ON CYCLIC CLASSES AND ATTRACTION CONES IN MAX ALGEBRA

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Abstract. In max algebra it is well-known that the sequence A^k, with A an irreducible square matrix, becomes periodic at sufficiently large k. This raises a number of questions on the periodic regime of A^k and $A^k \otimes x$, for a given vector x. Also, this leads to the concept of attraction spaces in max algebra, by which we mean spaces of vectors with prescribed orbit period.

This paper shows that some of these questions can be solved by matrix squaring (A,A^2,A^4, ...), analogously to recent findings of Semančíková [37, 38] concerning the orbit period in max-min algebra. Hence the computational complexity of such problems is $O(n^3 \log n)$. The main idea is to apply an appropriate diagonal similarity scaling $A \mapsto X^{-1}AX$, called visualization scaling, and to study the role of cyclic classes of the critical graph.

For powers of a visualized matrix in the periodic regime, we observe remarkable symmetry described by circulants and their rectangular generalizations. We exploit this symmetry to derive a system of equations for attraction space, and present an algorithm which computes the coefficients of the system.

1. Introduction

By *max algebra* we understand the analogue of linear algebra developed over the max-times semiring $\mathbb{R}_{\max,\times}$ which is the set of nonnegative numbers $\mathbb{R}_+$ equipped with the operations of "addition" $a \oplus b := \max(a, b)$ and the ordinary multiplication $a \otimes b := a \times b$. Zero and unity of this semiring coincide with the usual 0 and 1. The operations of the semiring are extended to the nonnegative matrices and vectors in the same way as in conventional linear algebra. That is if $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$ are matrices of compatible sizes with entries from $\mathbb{R}_+$, we write $C = A \oplus B$ if $c_{ij} = a_{ij} \oplus b_{ij}$ for all i, j and $C = A \otimes B$ if $c_{ij} = \sum_k^{\oplus} a_{ik}b_{kj} = \max_k(a_{ik}b_{kj})$ for all i, j. If A is a square matrix over $\mathbb{R}_+$ then the iterated product $A \otimes A \otimes ... \otimes A$ in which the symbol A appears k times will be denoted by A^k.

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The *max-plus semiring* $\mathbb{R}_{\max,+} = (\mathbb{R} \cup \{-\infty\}, \oplus = \max, \otimes = +)$, developed over the set of real numbers $\mathbb{R}$ with adjoined element $-\infty$ and the ordinary addition playing the role of multiplication, is another isomorphic "realization" of max algebra. In particular, $x \mapsto \exp(x)$ yields an isomorphism between $\mathbb{R}_{\max,+}$ and $\mathbb{R}_{\max,\times}$. In the max-plus setting, the zero element is $-\infty$ and the unity is 0.

The *min-plus semiring* $\mathbb{R}_{\min,+} = (\mathbb{R} \cup \{+\infty\}, \oplus = \min, \otimes = +)$ is also isomorphic to $\mathbb{R}_{\max,+}$ and $\mathbb{R}_{\max,\times}$. Another well-known semiring is the *max-min semiring* $\mathbb{R}_{\max,\min} = (\mathbb{R} \cup \{-\infty\} \cup \{\infty\}, \oplus = \max, \otimes = \min)$, see [24, 37, 38], but it is *not* isomorphic to any of the semirings above.

Max algebraic column spans of nonnegative matrices $A \in \mathbb{R}_+^{n\times n}$ are sets of max linear combinations of columns $\bigoplus_{i=1}^n \alpha_i A_{\cdot i}$ with nonnegative coefficients α_i. Such column spans are *max cones*, meaning that they are closed under componentwise maximum $\oplus$ and multiplication by nonnegative scalars. There are important analogies and links between max cones and convex cones [13, 16, 40, 39].

The maximum cycle geometric mean $\lambda(A)$, see below for exact definition, is one of the most important charasteristics of a matrix $A \in \mathbb{R}_+^{n\times n}$ in max algebra. In particular, it is the largest eigenvalue of the spectral problem $A \otimes x = \lambda x$. The cycles at which this maximum geometric mean is attained, are called *critical*. Further, one consideres the *critical graph* $C(A)$ which consists of all nodes and edges that belong to the critical cycles. This graph is crucial for the description of eigenvectors [3, 14, 25].

The well-known *cyclicity theorem* states that if A is irreducible, then the sequence A^k becomes periodic after some finite transient time, and that the ultimate period of A^k is equal to the cyclicity of the critical graph [3, 14, 25]. Generalizations to reducible case, computational complexity issues and important special cases of this result have been extensively studied in [15, 23, 24, 31, 32].

In this paper we study the behaviour of matrix powers and orbits $A^k \otimes x$ in the irreducible case in the periodic regime, i.e., after the periodicity is reached. One of the main ideas is to study the periodicity of *visualized* matrices, meaning matrices with all entries less than or equal to the maximum cycle geometric mean. This study provides a connection to the theory of Boolean matrices [6, 28].

In Boolean matrix algebra, one considers components of imprimitivity of a matrix [6, 28], or equivalently, cyclic classes of the associated digraph [4]. In max algebra, cyclic classes of the critical graph have been considered as an important tool in the proof of the cyclicity theorem mentioned above, see [25] Sect. 3.1. Recently, the cyclic classes appeared in max-min algebra [37, 38], where they were used to study the ultimate periods of orbits and other periodicity problems. It was shown that such questions can be solved by matrix squaring (A, A^2, A^4, A^8, ...), which yields computational complexity $O(n^3 \log n)$.

We show that the problems of computing ultimate period and matrix powers in the periodic regime can be solved by matrix squaring in *max algebra*, which yields the same complexity bound $O(n^3 \log n)$. This is achieved by exploiting visualization, and cyclic classes of the critical graph. Further it turns out that the periodic powers of visualized matrices have remarkable symmetry described by circulant matrices and their rectangular generalizations. We use this symmetry to derive a system of equations for *attraction cone*, meaning the max cone which consists of all vectors with prescribed orbit period. We also present an algorithm for computing the coefficients of this system.

The contents of the paper are as follows. In Section 2 we revise two important topics in max algebra, namely the spectral problem and Kleene stars. In Section 3, we speak of the visualization and the connection to the theory of Boolean matrices which it provides, see Propositions 3.1 and 3.3. In Section 4, we study basic properties of matrix powers in the periodic regime, see Propositions 4.4 – 4.6. The problems which can be solved by matrix squaring are described in Proposition 4.10. In Section 5 we observe circulant symmetries of periodic powers of visualized matrices, see Proposition 5.3, derive a system of equations for attraction space, see Proposition 5.6, and describe an algorithm which computes the coefficients of this system. We conclude with Section 6 which is devoted to numerical examples.

As $\mathbb{R}_{\max,+}$ and $\mathbb{R}_{\max,\times}$ are isomorphic, we use the possibility to switch between them, but only when it is really convenient. Thus, while the theoretical results are obtained over max-times semiring, which looks more natural in connection with diagonal matrix scaling and boolean matrices, the examples in Section 6 are written over *max-plus semiring*, where it is much easier to calculate.

We remark that some aspects of the theory of attraction spaces have been investigated in [5, 17, 29] in certain special cases. Also, the periodicity of max algebraic powers of matrices can be regarded from the viewpoint of max-plus semigroups as studied in [30].

2. Two topics in max algebra

2.1. **Spectral problem.** Let $A \in \mathbb{R}_+^{n\times n}$. Consider the problem of finding $\lambda \in \mathbb{R}_+$ and nonzero $x \in \mathbb{R}_+^n$ such that

$$A \otimes x = \lambda x. \tag{1}$$

If for some λ there exists a nonzero $x \in \mathbb{R}_+^n$ which satisfies (1), then λ is called a *max-algebraic eigenvalue* of A, and x is a *max-algebraic eigenvector* of A associated with λ. With the zero vector adjoined, the set of max-algebraic eigenvectors associated with λ forms a max cone, which is called the *eigencone* associated with λ.

The largest max-algebraic eigenvalue of $A \in \mathbb{R}_+^{n\times n}$ is equal to

$$\lambda(A) = \bigoplus_{k=1}^{n} (\mathrm{Tr}_{\oplus} A^k)^{1/k}, \tag{2}$$

where $\mathrm{Tr}_{\oplus}$ is defined by $\mathrm{Tr}_{\oplus}(A) := \bigoplus_{i=1}^{n} a_{ii}$ for any $A = (a_{ij}) \in \mathbb{R}_+^{n\times n}$. Further we explain the graph-theoretic meaning of (2), assumed that $\lambda(A) \neq 0$.

With $A = (a_{ij}) \in \mathbb{R}_+^{n\times n}$ we can associate the weighted digraph $D_A = (N(A), E(A))$, with the set of nodes $N(A) = \{1, \ldots, n\}$ and the set of edges $E(A) = \{(i, j) \mid a_{ij} \neq 0\}$ with weights $w(i, j) = a_{ij}$. Suppose that $\pi = (i_1, ..., i_p)$ is a path in D_A, then the *weight* of π is defined to be $w(\pi, A) = a_{i_1i_2}a_{i_2i_3}\ldots a_{i_{p-1}i_p}$ if $p > 1$, and 1 if $p = 1$. If $i_1 = i_p$ then π is called a cycle. One can check that

$$\lambda(A) = \max_{\sigma} \mu(\sigma, A),$$

where the maximization is taken over all cycles in D_A and

$$\mu(\sigma, A) = w(\sigma, A)^{1/k}$$

denotes the *geometric mean* of the cycle $\sigma = (i_1, ..., i_k, i_1)$. Thus $\lambda(A)$ is the *maximum cycle geometric mean* of D_A.

$A \in \mathbb{R}_+^{n\times n}$ is *irreducible* if for any nodes i and j there exists a path in D_A, which begins at i and ends at j. In this case A has a unique max-algebraic eigenvalue which equals $\lambda(A)$.

Note that $\lambda(\alpha A) = \alpha\lambda(A)$ and hence $\lambda(A/\lambda(A)) = 1$ if $\lambda(A) > 0$. Unless we need matrices with $\lambda(A) = 0$, we can always assume without loss of generality that $\lambda(A) = 1$. Such matrices will be called *definite*.

An important relaxation of (1) is

$$A \otimes x \leq \lambda x. \tag{3}$$

The nonzero vectors $x \in \mathbb{R}_+^n$ which satisfy (3) are called *subeigenvectors* associated with λ. With the zero vector adjoined, they form a max cone called *subeigencone*. This is a conventionally convex cone, meaning that it is closed under the *ordinary* addition. See [40] for more details.

The eigencone (resp. subeigencone) of A associated with $\lambda(A)$ will be denoted by $V(A)$ (resp. $V^*(A)$).

2.2. **Kleene stars.** Let $A \in \mathbb{R}_+^{n\times n}$. Consider the formal series

$$A^* = I \oplus A \oplus A^2 \oplus \ldots, \tag{4}$$

where I denotes the identity matrix with entries

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

Series (4) is a max-algebraic analogue of $(I - A)^{-1}$, and it converges to a matrix with finite entries if and only if $\lambda(A) \leq 1$ [3, 10]. In this case

$$A^* = I \oplus A \oplus A^2 \oplus \ldots \oplus A^{n-1}, \tag{5}$$

which is called the *Kleene star* of A.

