}^{s(i)} \text{span}(A^{(il)}),$$

where $A^{(il)} \in \mathbb{R}_+^{n \times n}$ are Kleene stars. Then we have that

$$(4.7) \quad (P_k \cdots P_1)^m y = \overline{(A^{(kl(k,m))})} \otimes' \dots \otimes' \overline{(A^{(ll(1,m))})} \otimes' \dots \otimes' \overline{(A^{(kl(k,1))})} \otimes' \dots \otimes' \overline{(A^{(ll(1,1))})} \otimes' y$$

for some index mappings $l(i, j)$.

It suffices to prove the stabilization of the sequence

$$(4.8) \quad B^{(m)} \otimes \dots \otimes B^{(1)} \otimes y,$$

where $B^{(i)} = ((A^{(kl(k,i))})^T \otimes \dots \otimes (A^{(1l(1,i))})^T)$. Note that the number of matrices $B^{(i)}$ is also finite. Since the spans of the matrices $A^{(1)}, \dots, A^{(k)}$ have a point in intersection, by Proposition 4.4 sequence (4.7) is bounded from below, and hence (4.8) is bounded from above.

Consider a finite product B of some matrices $B^{(i)}$, appearing in (4.8). If $\lambda(B) > 1$, then at least one of the matrices making this product will appear only a finite number of times. Otherwise the sequence will be unbounded, which is a contradiction.

Hence after some finite m the matrices $B^{(i)}$ appearing in the sequence will be such that $\lambda(B) \leq 1$ for any product B of no more than n of them.

After that, the finite convergence of alternating method is guaranteed by Proposition 4.6. \square

4.3. Bounds on the number of iterations. Now we examine the case when the system has no solution, i.e., when the max cones $\text{span}(A^{(1)}), \dots, \text{span}(A^{(k)})$ do not have nontrivial intersection. Here we will need the *total projective distance* between y^1, \dots, y^k , which is the sum of projective distances

$$(4.9) \quad \rho_{\Sigma}(y^1, \dots, y^k) = \rho_{\text{H}}(y^1, y^2) + \dots + \rho_{\text{H}}(y^k, y^1),$$

if y^1, \dots, y^k have equal supports, and $+\infty$ otherwise. Note that

$$(4.10) \quad \rho_{\Sigma}(y^1, \dots, y^k) = \rho_{\text{H}}(y^1, \dots, y^k) + \rho_{\text{H}}(y^k, \dots, y^1),$$

where ρ_{H} is the cyclic projective distance defined by (4.2). By analogy with (4.3),

$$(4.11) \quad \rho_{\Sigma}(y^1, \dots, y^k) = \log \inf \left\{ \prod_{i=1}^k \lambda_i \mu_i \mid y^i \leq \lambda_i y^{i+1}, y^{i+1} \leq \mu_i y^i, i \in [k] \right\},$$

where $y^{k+1} := y^1$. Like ρ_{H} , the total projective distance is stable under scalar multiplication of the arguments and their cyclic permutation.

Denote $\mathcal{S}_n := \{x \in \mathbb{R}_+^n \mid \sum_{i=1}^n x_i = 1\}$ and consider $\mathcal{S}_n^k := \overbrace{\mathcal{S}_n \times \dots \times \mathcal{S}_n}^k$ endowed with product topology. A function $\phi : \mathcal{S}_n^k \mapsto \mathbb{R}_+ \cup \{+\infty\}$ is called *lower semicontinuous* if the sublevel sets

$$(4.12) \quad \mathcal{S}_n^k(\phi, a) = \{(y^1, \dots, y^k) \in \mathcal{S}_n^k \mid \phi(y^1, \dots, y^k) \leq a\},$$

are closed for all $a \in \mathbb{R}_+$. The author gratefully acknowledges the idea of the proof of the following proposition to Stéphane Gaubert.

PROPOSITION 4.8. $\rho_{\Sigma}(y^1, \dots, y^k)$ and $\rho_{\text{H}}(y^1, \dots, y^k)$ are lower semicontinuous on \mathcal{S}_n^k .

