Max algebraic powers of irreducible matrices in the periodic regime: An application of cyclic classes[☆]

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ABSTRACT

In max algebra it is well known that the sequence of max algebraic powers A^k, with A an irreducible square matrix, becomes periodic after a finite transient time $T(A)$, and the ultimate period γ is equal to the cyclicity of the critical graph of A.

In this connection, we study computational complexity of the following problems: (1) for a given k, compute a periodic power A^r with $r \equiv k(\mathrm{mod}\ \gamma)$ and $r \geqslant T(A)$, (2) for a given x, find the ultimate period of $\{A^l \otimes x\}$. We show that both problems can be solved by matrix squaring in $O(n^3 \log n)$ operations. The main idea is to apply an appropriate diagonal similarity scaling $A \mapsto X^{-1}AX$, called visualization scaling, and to study the role of cyclic classes of the critical graph.

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1. Introduction

By *max algebra* we understand the analogue of linear algebra developed over the max-times semiring $\mathbb{R}_{\max,\times}$ which is the set of nonnegative numbers $\mathbb{R}_+$ equipped with the operations of "addition" $a \oplus b := \max(a, b)$ and the ordinary multiplication $a \otimes b := a \times b$. Zero and unity of this semiring coincide with the usual 0 and 1. The operations of the semiring are extended to the nonnegative matrices and vectors in the same way as in conventional linear algebra. That is if $A = (a_{ij}), B = (b_{ij})$ and $C = (c_{ij})$ are matrices of compatible sizes with entries from $\mathbb{R}_+$, we write $C = A \oplus B$ if $c_{ij} = a_{ij} \oplus b_{ij}$ for all i,j and $C = A \otimes B$ if $c_{ij} = \bigoplus_k a_{ik}b_{kj} = \max_k(a_{ik}b_{kj})$ for all i,j. If A is a square matrix over $\mathbb{R}_+$ then the iterated product $A \otimes A \otimes \cdots \otimes A$ in which the symbol A appears k times will be denoted by A^k.

The *max-plus semiring* $\mathbb{R}_{\max,+} = (\mathbb{R} \cup \{-\infty\}, \oplus = \max, \otimes = +)$, developed over the set of real numbers $\mathbb{R}$ with adjoined element $-\infty$ and the ordinary addition playing the role of multiplication, is another isomorphic "realization" of max algebra. In particular, $x \mapsto \exp(x)$ yields an isomorphism between $\mathbb{R}_{\max,+}$ and $\mathbb{R}_{\max,\times}$. In the max-plus setting, the zero element is $-\infty$ and the unity is 0.

The *min-plus semiring* $\mathbb{R}_{\min,+} = (\mathbb{R} \cup \{+\infty\}, \oplus = \min, \otimes = +)$ is also isomorphic to $\mathbb{R}_{\max,+}$ and $\mathbb{R}_{\max,\times}$. Another well-known semiring is the *max–min semiring* $\mathbb{R}_{\max,\min} = (\mathbb{R} \cup \{-\infty\} \cup \{\infty\}$, $\oplus = \max, \otimes = \min)$, see [21,35,36], but it is *not* isomorphic to any of the semirings above. For the basics of max algebra and its applications in scheduling and discrete event systems, see [3,12,21,22]. The current state of max algebra, idempotent analysis and related areas is represented in the recent collection of papers [26].

Max algebraic column spans of nonnegative matrices $A \in \mathbb{R}_+^{n \times n}$ are sets of max linear combinations of columns $\bigoplus_{i=1}^{n} \alpha_i A_{\cdot i}$ with nonnegative coefficients α_i. Such column spans are *max cones*, meaning that they are closed under componentwise maximum $\oplus$ and multiplication by nonnegative scalars. There are important analogies and links between max cones and convex cones [11,14,37,38].

The maximum cycle geometric mean $\lambda(A)$, see below for exact definition, is one of the most important characteristics of a matrix $A \in \mathbb{R}_+^{n \times n}$ in max algebra. In particular, it is the largest eigenvalue of the spectral problem $A \otimes x = \lambda x$. The cycles at which this maximum geometric mean is attained, are called *critical*. Further, one considers the *critical graph* $\mathcal{C}(A)$ which consists of all nodes and edges that belong to the critical cycles (such nodes and edges are also called critical). This graph is crucial for the description of eigenvectors [3,12,22].