For any $A \in \mathbb{R}_+^{n\times n}$,

$$A \text{ is a Kleene star } \Leftrightarrow \ A^2 = A,\ a_{ii} = 1\ \forall i. \tag{6}$$

The condition $\lambda(A) \leq 1$ suggests that there is a strong interplay between Kleene stars and spectral problems. To describe this in more detail, we need the following notions and notation.

A cycle σ in D_A is called *critical*, if $\mu(\sigma, A) = \lambda(A)$. Every node and edge that belongs to a critical cycle is called *critical*. The set of critical nodes is denoted by $N_c(A)$, the set of critical edges is denoted by $E_c(A)$. The *critical digraph* of A, further denoted by $C(A) = (N_c(A), E_c(A))$, is the digraph which consists of all critical nodes and critical edges of D_A. For definite $A \in \mathbb{R}_+^{n\times n}$, it follows that $a_{ij}a^*_{ji} \leq 1$ [3]. Further,

$$(i, j) \in E_c(A) \Leftrightarrow a_{ij}a^*_{ji} = 1. \tag{7}$$

For definite $A \in \mathbb{R}_+^{n\times n}$, the relation between Kleene star, critical graph and spectral problems is briefly as follows [3, 14, 40]:

$$V^*(A) = \operatorname{span}(A^*) = \left\{ \bigoplus_{i=1}^{n} \alpha_i A^*_{\cdot i},\ \alpha_i \in \mathbb{R}_+ \right\}, \tag{8}$$

$$V(A) = \left\{ \bigoplus_{i\in N_c(A)} \alpha_i A^*_{\cdot i},\ \alpha_i \in \mathbb{R}_+ \right\}, \tag{9}$$

$$x \in V^*(A),\, (i, j) \in E_c(A) \Rightarrow a_{ij}x_j = x_i. \tag{10}$$

Equation (8) means that $V^*(A)$ is the max-algebraic column span of Kleene star A^*, also called *Kleene cone*. This cone is convex in conventional sense. By (9), $V(A)$ is the max subcone of $V^*(A)$, spanned by the columns with critical indices. Implication (10) means that for any subeigenvector $x \in V^*(A)$ and $i \in N_c(A)$, the maximum in $\bigoplus_j a_{ij}x_j$ is attained at j such that $(i, j) \in E_c(A)$. In particular, $(A \otimes x)_i = x_i$ for all $x \in V^*(A)$ and $i \in N_c(A)$.

Not all columns in (8) and (9) are necessary. Let $C(A)$ have $c \in \{1, \ldots, n\}$ strongly connected components (s.c.c.) C_μ, for $\mu = 1, \ldots, c$. It follows from the definition of $C(A)$ that s.c.c. C_μ are disjoint. The corresponding node sets will be denoted by N_μ. Let m denote the number of non-critical nodes of D_A. It can be shown [3, 14] that if i, j belong to the same s.c.c. of $C(A)$, then the columns $A^*_{\cdot i}$ and $A^*_{\cdot j}$ are multiples of each other. The same holds for the rows $A^*_{i\cdot}$ and $A^*_{j\cdot}$. Hence

$$V^*(A) = \left\{ \bigoplus_{i \in K} \alpha_i A^*_{\cdot i},\ \alpha_i \in \mathbb{R}_+ \right\} \tag{11}$$

$$V(A) = \left\{ \bigoplus_{i \in N_c(A) \cap K} \alpha_i A^*_{\cdot i},\ \alpha_i \in \mathbb{R}_+ \right\}, \tag{12}$$

where K is any set of indices which contains all non-critical indices and for every C_μ there is a unique index of this component in K.

Consider A^*_{KK}, the principal submatrix of A^* extracted from the rows and columns with indices in K. Condition (6) implies that A^*_{KK} is itself a Kleene star. It follows from the maximality of C_μ that there is a unique permutation of K that has the greatest weight with respect to A^*_{KK}. The *weight of a permutation* π of $\{1, \ldots, n\}$ with respect to $A \in \mathbb{R}^{n\times n}_+$ is defined as $\prod_{i=1}^n a_{i\pi(i)}$. Thus A^*_{KK} is *strongly regular* in the sense of Butkovič [7]. From this it can be deduced that the columns of A^* with indices in K are *independent*, meaning that none of them can be expressed as a max combination of the other columns. In other words [9], the columns of A^* with indices in K (resp., in $N_c(A) \cap K$) form a *basis* of $V^*(A)$ (resp., of $V(A)$). This basis is essentially unique [9], meaning that any other basis can be obtained from it by scalar multiplication.

More precisely, the strong regularity of A^*_{KK} is equivalent to saying that this basis is *tropically independent*, hence the tropical rank of A^* is equal to $c + m$, see [2, 26, 27] for definitions and further details.

3. Visualization and Boolean matrices

3.1. Visualization. Consider a positive $x \in \mathbb{R}^n_+$ and define

$$X = \operatorname{diag}(x) := \begin{pmatrix} x_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & x_n \end{pmatrix} \tag{13}$$

The transformation $A \mapsto X^{-1}AX$ is called a *diagonal similarity scaling* of A. Such transformations do not change $\lambda(A)$ and $C(A)$ [20]. They commute with max-algebraic multiplication of matrices and hence with the operation of taking the Kleene star. Geometrically, they correspond to automorphisms of $\mathbb{R}^n_+$, both in the case of max algebra and in the case of nonnegative linear algebra. Further we define scalings which lead to particularly convenient forms of matrices in max algebra.

A definite matrix $A \in \mathbb{R}_+^{n\times n}$ is called *visualized*, if

$$a_{ij} \leq 1,\ \forall i,j = 1,\ldots,n \tag{14}$$

$$a_{ij} = 1,\ \forall (i,j) \in E_c(A) \tag{15}$$

A visualized matrix $A \in \mathbb{R}_+^{n\times n}$ is called *strictly visualized* if

$$a_{ij} = 1 \Leftrightarrow (i,j) \in E_c(A). \tag{16}$$

Visualization scalings were known already to Afriat [1] and Fiedler-Pták [22], and motivated extensive study of matrix scalings in nonnegative linear algebra, see e.g. [20, 21, 35, 36]. We remark that some constructions and facts related to application of visualization scaling in max algebra have been observed in connection with max algebraic power method [18, 19], behaviour of matrix powers [8] and max-balancing [35, 36].

Visualization scalings are described in [40] in terms of the subeigencone $V^*(A)$ and its relative interior. For the convenience of the reader, we show their existence for any definite $A \in \mathbb{R}_+^{n\times n}$. In the proposition stated below, the summation in part 2. is *conventional*.

Proposition 3.1. *Let $A \in \mathbb{R}_+^{n\times n}$ be definite and $X = \operatorname{diag}(x)$.*

1. *If $x = \bigoplus_{i=1}^n A^*_{\cdot i}$ then $X^{-1}AX$ is visualized.*
2. *If $x = \sum_{i=1}^n A^*_{\cdot i}$ then $X^{-1}AX$ is strictly visualized.*

Proof. 1. Observe that $x \in V^*(A)$ and x is positive. Then $a_{ij}x_j \leq x_i$ for all i,j implies $x_i^{-1}a_{ij}x_j \leq 1$, and by (10) $x_i^{-1}a_{ij}x_j = 1$ for all $(i,j) \in E_c(A)$.
2. Observe that x is positive, and that $x \in V^*(A)$ since $V^*(A)$ is convex. Hence $X^{-1}AX$ is visualized. It remains to check that $(i,j) \notin E_c(A)$ implies $a_{ij}x_j < x_i$. We need to find k such that $a_{ij}a^*_{jk} < a^*_{ik}$. But this is true for $k = i$, since $a^*_{ii} = 1$ and $a_{ij}a^*_{ji} < 1$ by (7). This completes the proof. □

More precisely [40], $A \in \mathbb{R}_+^{n\times n}$ can be visualized by any positive vector in $V^*(A)$, and it can be strictly visualized by any vector in the relative interior of $V^*(A)$.

3.2. **Max algebra and Boolean matrices.** Max algebra is related to the algebra of Boolean matrices. The latter algebra is defined over the Boolean semiring S which is the set $\{0,1\}$ equipped with logical operations "OR" $a \oplus b := a \vee b$ and "AND" $a \otimes b := a \wedge b$. Clearly, Boolean matrices can be treated as objects of max algebra, as a very special but crucial case.

For a strongly connected graph, its cyclicity is defined as the g.c.d. of the lengths of all cycles (or equivalently, all simple cycles). If the cyclicity is 1 then the graph is called *primitive*, otherwise it is called *imprimitive*. We will not distinguish between cyclicity (or primitivity) of a Boolean matrix A and the associated digraph D_A. Further we recall an important result of Boolean matrix theory.

Proposition 3.2 (Brualdi and Ryser [6]). *Let $A \in S^{n\times n}$ be irreducible, and let γ_A be the cyclicity of D_A (which is strongly connected). Then for each $k \geq 1$, there exists a permutation matrix P such that $P^{-1}A^kP$ has r irreducible diagonal blocks, where $r = \gcd(k, \gamma_A)$, and all elements outside these blocks are zero. The cyclicity of all these blocks is γ_A/r.*

In max algebra, let $A \in \mathbb{R}_+^{n\times n}$. Define the Boolean matrix $A^{[C]} = (a_{ij}^{[C]})$ by

$$a_{ij}^{[C]} = \begin{cases} 1, & (i,j) \in E_c(A) \\ 0, & (i,j) \notin E_c(A). \end{cases} \tag{17}$$

Let $A, B \in \mathbb{R}_+^{n\times n}$. Assume that $C(A)$ has c s.c.c. C_μ for $\mu = 1,\ldots,c$, with cyclicities γ_μ. Denote by $B_{\mu\nu}$ the block of B extracted from the rows with indices in N_μ and columns with indices in N_ν.

The following proposition can be seen as a corollary of Proposition 3.2. The idea of the proof given below is due to Hans Schneider. See also [25] Section 3.1 and [8] Theorem 2.3.

Proposition 3.3. *Let $A \in \mathbb{R}_+^{n\times n}$ and $\lambda(A) \neq 0$.*

1. $\lambda(A^k) = \lambda^k(A)$.
2. $(A^{[C]})^k = (A^k)^{[C]}$.
3. *For each $k \geq 1$, there exists a permutation matrix P such that $(P^{-1}A^kP)_{\mu\mu}^{[C]}$, for each $\mu = 1,\ldots,c$, has $r_\mu := \gcd(k, \gamma_\mu)$ irreducible blocks and all elements outside these blocks are zero. The cyclicity of all blocks in $(P^{-1}A^kP)_{\mu\mu}^{[C]}$ is equal to γ_μ/r_μ.*

Proof. We can assume that A is definite. Further, the diagonal similarity scaling commutes with max algebraic matrix multiplication and changes neither $\lambda(A)$ nor $C(A)$ [20], and by Proposition 3.1, part 2, there exists a strict visualization scaling. Hence we can assume that A is strictly visualized. In this case $A^{[C]} = A^{[1]}$, where $A^{[1]} = (a_{ij}^{[1]})$ is defined by

$$a_{ij}^{[1]} = \begin{cases} 1, & a_{ij} = 1, \\ 0, & a_{ij} < 1. \end{cases} \tag{18}$$

It is easily seen that $(A^{[1]})^k = (A^k)^{[1]}$. As $A^{[1]} = A^{[C]}$, all entries of $A^{[1]}$ outside the blocks $A_{\mu\mu}^{[1]}$ are zero, which assures that $(A^{[1]})_{\mu\mu}^k = (A_{\mu\mu}^{[1]})^k$.