PROOF. Consider sequences $\{y^{(m)i}, m \geq 1\} \subseteq \mathcal{S}_n$ converging to \bar{y}^i , for $i \in [k]$. We need to show that if $(y^{(m)1}, \dots, y^{(m)k}) \in \mathcal{S}_n^k(\rho_{\Sigma}, a)$ (resp. if $(y^{(m)1}, \dots, y^{(m)k}) \in \mathcal{S}_n^k(\rho_{\text{H}}, a)$) for all $m \geq 1$, then $(\bar{y}^1, \dots, \bar{y}^k) \in \mathcal{S}_n^k(\rho_{\Sigma}, a)$ (resp. $(\bar{y}^1, \dots, \bar{y}^k) \in \mathcal{S}_n^k(\rho_{\text{H}}, a)$). If $(y^{(m)1}, \dots, y^{(m)k}) \in \mathcal{S}_n^k(\rho_{\Sigma}, a)$ for all m , there exist $\lambda_i^{(m)}, \mu_i^{(m)} \in \mathbb{R}_+$ such that $y^{(m)i} \leq \lambda_i^{(m)} y^{(m)i+1}$ and $y^{(m)i+1} \leq \mu_i^{(m)} y^{(m)i}$ for all $i \in [k]$, and that $\prod_{i=1}^k \lambda_i^{(m)} \mu_i^{(m)} \leq a$. As $\sum_{l=1}^n y_l^{(m)i} = 1$ for all m and i , and $y^{(m)i} \leq \lambda_i^{(m)} y^{(m)i+1}$

and $y^{(m)i+1} \leq \mu_i^{(m)} y^{(m)i}$, we have that $\lambda_i^{(m)} \geq 1$ and $\mu_i^{(m)} \geq 1$. Using these inequalities and $\prod_{i=1}^k \lambda_i^{(m)} \mu_i^{(m)} \leq a$, we obtain that $1 \leq \lambda_i^{(m)} \leq a$ and $1 \leq \mu_i^{(m)} \leq a$ for all $i \in [k]$. Taking convergent subsequences if necessary, we can assume that $\lambda_i^{(m)} \rightarrow \lambda_i$ and $\mu_i^{(m)} \rightarrow \mu_i$ for $i \in [k]$. Then we have $\overline{y^i} \leq \lambda_i \overline{y^{i+1}}$ and $\overline{y^{i+1}} \leq \mu_i \overline{y^i}$ for all $i \in [k]$, and $\prod_{i=1}^k \lambda_i \mu_i \leq a$, which yields $(\overline{y^1}, \dots, \overline{y^k}) \in \mathcal{S}_n^k(\rho_\Sigma, a)$. The proof for the case of ρ_Σ is complete, the case of ρ_H is treated analogously. \square

By analogy with (4.4), the total projective distance between closed max cones V^1, \dots, V^k is defined by

$$(4.13) \quad \begin{aligned} \rho_\Sigma(V^1, \dots, V^k) &= \rho_H(V^1, V^2) + \dots + \rho_H(V^k, V^1) = \\ &= \min_{y^1 \in V^1, \dots, y^k \in V^k} \rho_\Sigma(y^1, \dots, y^k). \end{aligned}$$

Observe that $\rho_\Sigma(y^1, \dots, y^k) = 0$ if and only if y^1, \dots, y^k are multiples of each other. This is generalised in the following proposition.

PROPOSITION 4.9. *Let $V^1, \dots, V^k \subseteq \mathbb{R}_+^n$ be closed max cones. Then $\rho_\Sigma(V^1, \dots, V^k) = 0$ (equivalently, $\rho_H(V^1, \dots, V^k) = 0$) if and only if the intersection of V^1, \dots, V^k is nontrivial.*

PROOF. We show the “only if” part. The intersections of V^i and \mathcal{S}_n are closed sets. Let the sequences $\{y^{(m)i}, m \geq 1\}$, for $i \in [k]$ and $y^{(m)i} \in V^i \cap \mathcal{S}_n$, be such that $\lim_{m \rightarrow \infty} \rho_\Sigma(y^{(m)1}, \dots, y^{(m)k}) = 0$ (or $\lim_{m \rightarrow \infty} \rho_H(y^{(m)1}, \dots, y^{(m)k}) = 0$). As \mathcal{S}_n is compact, we can assume that $y^{(m)i} \rightarrow \overline{y^i}$ for $i \in [k]$, where $\overline{y^i} \in V^i \cap \mathcal{S}_n$ as $V^i \cap \mathcal{S}_n$ is closed. Proposition 4.8 implies that $\rho_\Sigma(\overline{y^1}, \dots, \overline{y^k}) = 0$ (resp. $\rho_H(\overline{y^1}, \dots, \overline{y^k}) = 0$). Hence $\overline{y^i}$, for $i \in [k]$, are proportional vectors contained in $V^1 \cap \dots \cap V^k$. The proof of the “only if” part is complete. The “if” part is obvious. \square

Let vector y and matrix A have finite entries. Denote

$$(4.14) \quad \|y\| = \log \bigoplus_{i,j} y_i y_j^{-1}, \quad \|A\| = \log \bigoplus_{i,j,k} a_{ik} a_{jk}^{-1}.$$

A vector $y = \bigwedge_{i=1}^n \lambda_i A_{.i}$, where $\lambda_i > 0$ for all $i \in [n]$, and \bigwedge denotes the component-wise minimum, will be called a *min combination* of the columns of A .