The well-known *cyclicity theorem* states that if A is irreducible, then the sequence A^k becomes periodic after some finite transient time $T(A)$, and that the ultimate period γ of A^k is equal to the cyclicity of the critical graph [3,12,22]. Generalizations to reducible case, computational complexity issues and important special cases of this result have been extensively studied in [13,20,21,27–29].

Results of this kind are studied with great precision in Boolean matrix algebra, where one considers components of imprimitivity of a matrix [5,25], or equivalently, cyclic classes of the associated digraph [4]. In max algebra, cyclic classes of the critical graph have been considered as an important tool in the proof of the cyclicity theorem [22, Section 3.1]. Recently, the cyclic classes appeared in max–min algebra [35,36], where they were used to study the ultimate periods of orbits and other periodicity problems. It was shown that such questions can be solved by matrix squaring $(A, A^2, A^4, A^8, \ldots)$, which yields computational complexity $O(n^3 \log n)$.

The chief aim of this paper is to study the behaviour of max algebraic powers A^k and orbits $A^k \otimes x$ in the irreducible case in the periodic regime, i.e., after the periodicity is reached. One of the main ideas is to study the periodicity of *visualized* matrices, meaning matrices with all entries less than or equal to the maximum cycle geometric mean, since this provides a better connection to the theory of Boolean matrices [5,25].

Our starting point is an observation [30] that the critical rows and columns of A^l become periodic after at most $l = n^2$. This observation is also crucial for the $O(n^5)$ complexity result of [27]. Using the spectral projector of [3,10], we show that the critical rows and columns of A^l for $l \geqslant n^2$ generate row and column spans of powers in the periodic regime (i.e., for $l \geqslant T(A)$). If A is visualized, then these critical rows and columns "permute" according to the structure of cyclic classes of the critical graph. Based on these observations, we study the computational complexity of the following problems, for irreducible $A \in \mathbb{R}_+^{n \times n}$: (1) for a given k, compute power A^r for $r \geqslant T(A)$ and $r \equiv k \pmod{\gamma}$, (2) for a given $x \in \mathbb{R}_+^n$, compute the ultimate period of $\{A^k \otimes x\}$. We show that, like in max–min algebra [35,36], these problems can be solved by matrix squaring in $O(n^3 \log n)$ time.

In the future we aim to apply this technique to attraction cones, by which we mean max cones consisting of vectors with prescribed ultimate orbit period. See [38, Section 5] for some results in this direction.

The contents of this paper are as follows. In Section 2 we revise two important topics in max algebra, namely the spectral problem and Kleene stars. In Section 3, we speak of the visualization and the connection to the theory of Boolean matrices which it provides, see Propositions 3.1 and 3.3. In Section 4, we start by basic observations on matrix powers in the periodic regime, see Propositions 4.4–4.7 and Eq. (33). Main results on the problems which can be solved by matrix squaring are stated

in Theorem 4.11. We conclude with Section 5 which contains a 9×9 example illustrating the results of Section 4.

As $\mathbb{R}_{\max,+}$ and $\mathbb{R}_{\max,\times}$ are isomorphic, we use the possibility to switch between them, but only when it is really convenient. Thus, while the theoretical results are obtained over max-times semiring, which looks more natural in connection with diagonal matrix scaling and Boolean matrices, the example in Section 6 is written over *max-plus semiring*, where it is usually much easier to calculate.

2. Two topics in max algebra

2.1. Spectral problem

Let $A \in \mathbb{R}_+^{n \times n}$. Consider the problem of finding $\lambda \in \mathbb{R}_+$ and nonzero $x \in \mathbb{R}_+^n$ such that

$$A \otimes x = \lambda x. \tag{1}$$

If for some λ there exists a nonzero $x \in \mathbb{R}_+^n$ which satisfies (1), then λ is called a *max-algebraic eigenvalue* of A, and x is a *max-algebraic eigenvector* of A associated with λ. With the zero vector adjoined, the set of max-algebraic eigenvectors associated with λ forms a max cone, which is called the *eigencone* associated with λ.