Proposition 3.2 implies that part 3. is true for $(A^{[1]})^k = (A^k)^{[1]}$. This implies that $P^{-1}(A^k)^{[1]}P$ has irreducible blocks and $\lambda(A^k) = 1$, which shows part 1. Also, $P^{-1}(A^k)^{[1]}P$

has block structure where all diagonal blocks are irreducible and all off-diagonal blocks are zero. This implies $(A^k)^{[C]} = (A^k)^{[1]}$, and parts 2. and 3. follow immediately. □

3.3. **Cyclic classes.** For a path P in a digraph $G = (N, E)$, where $N = \{1, \dots, n\}$, denote by $l(P)$ the length of P, i.e., the number of edges traversed by P.

Proposition 3.4 (Brualdi-Ryser [6])**.** *Let $G = (N, E)$ be a strongly connected digraph with cyclicity γ_G. Then the lengths of any two paths connecting $i \in N$ to $j \in N$ (with i, j fixed) are congruent modulo γ_G.*

Proposition 3.4 implies that the following *equivalence relation* can be defined: $i \sim j$ if there exists a path P from i to j such that $l(P) \equiv 0 (\mathrm{mod}\, \gamma_G)$. The equivalence classes of G with respect to this relation are called *cyclic classes* [4, 37, 38]. The cyclic class of i will be denoted by $[i]$.

Consider the following *access relations* between cyclic classes: $[i] \to_t [j]$ if there exists a path P from a node in $[i]$ to a node in $[j]$ such that $l(P) \equiv t (\mathrm{mod}\, \gamma_G)$. In this case, a path P with $l(P) \equiv t (\mathrm{mod}\, \gamma_G)$ exists between any node in $[i]$ and any node in $[j]$. Further, by Proposition 3.4 the length of any path between a node in $[i]$ and a node in $[j]$ is congruent to t, so the relation $[i] \to_t [j]$ is well-defined. Classes $[i]$ and $[j]$ will be called *adjacent* if $[i] \to_1 [j]$.

Cyclic classes can be computed in $O(|E|)$ time by Balcer-Veinott digraph condensation, where $|E|$ denotes the number of edges in G. At each step of this algorithm, we look for all edges which issue from a certain node i, and condense all end nodes of these edges into a single node. A precise description of this method can be found in [4, 6]. We give an example of its work, see Figures 1 and 2.

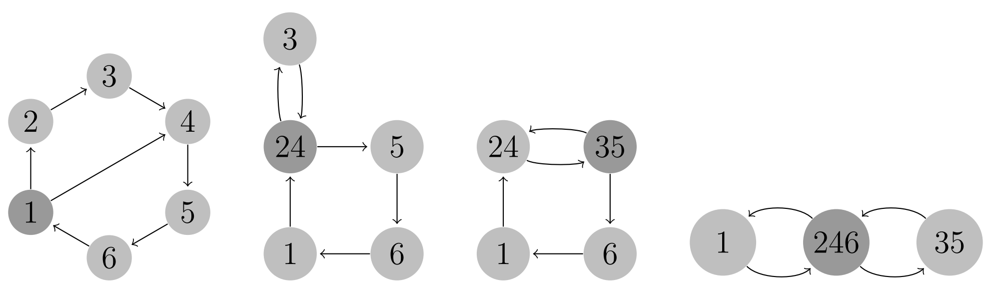

FIGURE 1. Balcer-Veinott algorithm

In this example, see Figure 1 at the left, we start by condensing nodes 2 and 4, which are "next to" node 1, into the node 24. Further we proceed with condensing nodes 3 and 5 into the node 35. In the end, see Figure 2 at the left, there are just two nodes 135 and 246. They correspond to two cyclic classes $\{1, 3, 5\}$ and $\{2, 4, 6\}$ of the initial graph, see Figure 2 at the right.

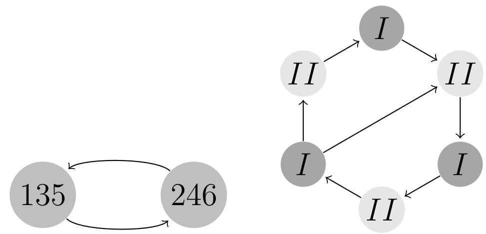

FIGURE 2. Result of the algorithm (left) and cyclic classes (right)

The notion of cyclic classes and access relations can be generalized to the case when G has c disjoint components G_μ with cyclicities γ_μ, for $\mu = 1, \ldots, c$ (just like the critical graph in max algebra). In this case we write $i \sim j$ if i, j belong to the same component and there exists a path P from i to j such that $l(P) \equiv 0 (\operatorname{mod} \gamma_\mu)$. If $l(P) \equiv t (\operatorname{mod} \gamma_\mu)$, then we write $[i] \to_t [j]$. In this case the *cyclicity* of G is $\gamma := \operatorname{lcm}\ \gamma_\mu,\ \mu = 1, \ldots, c$.

We will be interested in the cyclic classes of critical graphs, and below we also give an explanation of these, in terms of the Boolean matrix $A^{[C]}$. Let $A \in \mathbb{R}_+^{n \times n}$. Following Brualdi and Ryser [6] we can find such ordering of the indices that any submatrix $A^{[C]}_{\mu\mu}$, which corresponds to an imprimitive component C_μ of $C(A)$, will be of the form

$$\begin{pmatrix} \mathbf{0} & A^{[C]}_{s_1 s_2} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & A^{[C]}_{s_2 s_3} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & A^{[C]}_{s_{k-1} s_k} \\ A^{[C]}_{s_k s_1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{pmatrix}, \tag{19}$$

where k is the number of cyclic classes in C_μ. Indices s_i and s_{i+1} for $i = 1, \ldots, k-1$, and s_k and s_1 correspond to adjacent cyclic classes. By Proposition 3.3 part 2, when A is raised to power k, $A^{[C]}$ is also raised to the same power over the Boolean algebra. Any power of $A^{[C]}$ has a similar block-permutation form. In particular, $(A^{\gamma_\mu})^{[C]}_{\mu\mu}$ looks like

$$\begin{pmatrix} (A^{\gamma_\mu})^{[C]}_{s_1 s_1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & (A^{\gamma_\mu})^{[C]}_{s_2 s_2} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & (A^{\gamma_\mu})^{[C]}_{s_k s_k} \end{pmatrix} \tag{20}$$

Theorem 5.4.11 of [28] implies that the sequence $(A^k)^{[C]} = (A^{[C]})^k$ becomes periodic after $k \leq (n-1)^2 + 1$, with period $\gamma = \operatorname{lcm}(\gamma_\mu),\ \mu = 1, \ldots, c$. In the periodic regime, all entries of nonzero blocks are equal to 1.

4. Periodicity and complexity

4.1. **Spectral projector and matrix periodicity.** For a definite and irreducible A, consider the matrix $Q(A)$ with entries

$$q_{ij} = \bigoplus_{k \in N_c(A)} a^*_{ik} a^*_{kj}, \ \ i, j = 1, \ldots, n. \tag{21}$$

The max-linear operator whose matrix is $Q(A)$, is a max-linear *spectral projector* associated with A, in the sense that it projects $\mathbb{R}^n_+$ on the eigencone $V(A)$ [3].

This operator is closely related to the periodicity questions, as the following fact suggests.

Theorem 4.1 (Baccelli et al. [3], Theorem 3.109)**.** *Let $A \in \mathbb{R}^{n\times n}_+$ be irreducible and definite, and let all s.c.c. of $C(A)$ be primitive. Then there is an integer $T(A)$ such that $A^r = Q(A)$ for all $r \geq T(A)$.*

We will also need the following property of $Q(A)$ which follows directly from (21).

Proposition 4.2. *For $A \in \mathbb{R}^{n\times n}_+$ irreducible and definite, any critical column (or row) of $Q(A)$ is equal to the corresponding column (or row) of A^*.*

We also note that $Q(A)$ is important for the policy iteration algorithm of [12].

When $C(A)$ has imprimitive components, it follows from Proposition 3.3 part 3 that all components of $C(A^\gamma)$ are primitive, where γ is the cyclicity of $C(A)$. Hence, for any r great enough which is a multiple of γ, A^r is the matrix of the spectral projector onto the eigencone of A^γ. This also implies that for large enough r we have $A^r = A^{r+\gamma}$. The number r, after which this starts, is called the *transient* of $\{A^r\}$. It will be denoted by $T(A)$. Also, it is well-known that γ is the *ultimate period* of $\{A^r\}$, i.e., it is the least integer α such that $A^{r+\alpha} = A^r$ for all $r \geq T(A)$.

It is also important that the entries $a^{(r)}_{ij}$, where i or j are critical, become periodic much faster than the non-critical part of A. The following proposition is a known result, which is proved here for convenience of the reader. We recall that $N_c(A)$ denotes the set of critical nodes.

Proposition 4.3 (Nachtigall [33])**.** *Let $A \in \mathbb{R}^{n\times n}_+$ be a definite irreducible matrix. Critical rows and columns of A^r become periodic for $r \geq n^2$.*

Proof. We prove the claim for rows, and for columns everything is analogous. Let $i \in N_c(A)$. Then there is a critical cycle of length l_c to which i belongs. Hence $a^{(kl_c)}_{ii} = 1$ for $k \geq 1$. Since for all $m < k$ and any $t = 1, \ldots, n$ we have

$$a^{(ml_c)}_{is} = a^{((k-m)l_c)}_{ii} a^{(ml_c)}_{is} \leq a^{(kl_c)}_{is},$$

it follows that

$$a_{is}^{(kl_c)} = \bigoplus_{m=1}^{k} a_{is}^{(ml_c)}. \tag{22}$$

Entries $a_{is}^{(kl_c)}$ are maximal weights of paths of length k with respect to the matrix A^{l_c}. Since the weights of all cycles are less than or equal to 1 and all paths of length n are not simple, the maximum is achieved at $k \leq n$. Using (22) we obtain that $a_{is}^{((t+1)l_c)} = a_{is}^{(tl_c)}$ for all $t \geq n$. Further,

$$a_{is}^{(tl_c+d)} = \bigoplus_{k} a_{ik}^{(tl_c)} a_{ks}^{(d)},$$

and it follows that $a_{is}^{((t+1)l_c+d)} = a_{is}^{(tl_c+d)}$ for all $t \geq n$ and $0 \leq d \leq l_c - 1$. Hence $a_{is}^{(k)}$ is periodic for $k \geq nl_c$, and all these sequences, for any $i \in N_c(A)$ and any s, become periodic for $k \geq n^2$. $\square$

4.2. **The ultimate spans of matrices.** Max algebraic powers in the periodic regime have the following properties.