PROPOSITION 4.10. *Let $A \in \mathbb{R}_+^{n \times m}$ and $y \in \mathbb{R}_+^n$ have all entries positive. If y is a max combination or a min combination of the columns of A , then $\|y\| \leq \|A\|$.*

PROOF. Let $y = \bigoplus_j \lambda_j A_{.j}$, or let $y = \bigwedge_j \lambda_j A_{.j}$ with all $\lambda_j \neq 0$. Then

$$\begin{aligned} \exp(\|y\|) &= \bigoplus_{i,j} y_i y_j^{-1} = \bigoplus_{i,j} \left(\bigoplus_k \lambda_k a_{ik} \right) \cdot \left(\bigwedge_{l: \lambda_l \neq 0} \lambda_l^{-1} a_{jl}^{-1} \right) = \\ &= \bigoplus_{i,j,k} \lambda_k a_{ik} \cdot \left(\bigwedge_{l: \lambda_l \neq 0} \lambda_l^{-1} a_{jl}^{-1} \right) \leq \bigoplus_{i,j,k: \lambda_k \neq 0} a_{ik} a_{jk}^{-1} \leq \exp(\|A\|), \text{ or} \\ \exp(\|y\|) &= \bigoplus_{i,j} y_i y_j^{-1} = \bigoplus_{i,j} \left(\bigwedge_k \lambda_k a_{ik} \right) \cdot \left(\bigoplus_l \lambda_l^{-1} a_{jl}^{-1} \right) = \\ &= \bigoplus_{i,j,l} \lambda_l^{-1} a_{jl}^{-1} \cdot \left(\bigwedge_k \lambda_k a_{ik} \right) \leq \bigoplus_{i,j,l} a_{il} a_{jl}^{-1} \leq \exp(\|A\|), \end{aligned}$$

respectively. The claim follows by the monotonicity of the logarithm. \square

PROPOSITION 4.11. *Let $u \in \mathbb{R}_+^n$ be a positive vector, let $V \subseteq \mathbb{R}_+^n$ be a closed max cone and let $v = P_V(u)$. Then $\sum_{i=1}^n (\log u_i - \log v_i) \geq \rho_H(u, v)$.*

PROOF. As $v \leq u$ and $u_k = v_k$ for some k , we have that $\rho_H(u, v) = \max_{i=1}^n (\log u_i - \log v_i)$. As any sum of nonnegative numbers is greater than or equal to any of its terms, the claim follows. \square

PROPOSITION 4.12. *Suppose that $A \in \mathbb{R}_+^{n \times m}$, and suppose that $x^1, x^2 \in \mathbb{R}_+^m$ and $y^1, y^2 \in \mathbb{R}_+^n$ are positive and such that $y^1 \geq y^2$ with strict inequalities in at most n' coordinates, $x^1 \geq x^2$ and $A \otimes x^1 = y^1$, $A \otimes x^2 = y^2$. Then*

1. *there exists k such that $x_k^1 (x_k^2)^{-1} \geq \max_s y_s^1 (y_s^2)^{-1}$;*
2. *the inequality $\sum_{k=1}^m (\log x_k^1 - \log x_k^2) \geq \frac{1}{n'} \sum_{i=1}^n (\log y_i^1 - \log y_i^2)$ holds.*

PROOF. Let t be such that $\max_s y_s^1 (y_s^2)^{-1} = y_t^1 (y_t^2)^{-1}$ and define k such that $\max_s (a_{ts} x_s^1) = a_{tk} x_k^1 = y_t^1$. The inequalities $a_{tk} \neq 0$ and $a_{tk} x_k^2 \leq y_t^2$ imply part 1. To obtain part 2, we recall that any sum of nonnegative numbers is greater than or equal to any of its terms, and that the maximum is always greater than or equal to the arithmetic mean. \square

Now we obtain a bound for the number of iterations of the alternating method. For brevity, we denote $\rho_\Sigma(A^{(1)}, \dots, A^{(k)}) := \rho_\Sigma(\text{span}(A^{(1)}), \dots, \text{span}(A^{(k)}))$.