The largest max-algebraic eigenvalue of $A \in \mathbb{R}_+^{n \times n}$ is equal to

$$\lambda(A) = \bigoplus_{k=1}^{n} (\mathrm{Tr}_\oplus A^k)^{1/k}, \tag{2}$$

where $\mathrm{Tr}_\oplus$ is defined by $\mathrm{Tr}_\oplus(A) := \bigoplus_{i=1}^{n} a_{ii}$ for any $A = (a_{ij}) \in \mathbb{R}_+^{n \times n}$ [3,12,22]. Further we explain the graph-theoretic meaning of (2), assuming that $\lambda(A) \neq 0$.

With $A = (a_{ij}) \in \mathbb{R}_+^{n \times n}$ we can associate the weighted digraph $D_A = (N(A), E(A))$, with the set of nodes $N(A) = \{1, \ldots, n\}$ and the set of edges $E(A) = \{(i,j) \mid a_{ij} \neq 0\}$ with weights $w(i,j) = a_{ij}$. Suppose that $\pi = (i_1, \ldots, i_p)$ is a path in D_A, then the *weight* of π is defined to be $w(\pi, A) = a_{i_1 i_2} a_{i_2 i_3} \ldots a_{i_{p-1} i_p}$ if $p > 1$, and 1 if $p = 1$. If $i_1 = i_p$ then π is called a cycle. One can check that

$$\lambda(A) = \max_\sigma \mu(\sigma, A),$$

where the maximization is taken over all cycles in D_A,

$$\mu(\sigma, A) = w(\sigma, A)^{1/k}$$

denotes the *geometric mean* of the cycle $\sigma = (i_1, \ldots, i_k, i_1)$, and we assume that max $\emptyset = 0$. Thus $\lambda(A)$ is the *maximum cycle geometric mean* of D_A.

$A \in \mathbb{R}_+^{n \times n}$ is *irreducible* if for any nodes i and j there exists a path in D_A, which begins at i and ends at j. In this case A has a unique max-algebraic eigenvalue which equals $\lambda(A)$ [3,12,22].

Note that $\lambda(\alpha A) = \alpha \lambda(A)$ and hence $\lambda(A/\lambda(A)) = 1$ if $\lambda(A) > 0$. Unless we need matrices with $\lambda(A) = 0$, we can always assume without loss of generality that $\lambda(A) = 1$. Such matrices will be called *definite*.

An important relaxation of (1) is

$$A \otimes x \leqslant \lambda x. \tag{3}$$

The nonzero vectors $x \in \mathbb{R}_+^n$ which satisfy (3) are called *subeigenvectors* associated with λ. With the zero vector adjoined, they form a max cone called *subeigencone*. This is a conventionally convex cone, meaning that it is closed under the *ordinary* addition. See [39] for more details.

The eigencone (resp. subeigencone) of A associated with $\lambda(A)$ will be denoted by $V(A)$ (resp. $V^*(A)$).

2.2. Kleene stars

Let $A \in \mathbb{R}_+^{n \times n}$. Consider the formal series

$$A^* = I \oplus A \oplus A^2 \oplus \cdots, \tag{4}$$

where I denotes the identity matrix. Series (4) is a max-algebraic analogue of $(I - A)^{-1}$, and it converges to a matrix with finite entries if and only if $\lambda(A) \leqslant 1$ [3,9,12,22]. In this case

$$A^* = I \oplus A \oplus A^2 \oplus \cdots \oplus A^{n-1}, \tag{5}$$

which is called the *Kleene star* of A.

For any $A \in \mathbb{R}_+^{n \times n}$,

$$\begin{aligned} A \text{ is a Kleene star } &\Leftrightarrow A^2 = A, \quad a_{ii} = 1, \; \forall i \\ &\Leftrightarrow a_{ij}a_{jk} \leqslant a_{ik}, \quad a_{ii} = 1, \; \forall i, j, k. \end{aligned} \tag{6}$$

The condition $\lambda(A) \leqslant 1$ suggests that there is a strong interplay between Kleene stars and spectral problems. To describe this in more detail, we need the following notions and notation.

A cycle σ in D_A is called *critical*, if $\mu(\sigma, A) = \lambda(A)$. Every node and edge that belongs to a critical cycle is called *critical*. The set of critical nodes is denoted by $N_c(A)$, the set of critical edges is denoted by $E_c(A)$. The *critical digraph* of A, further denoted by $\mathcal{C}(A) = (N_c(A), E_c(A))$, is the digraph which consists of all critical nodes and critical edges of D_A. For definite $A \in \mathbb{R}_+^{n \times n}$, it follows that $a_{ij}a_{ji}^* \leqslant 1$, $\forall i, j$ [3]. Further,

$$(i, j) \in E_c(A) \Leftrightarrow a_{ij}a_{ji}^* = 1. \tag{7}$$

Further we always assume that $N_c(A)$ occupies the first c indices.