Proposition 4.4. *Let $A \in \mathbb{R}_+^{n\times n}$ be a definite and irreducible matrix, and let $t \geq 0$ be such that $t\gamma \geq T(A)$. Then for every integer $l \geq 0$*

$$A_{k\cdot}^{t\gamma+l} = \bigoplus_{i=1}^{c} a_{ki}^{(t\gamma)} A_{i\cdot}^{t\gamma+l},\ A_{\cdot k}^{t\gamma+l} = \bigoplus_{i=1}^{c} a_{ik}^{(t\gamma)} A_{\cdot i}^{t\gamma+l}, \quad 1 \leq k \leq n. \tag{23}$$

Proof. Due to Proposition 4.1, for $B = A^{\gamma}$ and any $r \geq T(B)$ we have

$$b_{kj}^{(r)} = \bigoplus_{i=1}^{c} b_{ki}^{*} b_{ij}^{*}, \quad 1 \leq k, j \leq n. \tag{24}$$

By Propositions 4.1 and 4.2, we have $b_{ki}^{*} = b_{ki}^{(r)} = a_{ki}^{(t\gamma)}$ and $b_{ij}^{*} = b_{ij}^{(r)} = a_{ij}^{(t\gamma)}$ for all $r \geq T(B)$ or equivalently $t\gamma \geq T(A)$, and any $i \leq c$. Hence

$$a_{kj}^{(t\gamma)} = \bigoplus_{i=1}^{c} a_{ki}^{(t\gamma)} a_{ij}^{(t\gamma)}, \quad 1 \leq k, j \leq n. \tag{25}$$

In the matrix notation, this is equivalent to:

$$A_{k\cdot}^{t\gamma} = \bigoplus_{i=1}^{c} a_{ki}^{(t\gamma)} A_{i\cdot}^{t\gamma},\ A_{\cdot k}^{t\gamma} = \bigoplus_{i=1}^{c} a_{ik}^{(t\gamma)} A_{\cdot i}^{t\gamma}, \quad 1 \leq k \leq n. \tag{26}$$

Multiplying (26) by any power A^{l}, we obtain (23). $\square$

In the proof of the next proposition we will use the following simple principle

$$a_{ij}^{(r)} a_{jk}^{(s)} \leq a_{ik}^{(r+s)}, \quad \forall i, j, k, r, s, \tag{27}$$

which holds for the matrix powers in max algebra.

Proposition 4.5. *Let $A \in \mathbb{R}_+^{n\times n}$ be a definite and irreducible matrix, and let $i, j \in N_c(A)$ be such that $[i] \to_l [j]$, for some $0 \le l < \gamma$.*

1. *For any $r \ge n^2$, there exists $t_1 \ge 0$ such that*

$$a_{ij}^{(t_1\gamma+l)} A^r_{\cdot i} = A^{(r+l)}_{\cdot j},\ a_{ij}^{(t_1\gamma+l)} A^r_{j\cdot} = A^{r+l}_{i\cdot}. \tag{28}$$

2. *If A is visualized, then for all $r \ge n^2$*

$$A^r_{\cdot i} = A^{r+l}_{\cdot j},\ A^r_{j\cdot} = A^{r+l}_{i\cdot}. \tag{29}$$

Proof. If $[i] \to_l [j]$ then $[j] \to_s [i]$ where $l + s = \gamma$. By the definition of access relations there exists a critical path of length $t_1\gamma + l$ connecting i to j, and a critical path of length $t_2\gamma + s$ connecting j to i. Hence $a_{ij}^{(t_1\gamma+l)} a_{ji}^{(t_2\gamma+s)} = 1$, and in the visualized case $a_{ij}^{(t_1\gamma+l)} = a_{ji}^{(t_2\gamma+s)} = 1$. Combining this with (27) we obtain

$$\begin{aligned}
A^r_{\cdot i} &= A^r_{\cdot i} a_{ij}^{(t_1\gamma+l)} a_{ji}^{(t_2\gamma+s)} \le A^{r+t_1\gamma+l}_{\cdot j} a_{ji}^{(t_2\gamma+s)} \le A^{r+(t_1+t_2+1)\gamma}_{\cdot i},\\
A^r_{j\cdot} &= A^r_{j\cdot} a_{ij}^{(t_1\gamma+l)} a_{ji}^{(t_2\gamma+s)} \le A^{r+t_1\gamma+l}_{i\cdot} a_{ji}^{(t_2\gamma+s)} \le A^{r+(t_1+t_2+1)\gamma}_{j\cdot}.
\end{aligned} \tag{30}$$

Since $r \ge n^2$, by Proposition 4.3 $A^r_{\cdot i} = A^{r+(t_1+t_2+1)\gamma}_{\cdot i}$ and $A^r_{j\cdot} = A^{r+(t_1+t_2+1)\gamma}_{j\cdot}$, hence all inequalities (30) are equalities. Multiplying them by $a_{ij}^{(t_1\gamma+l)}$ we obtain (28), which is (29) in the visualized case. ☐

Proposition 4.5 says that in any power A^r for $r \ge n^2$, the critical columns (or rows) can be obtained from the critical columns (or rows) of the spectral projector $Q(A^\gamma)$ via a permutation whose cycles are determined by the cyclic classes of $C(A)$. Proposition 4.4 adds to this that all non-critical columns (or rows) of any periodic power are in the max cone spanned by the critical columns (or rows). From this we conclude the following.

Proposition 4.6. *All powers A^r for $r \ge T(A)$ have the same column span, which is the eigencone $V(A^\gamma)$.*

Proposition 4.6 enables us to say that $V(A^\gamma)$ is the *ultimate column span* of A. Similarly, we have the *ultimate row span* which is $V((A^T)^\gamma)$. These cones are generated by critical columns (or rows) of the Kleene star $(A^\gamma)^*$. For a basis of this cone, we can take any set of columns $(A^\gamma)^*$ (equivalently $Q(A^\gamma)$ or A^r for $r \ge T(A)$), whose indices form a minimal set of representatives of all cyclic classes of $C(A)$. This basis is tropically independent in the sense of [2, 27, 26].

4.3. **Solving periodicity problems by square multiplication.** Let $A \in \mathbb{R}_+^{n\times n}$ and $\lambda(A) = 1$. The *t-attraction cone* $\mathrm{Attr}(A, t)$ is the max cone which consists of all vectors x, for which there exists an integer r such that $A^r \otimes x = A^{r+t} \otimes x$, and hence this is also true for all integers greater than or equal to r. Actually we may speak of any $r \ge T(A)$, due to the following observation.

Proposition 4.7. *Let A be irreducible and definite. The systems $A^r \otimes x = A^{r+t} \otimes x$ are equivalent for all $r \geq T(A)$.*

Proof. Let x satisfy $A^s \otimes x = A^{s+t} \otimes x$ for some $s \geq T(A)$, then it also satisfies this system for all greater s. Due to the periodicity, for all k from $T(A) \leq k \leq s$ there exists $l > s$ such that $A^k = A^l$. Hence $A^k \otimes x = A^{k+t} \otimes x$ also hold for $T(A) \leq k \leq s$. □

Corollary 4.8. $\mathrm{Attr}(A,t) = \mathrm{Attr}(A^t, 1)$.

Proof. By Proposition 4.7, $\mathrm{Attr}(A,t)$ is solution set to the system $A^r \otimes x = A^{r+t} \otimes x$ for any $r \geq T(A)$ which is a multiple of t, which proves the statement. □

A component (that is, equation) of $A^r \otimes x = A^{r+t} \otimes x$ with index in $N_c(A)$ will be called *critical*, and the subsystem of components with indices in $N_c(A)$ will be called the *critical subsystem.*

Proposition 4.9. *Let A be irreducible and definite and let $r \geq T(A)$. Then $A^r \otimes x = A^{r+t} \otimes x$ is equivalent to its critical subsystem.*

Proof. Consider a non-critical component $A^r_{k\cdot} \otimes x = A^{r+t}_{k\cdot} \otimes x$. Using (23) it can be written as

$$\bigoplus_{i \in N_c(A)} a_{ki}^{(r)} A^r_{i\cdot} \otimes x = \bigoplus_{i \in N_c(A)} a_{ki}^{(r)} A^{r+t}_{i\cdot} \otimes x, \tag{31}$$

hence it is a max combination of equations in the critical subsystem. □

Next we give a bound on the computational complexity of deciding whether $x \in \mathrm{Attr}(A,t)$, as well as other related problems which we formulate below.

P1. For a given x, decide whether $x \in \mathrm{Attr}(A,t)$.
P2. For a given $k:\ 0 \leq k < \gamma$, compute periodic power A^r where $r \equiv k (\mathrm{mod}\ \gamma)$.
P3. For a given x compute the ultimate period of $\{A^r \otimes x,\ r \geq 0\}$, meaning the least integer α such that $A^{r+\alpha} \otimes x = A^r \otimes x$ for all $r \geq T(A)$.

The following proposition is analogous to the results of Semančíková [37, 38].

Proposition 4.10. *For any irreducible matrix $A \in \mathbb{R}_+^{n\times n}$, the problems P1-P3 can be solved in $O(n^3 \log n)$ time.*

Proof. First note that we can compute both $\lambda(A)$ and a subeigenvector, and identify all critical nodes in no more than $O(n^3)$ operations, which is done essentially by Karp and Floyd-Warshall algorithms [34]. Further we can identify all cyclic classes of $C(A)$ by Balcer-Veinott condensation in $O(n^2)$ operations.

By Proposition 4.3 the critical rows and columns become periodic for $r \geq n^2$. To know the critical rows and columns of a given power $r' \geq T(A)$, it suffices to compute

A^r for arbitrary $r \geq n^2$ which can be done in $O(\log n)$ matrix squaring (A, A^2, A^4, ...) and takes $O(n^3 \log n)$ time, and to apply the corresponding permutation on cyclic classes which takes $O(n^2)$ overrides. By Proposition 4.9 we readily solve P1 by the verification of the critical subsystem of $A^{r'} \otimes x = A^{r'+t} \otimes x$ which takes $O(n^2)$ operations. Using linear dependence (23) the remaining non-critical submatrix of A^r, for any $r \geq T(A)$ such that $r \equiv k (\text{mod } \gamma)$, can be computed in $O(n^3)$ time. This solves P2.

As the non-critical rows of A are generated by the critical rows, the ultimate period of $\{A^r \otimes x\}$ is determined by the critical components. For visualized matrix we know that $A^{r+t}_{i\cdot} = A^r_{j\cdot}$ for all i, j such that $[i] \to_t [j]$. This implies $(A^{r+t} \otimes x)_i = (A^r \otimes x)_j$ for $[i] \to_t [j]$, meaning that, to determine the period we need only the critical subvector of $A^r \otimes x$ for any fixed $r \geq n^2$. Indeed, for any $i \in N_c(A)$ and $r \geq n^2$ the sequence $\{(A^{r+t} \otimes x)_i,\ t \geq 0\}$ can be represented as a sequence of critical coordinates of $A^r \otimes x$ determined by a permutation on γ_μ cyclic classes of the s.c.c. to which i belongs. To compute the period, we take a sample of γ_μ numbers appearing consecutively in the sequence, and check all possible periods, which takes no more than γ_μ^2 operations. The period of $A^r \otimes x$ appears as the l.c.m. of these periods. It remains to note that all operations above do not require more than $O(n^3)$ time. This solves P3. $\square$

5. Circulants and attraction cones

5.1. **Rectangular circulants.** Matrix $A \in \mathbb{R}_+^{n \times n}$ is called a *circulant* if there exist scalars $\alpha_1, \ldots, \alpha_n$ such that $a_{ij} = \alpha_d$ whenever $j - i = d (\text{mod } n)$. This looks like

$$A = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_n \\ \alpha_n & \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} \\ \alpha_{n-1} & \alpha_n & \alpha_1 & \cdots & \alpha_{n-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \alpha_2 & \alpha_3 & \ldots & \ldots & \alpha_1 \end{pmatrix} \tag{32}$$

We also consider the following generalizations of this notion.