THEOREM 4.13. *Suppose that $A^{(1)} \in \mathbb{R}_+^{n \times m_1}, \dots, A^{(k)} \in \mathbb{R}_+^{n \times m_k}$, that $A^{(k)}$ has all entries positive, and that $\text{span}(A^{(1)}) \cap \dots \cap \text{span}(A^{(k)}) = \{0\}$. Then after no more than*

$$(4.15) \quad 2(n-1) \min(\|A^{(k)}\|, (m_k-1)\|A^{(k)T}\|) / \rho_\Sigma(A^{(1)}, \dots, A^{(k)})$$

iterations the alternating method will terminate.

PROOF. Let the sequences $\{y^{(l)s}, l \geq 1\}$ and $\{x^{(l)s}, l \geq 1\}$, for $s \in [k]$, be as in the formulation of the alternating method. Using Proposition 4.11, we obtain the following lower bound for the total sum of logarithmic coordinate losses of $y^{(l)}$ at each iteration:

$$(4.16) \quad \sum_{i=1}^n (\log y_i^{(l+1)} - \log y_i^{(l)}) = \sum_{s=0}^{k-1} \sum_{i_s=1}^n (\log y_{i_s}^{(l)s+1} - \log y_{i_s}^{(l)s}) \geq \\ \geq \rho_\Sigma(y^{(l)1}, \dots, y^{(l)k}) \geq \rho_\Sigma(A^{(1)}, \dots, A^{(k)}).$$

Using Proposition 4.12, we also obtain that

$$(4.17) \quad \sum_{i=1}^n (\log x_i^{(l+1)} - \log x_i^{(l)}) \geq \frac{1}{n-1} \sum_{i=1}^n (\log y_i^{(l+1)} - \log y_i^{(l)}) \geq \\ \geq \frac{1}{n-1} \rho_\Sigma(A^{(1)}, \dots, A^{(k)}).$$

Let j be a temporary sleeper for $\{x^{(l)}\}$ and let i be a temporary sleeper for $\{y^{(l)}\}$. The existence of temporary sleepers was shown in Proposition 4.4. Thus the total sum of all logarithmic coordinate losses of $y^{(l)}$ at each iteration is at least $\rho_\Sigma(A^{(1)}, \dots, A^{(k)})$, while the i th coordinate of $y^{(l)}$ is a sleeper, and the total sum of all logarithmic coordinate losses of $x^{(l)}$ is at least $\frac{1}{n-1} \rho_\Sigma(A^{(1)}, \dots, A^{(k)})$ while the j th coordinate of $x^{(l)}$ is a sleeper. This will stop the alternating method. Indeed,

we repeatedly apply $P_k \cdots P_1$ and stop when all coordinates of $y^{(l)}$ decrease with respect to that of $y^{(0)}$. As $y^{(l)}$, for $l \geq 1$, is a max combination of the columns of $A^{(k)}$, by Proposition 4.10 we have that $\log y_t^{(1)} - \log y_i^{(1)} \leq \|y^{(1)}\| \leq \|A^{(k)}\|$ for all $t \in [n]$. Lower bound (4.16) for the total sum of logarithmic coordinate losses of $y^{(l)}$ at each iteration implies that after at most $2(n-1)\|A^{(k)}\|/\rho_\Sigma(A^{(1)}, \dots, A^{(k)})$ iterations there will be t such that $\log y_i^{(l)} - \log y_t^{(l)} > \|A^{(k)}\|$, if the method does not stop, and this contradicts Proposition 4.10. Hence, after at most that number of iterations all coordinates will have to fall in value with respect to the coordinates of the initial vector. Now, as $x^{(l)}$, for $l \geq 1$, is a min combination of the columns of $\overline{A^{(k)}}$, by Proposition 4.10 we have that $\log x_t^{(1)} - \log x_i^{(1)} \leq \|x^{(1)}\| \leq \|A^{(k)T}\|$ for all $t \in [m_k]$ (note that $\|\overline{A}\| = \|A^T\|$ for any positive matrix A). Using (4.17) instead of (4.16) and arguing as above, we obtain the upper bound $2(m_k - 1)\|A^{(k)T}\|/(\frac{1}{n-1}\rho_\Sigma(A^{(1)}, \dots, A^{(k)}))$ on the number of iterations, and this proves the claim. \square

If there is more than one matrix with all entries positive, then bound (4.15) can be improved.

THEOREM 4.14. *Suppose that $A^{(1)} \in \mathbb{R}_+^{n \times m_1}, \dots, A^{(k)} \in \mathbb{R}_+^{n \times m_k}$, that $A^{(r_1)}, \dots, A^{(r_s)}$ have all entries positive, and that $\text{span}(A^{(1)}) \cap \dots \cap \text{span}(A^{(k)}) = \{0\}$. Then after no more than*

$$(4.18) \quad 2(n-1) \min_{i=1}^s \min(\|A^{(r_i)}\|, (m_{r_i} - 1)\|A^{(r_i)T}\|) / \rho_\Sigma(A^{(1)}, \dots, A^{(k)})$$

iterations the alternating method will terminate.