Matrix $A \in \mathbb{R}_+^{m \times n}$ will be called a *rectangular circulant* when $a_{ij} = a_{ps}$ if $(p-i) \bmod m = (s-j) \bmod n$.

Matrix $A \in \mathbb{R}_+^{m \times n}$ will be called a *block circulant* when there exist scalars $\alpha_1, \ldots, \alpha_k$ and a block decomposition $A = (A_{ij}),\ i, j = 1, \ldots, k$ such that $A_{ij} = \alpha_d E_{ij}$ if $j - i = d (\text{mod } k)$, where all entries of blocks E_{ij} are equal to 1.

A rectangular circulant $A \in \mathbb{R}_+^{m \times n}$ is called *d-periodic* when $a_{ij} = a_{is}$ if $(s-j) \bmod n$ is a multiple of d.

Proposition 5.1. *Let $A \in \mathbb{R}_+^{m \times n}$ be a k-periodic rectangular circulant, for any integer $k = k_1, \ldots, k_l$, and let $d = g.c.d.(k_1, \ldots, k_l, m, n)$.*

1. *A is d-periodic.*
2. *There exist permutation matrices P and Q such that PAQ is block $d \times d$ circulant.*

Proof. 1. There are integers $(t_1, \ldots, t_l, t_m, t_n)$ such that $d = t_1 k_1 + \ldots + t_l k_l + t_m m + t_n n$. Using the definitions we obtain that $a_{ij} = a_{is}$ for $s - j = t_1 k_1 (\operatorname{mod} n)$. Proceeding with $t_2 k_2$ and other terms we obtain $a_{ij} = a_{is}$ for $s - j = d (\operatorname{mod} n)$.
2. Both m and n can be divided into d classes in such a way that $a_{ij} = a_{is}$ if j and s are in the same class and $a_{ij} = a_{pj}$ if i and p are in the same class. To obtain a block circulant form, it amounts to find such permutations P and Q which put the elements of these classes together. □

Corollary 5.2. *Let $A \in \mathbb{R}_+^{m \times n}$ be a rectangular circulant and let $d := g.c.d.(m, n)$.*

1. *A is d-periodic.*
2. *There are permutation matrices P and Q such that PAQ is block $d \times d$ circulant.*

Proof. Observe that any rectangular circulant is both m- and n-periodic. □

We give an example of 6×9 rectangular circulant A and the corresponding 3×3 block form B:

$$A = \begin{pmatrix} 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\ 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 \\ 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\ 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 \\ 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\ 2 & 2 & 2 & 0 & 0 & 0 & 1 & 1 & 1 \\ 2 & 2 & 2 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 2 & 2 & 0 & 0 & 0 \\ 1 & 1 & 1 & 2 & 2 & 2 & 0 & 0 & 0 \end{pmatrix}$$

5.2. **The block structure of periodic powers.** Let $A \in \mathbb{R}_+^{n \times n}$ and $C(A)$ consist of c s.c.c. C_μ with cyclicities γ_μ, for $\mu = 1, \ldots, c$. Let $\gamma = \operatorname{lcm}\ \gamma_\mu$ and m be the number of non-critical nodes. Further it will be convenient (though artificial) to consider, together with these components, also "non-critical components" C_μ for $\mu = c+1, \ldots, c+m$, whose node sets N_μ consist of just one non-critical node, whose set of edges is empty.

Consider the block decomposition $A^r = (A_{\mu\nu}^{(r)}),\ \mu, \nu = 1, \ldots, c+m$, where each block $A_{\mu\nu}^{(r)}$ is extracted from the rows with indices in N_μ and the columns with indices in N_ν.

Applying a similarity scaling $P^{-1}AP$ with a permutation matrix P if necesasary, we can assume that $A^{(r)}_{\mu\nu}$ has a block structure

$$A^{(r)}_{\mu\nu} = \begin{pmatrix} A^{(r)}_{s_1t_1} & \cdots & A^{(r)}_{s_1t_l} \\ \vdots & \ddots & \vdots \\ A^{(r)}_{s_kt_1} & \cdots & A^{(r)}_{s_kt_l} \end{pmatrix}, \tag{33}$$

where $k = \gamma_\mu$ and $l = \gamma_\nu$, and the indices

$$s_i = \sum_{\rho<\mu} \gamma_\rho + i,$$
$$t_j = \sum_{\rho<\nu} \gamma_\rho + j.$$

correspond to cyclic classes $[u_i]$ of C_μ and, respectively, to cyclic classes $[v_j]$ of C_ν numbered so that

$$[u_1] \to_1 [u_2] \to_1 \ldots \to_1 [u_k] \to_1 [u_1],$$
$$[v_1] \to_1 [v_2] \to_1 \ldots \to_1 [v_l] \to_1 [v_1].$$

In the case of non-critical μ (resp. ν) we have (33) with $k = 1$ (resp. $l = 1$).

Taking $r \geq T(A)$ and $t = 0$ in (29), we obtain that all rows of A^r with indices in the same cyclic class coincide, as well as all columns of A^r with indices in the same cyclic class. In terms of block decomposition (33), we obtain $A^{(r)}_{s_it_j} = \tilde{a}^{(r)}_{s_it_j} E_{s_it_j}$, where $\tilde{a}^{(r)}_{s_it_j}$ are scalars and $E_{s_it_j}$ are matrices, which are of the same dimension as $A^{(r)}_{s_it_j}$ and have all entries equal to 1. This observation can be found in [18, 19].

We define the matrix $\tilde{A}^{(r)} \in \mathbb{R}^{p\times p}_+$, where p is the total number of cyclic classes plus the number of non-critical nodes, as the matrix with entries $\tilde{a}^{(r)}_{s_it_j}$. It has blocks $\tilde{A}^{(r)}_{\mu\nu}$, the entries of which correspond to cyclic classes in C_μ and C_ν, and namely:

$$\tilde{A}^{(r)}_{\mu\nu} = \begin{pmatrix} \tilde{a}^{(r)}_{s_1t_1} & \cdots & \tilde{a}^{(r)}_{s_1t_m} \\ \vdots & \ddots & \vdots \\ \tilde{a}^{(r)}_{s_kt_1} & \cdots & \tilde{a}^{(r)}_{s_kt_m} \end{pmatrix}. \tag{34}$$

Proposition 5.3. *Let $A \in \mathbb{R}^{n\times n}_+$ be a visualized matrix which admits block decomposition (33), and $r \geq T(A)$. Let C_μ, C_ν be two (possibly equal) components of $C(A)$, and $d = g.c.d.(\gamma_\mu, \gamma_\nu)$.*

1. *$\tilde{A}^{(r)}_{\mu\nu}$ is a rectangular circulant.*
2. *For any critical μ and ν, there is a permutation P such that $(P^T\tilde{A}P)^{(r)}_{\mu\nu}$ is a block $d\times d$ circulant matrix.*
3. *If r is a multiple of γ, then $\tilde{A}^{(r)}_{\mu\mu}$ are circulant strongly regular Kleene stars.*

Proof. 1.: Using Eqn. (29) we see that for all (i,j) and (k,l) such that $k = i + t(\operatorname{mod} \gamma_\mu)$ and $l = j + t(\operatorname{mod} \gamma_\nu)$,

$$\tilde{a}^{(r)}_{s_k t_l} = \tilde{a}^{(r+t)}_{s_i t_l} = \tilde{a}^{(r)}_{s_i t_j}.$$

2.: Follows from Corollary 5.2.

3.: If $r \geq T(A)$ and r is a multiple of γ then the critical submatrix of A^r coincides with the same submatrix of $(A^\gamma)^*$. Hence the critical submatrices of both A^r and $\tilde{A}^{(r)}$ are Kleene stars. The critical submatrix of $\tilde{A}^{(r)}$ can be viewed as a submatrix of A^r, and hence of $(A^\gamma)^*$, extracted from rows and columns whose indices form a minimal representing family of cyclic classes of $C(A)$. As the cyclic classes are the node sets of strongly connected components of $C(A^\gamma)$, the critical submatrix of $\tilde{A}^{(r)}$ is strongly regular. Thus $\tilde{A}^{(r)}_{\mu\mu}$ are strongly regular Kleene stars. Part 1. adds to this that they are circulants. □

We note that the circulant Kleene stars which appear in Proposition 5.3 can be defined for any irreducible matrix. Indeed, for any subeigenvector x of A and any indices i,j in the same N_μ, the ratio $x_i x_j^{-1}$ is equal to $w(\pi, A) = a_{i_1 i_2} \ldots a_{i_{k-1} i_k}$ for any critical path $\pi = (i_1, \ldots, i_k)$ with $i_1 = i$ and $i_k = j$. Hence the (i,j) entries of $(X^{-1}AX)^r$, for $i,j \in N_\mu$ do not depend on the choice of the visualization scaling $X = \operatorname{diag}(x)$. Therefore, for any irreducible matrix A we can introduce the set of its *principal circulants*, as the circulant Kleene stars of Proposition 5.3 part 3, for any visualized form of A.

For any visualized $A \in \mathbb{R}^{n\times n}_+$, we can define the corresponding *core matrix* $A^C = (\alpha_{\mu\nu})$, $\mu,\nu = 1,\ldots,c+m$ by

$$\alpha_{\mu\nu} = \max\{a_{ij} \mid i \in N_\mu,\ j \in N_\nu\}. \tag{35}$$

The entries of $(A^C)^*$ will be denoted by $\alpha^*_{\mu\nu}$. Their role is investigated in the next proposition.

Proposition 5.4. *Let $A \in \mathbb{R}^{n\times n}_+$ be a visualized matrix and $r \geq T(A)$. Let $\mu,\nu = 1,\ldots,c+m$ be such that at least one of these indices is critical. Then the maximal entry of the block $A^{(r)}_{\mu\nu}$ (or $\tilde{A}^{(r)}_{\mu\nu}$) is equal to $\alpha^*_{\mu\nu}$.*

Proof. The entry $\alpha^*_{\mu\nu}$ is the maximal weight over paths from μ to ν, with respect to the matrix A^C. We take such a path $(\mu_1,\ldots,\mu_l)$ with maximal weight, where $\mu_1 := \mu$ and $\mu_l = \nu$. To this path we can associate a path π defined by $\pi = \tau_1 \circ \sigma_1 \circ \tau_2 \circ \ldots \circ \sigma_{l-1} \circ \tau_l$, where τ_i are critical paths which entirely belong to the components C_{μ_i}, and σ_i are edges with maximal weight connecting C_{μ_i} to $C_{\mu_{i+1}}$. Such a path exists since any two nodes in the same component C_μ can be connected to each other by critical paths if μ is critical, and if μ is non-critical then C_μ consists just of one node. The weights of τ_i are equal to 1, hence the weight of π is equal to $\alpha^*_{\mu\nu}$. It follows from the definition of $\alpha_{\mu\nu}$ and $\alpha^*_{\mu\nu}$ that this weight is maximal over all paths which connect nodes in C_μ to nodes in C_ν. As at least one of the indices μ,ν is critical, there is freedom in the choice of the paths τ_1 or τ_l which can be of arbitrary length. Hence, for any $r \geq T(A)$, any block $A^{(r)}_{\mu\nu}$ with μ or ν critical, contains an entry equal to $\alpha^*_{\mu\nu}$ which is the greatest entry of the block. □

5.3. **A system for attraction cone.** Next we show how the specific circulant structure of A^r at $r \geq T(A)$ can be exploited, to derive a more concise system of equations for the attraction cone $\operatorname{Attr}(A,1)$. Due to Proposition 5.4 the core matrix $A^C = \{\alpha_{\mu\nu} \mid \mu,\nu = 1,\ldots,c\}$, and its Kleene star $(A^C)^* = \{\alpha^*_{\mu\nu} \mid \mu,\nu = 1,\ldots,c\}$ will be of special importance. We introduce the notation

$$
\begin{aligned}
&S^{(r)}_{\mu\nu}(i) = \{j \in N_\nu \mid a^{(r)}_{ij} = \alpha^*_{\mu\nu}\},\ \forall i \in N_\mu,\ \forall \nu : C_\nu \neq C_\mu,\\
&T^{(r)}_{\mu}(i) = \{t \notin N_c(A) \mid a^{(r)}_{it} = \alpha^*_{\mu\nu(t)}\},\ \forall i \in N_\mu,
\end{aligned}
\tag{36}
$$

where C_μ and C_ν are s.c.c. of $C(A)$, N_μ and N_ν are their node sets, and $\nu(t)$ in the second definition denotes the index of the non-critical component which consists of the node t.

The results of Subsect. 5.2 lead to the following properties of $S^{(r)}_{\mu\nu}$ and $T^{(r)}_\mu(i)$.

Proposition 5.5. *Let $r \geq T(A)$.*

1. *If $[i] \to_t [j]$ and $i,j \in N_\mu$ then $S^{(r+t)}_{\mu\nu}(i) = S^{(r)}_{\mu\nu}(j)$ and $T^{(r+t)}_\mu(i) = T^{(r)}_\mu(j)$.*
2. *$S^{(r)}_{\mu\nu}(i)$ are composed of cyclic classes of C_ν.*
3. *Let $d = g.c.d.(\gamma_\mu,\gamma_\nu)$. Then, if $[p] \in S^{(r)}_{\mu\nu}(i)$ and $[p] \to_d [s]$ then $[s] \in S^{(r)}_{\mu\nu}(i)$.*
4. *Let $i,j \in N_\mu$ and $p,s \in N_\nu$. Let $[i] \to_t [j]$ and $[p] \to_t [s]$. Then $p \in S^{(r)}_{\mu\nu}(i)$ if and only if $s \in S^{(r)}_{\mu\nu}(j)$.*

Next we recall some max-algebraic cancellation rules which will enable us to write out a concise system of equations for the attraction cone $\operatorname{Attr}(A,1)$.

If $a < c$, then

$$ax \oplus b = cx \oplus d \Leftrightarrow b = cx \oplus d. \tag{37}$$

Now consider a system of equations over max algebra:

$$\bigoplus_{i=1}^{n} a_{1i}x_i \oplus c_1 = \bigoplus_{i=1}^{n} a_{2i}x_i \oplus c_2 = \ldots = \bigoplus_{i=1}^{n} a_{ni}x_i \oplus c_n. \tag{38}$$

Suppose that $\alpha_1,\ldots,\alpha_n \in \mathbb{R}_+$ are such that $a_{li} \leq \alpha_i$ for all l and i, and $S_l = \{i \mid a_{li} = \alpha_i\}$ for $l = 1,\ldots,n$. Let S_l be such that $\bigcup_{l=1}^n S_l = \{1,\ldots,n\}$. Repeatedly applying the elementary cancellation law described above, we obtain that (38) is equivalent to

$$\bigoplus_{i\in S_1} \alpha_i x_i \oplus c_1 = \bigoplus_{i\in S_2} \alpha_i x_i \oplus c_2 = \ldots = \bigoplus_{i\in S_n} \alpha_i x_i \oplus c_n. \tag{39}$$

We will refer to the equivalence between (38) and (39), which we acknowledge to Dokka [17], as to *(multisided) cancellation.*

Using notation (36) and Proposition 5.5 we can formulate the following.

Proposition 5.6. *Let $A \in \mathbb{R}_+^{n\times n}$ be a visualized matrix and and $r \geq T(A)$ be a multiple of γ. Then the system $A^r \otimes x = A^{r+1} \otimes x$ is equivalent to*

$$\begin{aligned} &\bigoplus_{k\in[i]} x_k \oplus \bigoplus_{C_\nu\neq C_\mu} \alpha^*_{\mu\nu}\left(\bigoplus_{k\in S^{(r)}_{\mu\nu}(i)} x_k\right) \oplus \bigoplus_{t\in T^{(r)}_\mu(i)} \alpha^*_{\mu\nu(t)} x_t = \\ &= \bigoplus_{k\in[j]} x_k \oplus \bigoplus_{C_\nu\neq C_\mu} \alpha^*_{\mu\nu}\left(\bigoplus_{k\in S^{(r)}_{\mu\nu}(j)} x_k\right) \oplus \bigoplus_{t\in T^{(r)}_\mu(j)} \alpha^*_{\mu\nu(t)} x_t, \end{aligned} \tag{40}$$

where C_μ is the component of $C(A)$ which contains both $[i]$ and $[j]$, and $[i]$ and $[j]$ range over all pairs of cyclic classes such that $[i] \to_1 [j]$.

Proof. By Proposition 4.9 $A^r \otimes x = A^{r+1} \otimes x$ is equivalent to its critical subsystem. Consider a critical component of $A^r \otimes x = A^{r+1} \otimes x$:

$$\bigoplus_k a^{(r)}_{ik} x_k = \bigoplus_k a^{(r+1)}_{ik} x_k,\ i \in N_c(A). \tag{41}$$

Consider j such that $[i] \to_1 [j]$. Then by Proposition 4.5, $a^{(r+1)}_{ik} = a^{(r)}_{jk}$, hence the critical subsystem of $A^r \otimes x = A^{r+1} \otimes x$ is as follows:

$$\bigoplus_k a^{(r)}_{ik} x_k = \bigoplus_k a^{(r)}_{jk} x_k,\ \forall i,j:\ [i] \to_1 [j]. \tag{42}$$

Proposition 5.3, part 3, implies that, after applying an appropriate permutation scaling, all principal submatrices of A^r extracted from critical components have a circulant block structure. In this structure, all entries of the diagonal blocks are equal to 1, and the entries of all off-diagonal blocks are strictly less than 1. Hence we can apply the cancellation (equivalence between (38) and (39)) and obtain the first terms on both sides of (40). By Proposition 5.4 each block $A_{\mu\nu}$ contains an entry equal to $\alpha^*_{\mu\nu}$. For a non-critical $\nu(t)$, this readily implies that the corresponding "subcolumn" $A_{\mu\nu(t)}$ contains an entry $\alpha^*_{\mu\nu(t)}$. Applying the cancellation we obtain the last terms on both sides of (40). Due to the block circulant structure of $A_{\mu\nu}$ with both μ and ν critical, see Proposition 5.3 and Proposition 5.5, we see that each column of such block also contains an entry equal to $\alpha^*_{\mu\nu}$. Applying the cancellation we obtain the remaining terms in (40). $\square$

As $\mathrm{Attr}(A,t) = \mathrm{Attr}(A^t,1)$, system (40) also describes more general attraction cones, it only amounts to substitute $C(A^t)$ for $C(A)$ and the entries of $((A^t)^C)^*$ for $\alpha^*_{\mu\nu}$(the dimension of this matrix is different, in general, see Proposition 3.3 part 3).

5.4. **When $C(A)$ is strongly connected.** Coefficients of the system of equations which defines attraction cone are determined by the entries of $(A^C)^*$ which can be found in $O(m^3)$ operations. However it remains to find the places where these coefficients appear, i.e., the sets $S^{(r)}_{\mu\nu}(i)$ and $T_\nu(i)$. Defining this, we can get a polynomial method for computing the coefficients of (40) which requires $O(m^3)$ operations with real numbers.

Here we restrict our attention to the case when $C(A)$ is strongly connected. In this case there are no second terms on both sides of (40) and we need only $T^{(r)}_\nu(i)$. The digraph $D(A^C)$ associated with the matrix A^C consists of one critical node which corresponds to the whole $C(A)$ and will be denoted by μ, and m non-critical nodes $\nu(t)$, for $t \notin N_c(A)$. The cyclicity of $C(A)$ is γ. The entries of A^C are given by

$$\begin{aligned} \alpha_{\mu\mu} &= 1, \\ \alpha_{\mu\nu(t)} &= \max_{k\in N_c(A)} a_{kt},\ \alpha_{\nu(t)\mu} = \max_{k\in N_c(A)} a_{tk},\ t\notin N_c(A), \\ \alpha_{\nu(s)\nu(t)} &= a_{st},\ s\notin N_c(A),\ t\notin N_c(A). \end{aligned} \tag{43}$$

$C(A)$ has γ cyclic classes $[s_1],\ldots,[s_\gamma]$, which we assume to be numbered in such a way that $[s_i]\to_1 [s_{i+1}]$. We also put $[s_{i+l}] = [s_k]$ where $k = i+l (\mathrm{mod}\ \gamma)$. For each $t \notin N_c(A)$, we initialize boolean γ-vectors P_t by

$$P_t(i) = \begin{cases} 1, & \text{if } [s_{i-1}] \cap \arg\max_{k\in N_c(A)} a_{kt} \neq \emptyset \\ 0, & \text{otherwise.} \end{cases} \tag{44}$$

Initialized in this way, P_t contain information on starting cyclic classes of paths with maximal weights among those, which connect the nodes in $N_c(A)$ directly to node t and whose length is a multiple of γ. By saying "directly" we mean that only the end node of the path is non-critical.

Further we compute the Kleene star of the non-critical submatrix $B := A_{MM}$, where M denotes the set of non-critical nodes, and store the information on the lengths of paths with maximal weight in boolean m-vectors U_{st} associated to each entry of B. We recall the max-algebraic version of Schur complement [10], which computes

$$(A^C)^* = \begin{pmatrix} 1 & h^T \\ g & B \end{pmatrix}^* = \begin{pmatrix} 1 & h^TB^* \\ B^*g & B^*\oplus B^*gh^TB^* \end{pmatrix}, \tag{45}$$

where $h, g \in \mathbb{R}^m_+$. Note that all information that we need for system (40), is the entries of $h^T \otimes B^*$ and the indices of equations of the system where the entries of $h^T\otimes B^*$ appear. Computing $(h^T\otimes B^*)_i$ means in particular obtaining the "winning" indices

$$W_t = \arg\max_{s\in M} h_s b^*_{st}. \tag{46}$$

After that, the idea is to combine P_p with U_{pt} for all $p \in W_t$ and unite the obtained indices in P_t^1. More precisely, for each number m stored in U_{pt} we define the shifted set $P_p^{\to m}$ by

$$j \in P_p^{\to m} \Leftrightarrow i \in P_p \text{ and } j - i = m (\mathrm{mod}\ \gamma). \tag{47}$$

The set P_t^1 is computed by

$$P_t^1 := \bigcup_{p \in W_t} \bigcup_{m \in U_{pt}} P_p^{\to m}. \tag{48}$$

This set encodes information on starting cyclic classes of the paths whose weight is maximal among those, which connect the nodes in $N_c(A)$ to t and whose length is a multiple of γ. These paths are absolute winners, with no restriction on non-critical nodes that they pass through. Therefore, the sets $T_i^{(r)}$ defined by (36) with $r \geq T(A)$, can now be computed by

$$t \in T_i^{(r)} \Leftrightarrow \exists k \in P^1_{\mu\nu(t)} \text{ s.t. } [i] = [s_k]. \tag{49}$$

Summarizing above said, we have the following algorithm for computing the coefficients of (40) in the case when $C(A)$ is strongly connected. Recall that in this case there is no second term on both sides of (40). The computation of coefficients of the third term includes the computation of $h^T B^*$ and the sets P_t^1 for each $t \notin N_c(A)$.