PROOF. Applying the argument of Theorem 4.13 and using the fact that ρ_Σ , like ρ_H , is stable under the cyclic permutations of its arguments, we obtain that for any $t = 1, \dots, s$, after at most

$$(4.19) \quad l = 2(n-1) \min(\|A^{(r_t)}\|, (m_{r_t} - 1)\|A^{(r_t)T}\|) / \rho_\Sigma(A^{(1)}, \dots, A^{(k)})$$

iterations all coordinates of $y^{(l)r_t}$ have to fall with respect to the coordinates of $y^{(1)r_t}$. This means that there is a $\mu < 1$ such that $y^{(l)r_t} \leq \mu y^{(1)r_t}$. As all projectors are homogeneous and order preserving, we also have that $y^{(l)} \leq \mu y^{(1)}$. Therefore all the coordinates of $y^{(l)}$ decrease with respect to that of $y^{(1)}$, and hence to that of $y^{(0)}$, and the alternating method stops with negative answer. So the number of iterations does not exceed (4.19) for each r_t , and hence it does not exceed the minimum of these, which is (4.18). \square

Now we show that the techniques developed above apply to the case of integer matrices over the max-plus semiring $\mathbb{R}_{\max,+} = (\mathbb{R} \cup \{-\infty\}, \oplus = \max, \otimes = +)$ investigated by Cuninghame-Green and Butkovič [20]. In what follows, we switch to the matrix algebra over the max-plus semiring and to the alternating method formulated over that semiring.

First note that if $y \in \mathbb{R}^n$ is a max-plus or min-plus combination of columns of a matrix $A \in \mathbb{R}^{n \times m}$ with real entries, then $\|y\| \leq \|A\|$, where like in (4.14) but without logarithm, the norms are defined by

$$(4.20) \quad \|y\| = \max_{i,j} (y_i - y_j), \quad \|A\| = \max_{i,j,k} (a_{ik} - a_{jk}).$$

THEOREM 4.15. *Suppose that $A^{(1)} \in \mathbb{R}^{n \times m_1}, \dots, A^{(k)} \in \mathbb{R}^{n \times m_k}$ have all entries integer. Then after no more than*

$$(4.21) \quad 2 \min_{i=1}^k \min((n-1) \frac{k-1}{k} \|A^{(i)}\|, (m_i-1) \|A^{(i)T}\|)$$

iterations the alternating method will terminate.

PROOF. We are in almost the same situation as in Theorem 4.14: for all $x^{(l)s}$ and $y^{(l)s}$ there exist temporary sleepers, the norms $\|y^{(l)s}\|$ do not exceed $\|A^{(s)}\|$ and the norms $\|x^{(l)s}\|$ do not exceed $\|A^{(s)T}\|$. It remains to give bounds for the total sum of coordinate losses for $x^{(l)s}$ and $y^{(l)s}$ at each iteration. As everything is integer, the total sum of losses for both $x^{(l)s}$ and $y^{(l)s}$ is not less than 1. The multiple $\frac{k-1}{k}$ at $\|A^{(i)}\|$, which may be important only if k is small, is due to the observation that if we apply P_1, \dots, P_{k-1} to $y^{(l)} \in A^{(k)}$ and do not see any fall in coordinates, then $y^{(l)}$ is in the intersection and the method immediately stops, hence during the run of the algorithm, after at most $k-1$ actions (not k but $k-1$) of the sole projectors at least one coordinate of y has to fall. The claim now follows by the same argument as in Theorems 4.13 and 4.14. \square

The bounds on number of iterations in [20], obtained in the case $k=2$, are in the same vein as (4.21). The only bound on number of iterations in [20] which does not depend on the choice of initial vector would read in our terms essentially as $2 \min_{l=1}^k ((m_l-1) \max_{i,j} (|a_{ij}^{(l)}|))$, where $|\cdot|$ denotes the modulus of an entry. The bound of (4.21) is expressed in terms of projective norms of rows and columns of the matrices, which makes it more precise.

5. Acknowledgement

The author is grateful to the anonymous referee for careful reading and useful remarks, and wishes to thank Peter Butkovič, Stéphane Gaubert, Alexander Guterman, Grigori Litvinov, Victor Maslov, Alexander Mikhalev, Hans Schneider and Andrej Sobolevskii, whose help, interest, valuable ideas and stimulating discussions have been very important for this work.

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