ALGORITHM *Compute the coefficients of* (40) *if* $C(A)$ *is strongly connected.*

Input. Visualized matrix A, critical graph $C(A)$ which is strongly connected and the cyclic classes of $C(A)$.
1. Compute h and initialize P_t for $t \in M$. This takes $m(n-m)$ operations both with real numbers and integers.
2. Compute B^* and initialize the boolean vectors U_{st} for all $s, t \in M$. It takes $O(m^3)$ operations both with real numbers and integers.
3. Compute $h^T \otimes B^*$ and initialize P_t^1 for $t \in M$, by (46), (47) and (48). Computation of $h^T \otimes B^*$ and W_t by (46) requires m^2 operations both with real numbers and integers, computation of shifted vectors P_p requires γm^2 operations with integers and booleans, and the union (48) takes γm^3 operations with booleans.
4. Compute $T_i^{(r)}$ by (49). This requires $m(n-m)$ operations with booleans.

We conclude the following.

Theorem 5.7. *Let $A \in \mathbb{R}_+^{n \times n}$ be visualized, $C(A)$ be strongly connected, m be the number of non-critical nodes, and suppose we know $C(A)$ and all γ cyclic classes. Then there is algorithm which decides, whether or not a given vector belongs to* $\mathrm{Attr}(A,1)$, *in*

$O(m^3)+O(n^2)$ operations with real numbers and up to $O(m^3\gamma)$ operations with integers and booleans.

It is also important that the eigenvalue and an eigenvector of irreducible matrix can be computed by the policy iteration algorithm of [11], which is experimentally very fast. After that, $C(A)$ and the cyclic classes can be computed in $O(n^2)$ time. Thus we are led to an efficient method of solving the reachability problem in the case when A is irreducible and $C(A)$ is strongly connected, especially in the case when the number of non-critical nodes is small. Note that the case of irreducible A and strongly connected $C(A)$ is generic when matrices A are real and generated at random. Also, in this generic case it almost never happens that maxima in blocks or among the weights of paths are achieved twice, which means that we do not need to assign boolean vectors to each entry, and reduces the total number of operations after the visualization, both with integers and with reals, to $O(m^3)+O(n^2)$, where m is the number of *non-critical* nodes.

6. Examples

6.1. **Matrix squaring.** In this subsection we will examine the problems that can be solved by matrix squaring on 9×9 real matrix over the *max-plus semiring*:

$$A=\begin{pmatrix} -1 & 0 & -1 & -1 & -9 & -7 & -10 & -4 & -8\\ 0 & -1 & 0 & -1 & -10 & -1 & -10 & -9 & -4\\ -1 & -1 & -1 & 0 & -2 & -3 & -2 & -6 & -6\\ 0 & -1 & -1 & -1 & -10 & -6 & -10 & -6 & -1\\ -10 & -2 & -8 & -1 & -1 & 0 & -1 & -10 & -1\\ -5 & -5 & -10 & -9 & -1 & -1 & 0 & -3 & -6\\ -9 & -10 & -7 & -10 & 0 & -1 & -1 & -8 & -8\\ -75 & -80 & -77 & -83 & -80 & -77 & -82 & -2 & -0.5\\ -84 & -81 & -77 & -80 & -78 & -77 & -78 & -0.5 & -2 \end{pmatrix}$$

The corresponding max-times example is obtained by, e.g., taking *exponents* of the entries.

The critical graph of A, see Figure 3, has two s.c.c.: C_1 with nodes $N_1=\{1,2,3,4\}$ and C_2 with nodes $N_2=\{5,6,7\}$. The cyclicity of C_1 is $\gamma_1=2$ and the cyclicity of C_2 is $\gamma_2=3$, so the cyclicity of $C(A)$ is $\gamma=\mathrm{lcm}(2,3)=2\times 3=6$.

The matrix can be decomposed into blocks

$$A=\begin{pmatrix} A_{11} & A_{12} & A_{1M}\\ A_{21} & A_{22} & A_{2M}\\ A_{M1} & A_{M1} & A_{MM} \end{pmatrix},$$

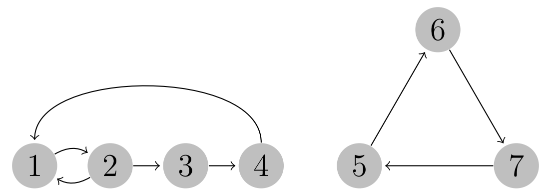

FIGURE 3. The critical graph of A

where the submatrices A_{11} and A_{22} correspond to two s.c.c. C_1 and C_2 of $C(A)$, see Figure 3. They equal

$$A_{11} = \begin{pmatrix} -1 & 0 & -1 & -1 \\ 0 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 \\ 0 & -1 & -1 & -1 \end{pmatrix}, \quad A_{22} = \begin{pmatrix} -1 & 0 & -1 \\ -1 & -1 & 0 \\ 0 & -1 & -1 \end{pmatrix},$$

and A_{MM} is the non-critical principal submatrix

$$A_{MM} = \begin{pmatrix} -2 & -0.5 \\ -0.5 & -2 \end{pmatrix}.$$

The submatrices A_{12}, A_{21}, A_{1M} and A_{2M} are composed of randomly taken numbers from -1 to -10, and A_{M1} and A_{M2} are composed of randomly taken numbers from -75 to -85.

It can be checked that the powers of A become periodic after $T(A) = 154$.

We will consider the following instances of problems P2 and P3.

P2. Compute A^r for $r \geq T(A)$ and $r \equiv 2 (\text{mod } 6)$.

P3. For given $x \in \mathbb{R}^9_+$, find ultimate orbit period of $A^k \otimes x$.

Solving P2. Using the idea of Proposition 4.10, we perform 7 squarings $A, A^2, A^4, \ldots$ to raise A to the power $128 > 9 \times 9$. This brings us to the matrix

$$A^{128} = \begin{pmatrix} A_{11}^{(128)} & A_{12}^{(128)} & A_{1M}^{(128)} \\ A_{21}^{(128)} & A_{22}^{(128)} & A_{2M}^{(128)} \\ A_{M1}^{(128)} & A_{M2}^{(128)} & A_{MM}^{(128)} \end{pmatrix},$$

where

$$A_{11}^{(128)} = \begin{pmatrix} 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 \end{pmatrix}, \quad A_{22}^{(128)} = \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ -1 & 0 & -1 \end{pmatrix},$$

all entries of $A_{12}^{(128)}$ and $A_{21}^{(128)}$ are -1 and

$$A_{1M}^{(128)} = \begin{pmatrix} -2.5 & -1 \\ -1.5 & -2 \\ -2.5 & -1 \\ -1.5 & -2 \end{pmatrix}, \quad A_{2M}^{(128)} = \begin{pmatrix} -1.5 & -2 \\ -2.5 & -2 \\ -2.5 & -1 \end{pmatrix}$$

$$A_{M1}^{(128)} = \begin{pmatrix} -76 & -75.5 \\ -75 & -76.5 \\ -76 & -75.5 \\ -75 & -76.5 \end{pmatrix}^T, \quad A_{M2}^{(128)} = \begin{pmatrix} -76 & -76.5 \\ -76 & -76.5 \\ -76 & -76.5 \end{pmatrix}^T$$

We are lucky since $128 \equiv 2 (\mathrm{mod}\ 6)$, as we already have true critical columns and rows of A^r. However, the non-critical principal submatrix of A^{128} is

$$A_{MM}^{(128)} = \begin{pmatrix} -64 & -65.5 \\ -65.5 & -64 \end{pmatrix}.$$

It can be checked that this is *not* the non-critical submatrix of A^r that we seek (recall that $T(A) = 154$). Hence, it remains to compute the principal non-critical submatrix $A_{MM}^{(r)}$.

We note that A^{132} has critical rows and columns of the spectral projector $Q(A)$, since 132 is a multiple of $\gamma = 6$. In A^{132}, the critical rows and columns $1-4$ (in C_1) are the same as that of A^{128}, since $\gamma_1 = 2$ and both 128 and 132 are even. The critical rows $5-7$ (in C_2) can be computed from those of A^{128} by cyclic permutation $(5,6,7) \to (7,5,6)$, and the critical rows $5-7$ can be computed by the inverse permutation $(5,6,7) \to (6,7,5)$. This implies that all blocks in A^{132} are the same as in A^{128} above (in the analogous block decomposition of A^{132}), except for

$$A_{22}^{(132)} = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}, \quad A_{2M}^{(132)} = \begin{pmatrix} -2.5 & -2 \\ -2.5 & -1 \\ -1.5 & -2 \end{pmatrix}.$$

Now the remaining non-critical submatrix of A^r can be computed using linear dependence (23), which specifies to

$$A_{\cdot k}^{(r)} = \bigoplus_{i=1}^{7} a_{ik}^{(132)} A_{\cdot i}^{(128)}, \quad k = 8, 9.$$

This yields

$$A_{MM}^{(r)} = \begin{pmatrix} -76.5 & -77 \\ -78 & -76.5 \end{pmatrix}$$

Solving P3 We examine the orbit period of $A^k x$ for $x = x^1, x^2, x^3, x^4$, where

$$x^1 = [1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9],$$

$$x^2 = [1 \quad 2 \quad 3 \quad 4 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0],$$

$$x^3 = [0 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1 \quad 1],$$

$$x^4 = [0 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0].$$

We compute $y = A^{128}x$ for $x = x^1, x^2, x^3, x^4$:

$$y^1 = A^{128} \otimes x^1 = [8 \quad 7 \quad 8 \quad 7 \quad 7 \quad 7 \quad 8 \quad \times \quad \times],$$

$$y^2 = A^{128} \otimes x^2 = [3 \quad 4 \quad 3 \quad 4 \quad 3 \quad 3 \quad 3 \quad \times \quad \times],$$

$$y^3 = A^{128} \otimes x^3 = [1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 0 \quad 0 \quad \times \quad \times],$$

$$y^4 = A^{128} \otimes x^4 = [1 \quad 1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0 \quad \times \quad \times].$$

Here $\times$ correspond to non-critical entries which we do not need. The cyclic classes of C_1 are $\{1,3\}$, $\{2,4\}$, and the cyclic classes of C_2 are $\{5\}$, $\{6\}$ and $\{7\}$. From the considerations of Proposition 4.10, it follows that the coordinate sequences $\{(A^r x)_i,\ r \geq T(A)\}$ are

$$y_1,\ y_2,\ y_1,\ y_2, \ldots, \text{ for } i = 1,2,3,4,$$

$$y_5,\ y_6,\ y_7,\ y_5,\ y_6,\ y_7, \ldots, \text{ for } i = 5,6,7.$$

Looking at $y^1, \ldots, y^4$ above, we conclude that the orbit of x^1 is of the largest possible period 6, the orbit of x^2 is of the period 2 (in other words, $x^2 \in \mathrm{Attr}(A,2)$), the orbit of x^3 is of the period 3 (i.e., $x^3 \in \mathrm{Attr}(A,3)$), and the orbit of x^4 is of the period 1 (i.e., $x^1 \in \mathrm{Attr}(A,1)$).

6.2. **Circulants.** Here we consider another 9×9 example

$$A = \begin{pmatrix} -8 & 0 & -1 & -8 & -8 & -9 & -4 & -5 & -1 \\ -4 & -5 & 0 & -2 & -6 & 0 & -7 & -3 & -9 \\ -7 & -9 & -8 & 0 & -8 & -4 & -6 & -9 & -10 \\ -8 & -8 & -10 & -7 & 0 & -4 & -6 & -10 & -1 \\ -2 & -8 & -7 & -4 & -8 & 0 & -3 & -1 & -10 \\ 0 & -1 & -2 & -7 & -10 & -6 & -3 & -6 & -1 \\ -10 & -7 & -7 & -7 & -6 & -1 & -5 & 0 & -9 \\ -8 & -3 & -6 & -8 & -6 & -8 & -5 & -10 & 0 \\ -4 & -3 & -5 & -6 & -6 & -10 & 0 & -6 & -9 \end{pmatrix} \tag{50}$$

The critical graph of this matrix consists of two s.c.c. comprising 6 and 3 nodes respectively. They are shown in Figures 4 and 5, together with their cyclic classes.

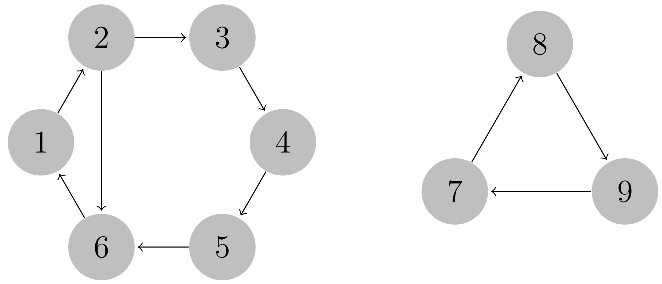

FIGURE 4. Critical graph of (50)

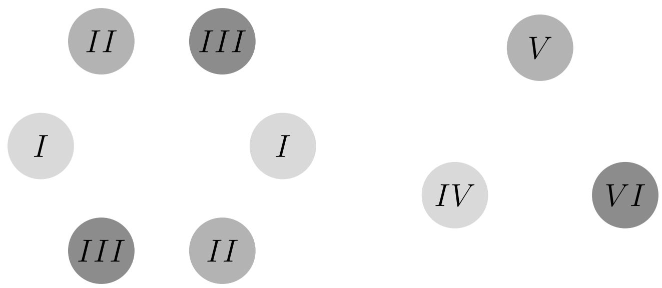

FIGURE 5. Cyclic classes of the critical graph

The components of $C(A)$ induce block decomposition

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \tag{51}$$

where

$$A_{11} = \begin{pmatrix} -8 & 0 & -1 & -8 & -8 & -9 \\ -4 & -5 & 0 & -2 & -6 & 0 \\ -7 & -9 & -8 & 0 & -8 & -4 \\ -8 & -8 & -10 & -7 & 0 & -4 \\ -2 & -8 & -7 & -4 & -8 & 0 \\ 0 & -1 & -2 & -7 & -10 & -6 \end{pmatrix}, \quad A_{22} = \begin{pmatrix} -5 & 0 & -9 \\ -5 & -10 & 0 \\ 0 & -6 & -9 \end{pmatrix} \tag{52}$$

The core matrix and its Kleene star are equal to

$$A^C = (A^C)^* = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}. \tag{53}$$

The powers of A become periodic after $T(A) = 6$. With the block decomposition of A^6 analogous to (51),

(54)
$$A_{11}^{(6)} = \begin{pmatrix} 0 & -1 & -2 & 0 & -1 & -2 \\ -2 & 0 & -1 & -2 & 0 & -1 \\ -1 & -2 & 0 & -1 & -2 & 0 \\ 0 & -1 & -2 & 0 & -1 & -2 \\ -2 & 0 & -1 & -2 & 0 & -1 \\ -1 & -2 & 0 & -1 & -2 & 0 \end{pmatrix}, \quad A_{12}^{(6)} = \begin{pmatrix} -2 & -1 & -1 \\ -1 & -2 & -1 \\ -1 & -1 & -2 \\ -2 & -1 & -1 \\ -1 & -2 & -1 \\ -1 & -1 & -2 \end{pmatrix},$$
$$A_{21}^{(6)} = \begin{pmatrix} -3 & -1 & -2 & -3 & -1 & -2 \\ -2 & -3 & -1 & -2 & -3 & -1 \\ -1 & -2 & -3 & -1 & -2 & -3 \end{pmatrix}, \quad A_{22}^{(6)} = \begin{pmatrix} 0 & -3 & -2 \\ -2 & 0 & -3 \\ -3 & -2 & 0 \end{pmatrix}.$$

The corresponding blocks of "reduced" power $\tilde{A}^{(6)}$ are

(55)
$$\tilde{A}_{11}^{(6)} = \begin{pmatrix} 0 & -1 & -2 \\ -2 & 0 & -1 \\ -1 & -2 & 0 \end{pmatrix}, \quad \tilde{A}_{12}^{(6)} = \begin{pmatrix} -2 & -1 & -1 \\ -1 & -2 & -1 \\ -1 & -1 & -2 \end{pmatrix},$$
$$\tilde{A}_{21}^{(6)} = \begin{pmatrix} -3 & -1 & -2 \\ -2 & -3 & -1 \\ -1 & -2 & -3 \end{pmatrix}, \quad \tilde{A}_{22}^{(6)} = \begin{pmatrix} 0 & -3 & -2 \\ -2 & 0 & -3 \\ -3 & -2 & 0 \end{pmatrix}.$$

All blocks in these decompositions are circulants. Note that $\tilde{A}_{11}^{(6)}$ and $\tilde{A}_{22}^{(6)}$ are strongly regular Kleene stars, which we called the principal circulants associated with A.

Specializing system (40) to our case, we see that this system of equations for the attraction cone $\text{Attr}(A, 1)$ consists of two chains of equations, namely

(56)
$$\begin{aligned} & x_1 \oplus x_4 \oplus (x_8 - 1) \oplus (x_9 - 1) = \\ & \qquad = x_2 \oplus x_5 \oplus (x_7 - 1) \oplus (x_9 - 1) = x_3 \oplus x_6 \oplus (x_7 - 1) \oplus (x_8 - 1), \\ & (x_2 - 1) \oplus (x_5 - 1) \oplus x_7 = \\ & \qquad = (x_3 - 1) \oplus (x_6 - 1) \oplus x_8 = (x_1 - 1) \oplus (x_4 - 1) \oplus x_9. \end{aligned}$$

Note that only the coefficients of $(A^C)^*$ (which is equal to A^C in our example) appear in this system.

6.3. **Algorithm for the strongly connected case.** Here we consider a 6×6 max-plus example

$$A=\begin{pmatrix}-3 & 0 & -1 & -19 & -7 & -3\\ -3 & -4 & 0 & -10 & -19 & -16\\ 0 & -3 & -1 & -10 & -8 & -8\\ -1 & -4 & -4 & -1 & -1 & -3\\ -1 & -1 & -4 & -2 & -4 & -1\\ -4 & -2 & -4 & -1 & -4 & -1\end{pmatrix}, \tag{57}$$

and apply to it the algorithm described in Subsect. 5.4. The critical graph of this matrix consists just of one cycle of length 3, and there are 3 non-critical nodes.

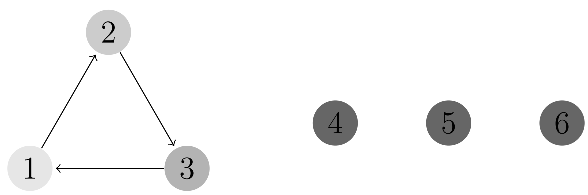

FIGURE 6. Critical graph and non-critical nodes of (57)

The core matrix in this case is equal to

$$A^C=\begin{pmatrix}0 & -10 & -7 & -3\\ -1 & -1 & -1 & -3\\ -1 & -2 & -4 & -1\\ -2 & -1 & -4 & -1\end{pmatrix}$$

Vector $h=(-10\ -7\ -3)^T$, whose components are computed by

$$h_i=\bigoplus_{k=1}^{3} a_{ki},\ \text{for } i=4,5,6, \tag{58}$$

comprises $2,3,4$-components of the first row of A^C. The arguments of maxima in (58) give, after the cyclic shift by one position, the boolean vectors

$$P_4=(1\ 0\ 1),\ P_5=(0\ 1\ 0),\ P_6=(0\ 1\ 0). \tag{59}$$

This vectors encode, for the corresponding non-critical nodes $t=4,5,6$, the starting cyclic classes (here, just critical nodes!) of paths which go from $C(A)$ directly to t and whose length is 3.

The non-critical principal submatrix of A and its Kleene star are equal to

$$B = \begin{pmatrix} -1 & -1 & -3 \\ -2 & -4 & -1 \\ -1 & -4 & -1 \end{pmatrix}, \quad B^* = \begin{pmatrix} 0 & -1 & -2 \\ -2 & 0 & -1 \\ -1 & -2 & 0 \end{pmatrix}$$

The lengths of optimal non-critical paths (whose weights are entries of B^*) can be written in the matrix

$$U = \begin{pmatrix} 0 & 1 & 2 \\ \{1,2\} & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix} \tag{60}$$

Further we compute

$$h^T \otimes B^* = (-10 \ -7 \ -3) \otimes \begin{pmatrix} 0 & -1 & -2 \\ -2 & 0 & -1 \\ -1 & -2 & 0 \end{pmatrix} = (-4 \ -5 \ -3)$$

The maxima in $\bigoplus_t h_t b^*_{ti}$ for all i are achieved only at $t = 6$, so $W_4 = W_5 = W_6 = \{6\}$. Hence P^1_4, P^1_5 and P^1_6 are shifted P_6 and the shift is determined by the components in the last row of U which is $(1\ 2\ 0)$. From $P_6 = (0\ 1\ 0)$ we conclude that

$$P^1_4 = (0\ 0\ 1),\ P^1_5 = (1\ 0\ 0),\ P^1_6 = (0\ 1\ 0).$$

Using this information and the vector of coefficients $h^T \otimes B^* = (-4 \ -5 \ -3)$, we can write out the system for attraction cone

$$x_1 \oplus (x_5 - 5) = x_2 \oplus (x_6 - 3) = x_3 \oplus (x_4 - 4). \tag{61}$$

On the other hand, in our case $T(A) = 8$ and

$$A^8 = \begin{pmatrix} -1 & -1 & 0 & -4 & -6 & -4 \\ 0 & -1 & -1 & -5 & -5 & -4 \\ -1 & 0 & -1 & -5 & -6 & -3 \\ -2 & -1 & -2 & -6 & -1 & -4 \\ -2 & -1 & -1 & -5 & -7 & -4 \\ -2 & -3 & -2 & -6 & -7 & -6 \end{pmatrix}$$

$$A^9 = \begin{pmatrix} 0 & -1 & -1 & -5 & -5 & -4 \\ -1 & 0 & -1 & -5 & -6 & -3 \\ -1 & -1 & 0 & -4 & -6 & -4 \\ -2 & -2 & -1 & -5 & -7 & -5 \\ -1 & -2 & -1 & -5 & -6 & -5 \\ -2 & -2 & -3 & -7 & -7 & -5 \end{pmatrix}$$

Applying cancellation to the critical subsystem of $A^8 \otimes x = A^9 \otimes x$, we obtain (61).

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