For definite $A \in \mathbb{R}_+^{n \times n}$, the relation between Kleene star, critical graph and spectral problems is briefly as follows [3,12,39]:

$$V^*(A) = \text{span}(A^*) = \left\{ \bigoplus_{i=1}^n \alpha_i A_{\cdot i}^*, \; \alpha_i \in \mathbb{R}_+ \right\}, \tag{8}$$

$$V(A) = \left\{ \bigoplus_{i=1}^c \alpha_i A_{\cdot i}^*, \; \alpha_i \in \mathbb{R}_+ \right\}, \tag{9}$$

$$x \in V^*(A), \quad (i, j) \in E_c(A) \Rightarrow a_{ij}x_j = x_i. \tag{10}$$

Eq. (8) means that $V^*(A)$ is the max-algebraic column span of Kleene star A^*, also called *Kleene cone*. This cone is convex in conventional sense. By (9), $V(A)$ is the max subcone of $V^*(A)$, spanned by the columns with critical indices. Implication (10) means that for any subeigenvector $x \in V^*(A)$ and $i \in N_c(A)$, the maximum in $\bigoplus_j a_{ij}x_j$ is attained at j if $(i, j) \in E_c(A)$. In particular, $(A \otimes x)_i = x_i$ for all $x \in V^*(A)$ and $i \leqslant c$.

Not all columns in (8) and (9) are necessary. Let $\mathcal{C}(A)$ have $n_c \in \{1, \ldots, n\}$ strongly connected components (s.c.c.) $\mathcal{C}_\mu$, for $\mu = 1, \ldots, n_c$. It follows from the definition of $\mathcal{C}(A)$ that s.c.c. $\mathcal{C}_\mu$ are disjoint. The corresponding node sets will be denoted by N_μ. Let m denote the number of non-critical nodes of D_A. It can be shown [3,12,22] that if i, j belong to the same s.c.c. of $\mathcal{C}(A)$, then the columns $A_{\cdot i}^*$ and $A_{\cdot j}^*$ are multiples of each other. The same holds for the rows $A_{i \cdot}^*$ and $A_{j \cdot}^*$. Hence

$$V^*(A) = \left\{ \bigoplus_{i \in K} \alpha_i A_{\cdot i}^*, \; \alpha_i \in \mathbb{R}_+ \right\}, \tag{11}$$

$$V(A) = \left\{ \bigoplus_{i \leqslant c, \; i \in K} \alpha_i A_{\cdot i}^*, \; \alpha_i \in \mathbb{R}_+ \right\}, \tag{12}$$

where K is any set of indices which contains all non-critical indices and for every $\mathcal{C}_\mu$ there is a unique index of this component in K.

Consider A_{KK}^*, the principal submatrix of A^* extracted from the rows and columns with indices in K. Condition (6) implies that A_{KK}^* is itself a Kleene star. It follows from the maximality of $\mathcal{C}_\mu$ that there is a unique permutation of K that has the greatest weight with respect to A_{KK}^*. The *weight of a permutation* π of $\{1, \ldots, n\}$ with respect to $A \in \mathbb{R}_+^{n \times n}$ is defined as $\prod_{i=1}^n a_{i\pi(i)}$. Thus A_{KK}^* is *strongly*

regular in the sense of Butkovič [6]. From this it can be deduced that the columns of A^* with indices in K are *independent*, meaning that none of them can be expressed as a max combination of the other columns. In other words [8], the columns of A^* with indices in K (resp., in K and less than or equal to c) form a *basis* of $V^*(A)$ (resp., of $V(A)$). This basis is essentially unique [8], meaning that any other basis can be obtained from it by scalar multiplication.

More precisely, the strong regularity of A^*_{KK} is equivalent to saying that this basis is *tropically independent*, hence the tropical rank of A^* is equal to $n_c + m$, see [2,23,24] for definitions and further details.

3. Visualization and Boolean matrices

3.1. Visualization

Consider a positive $x \in \mathbb{R}^n_+$ and define

$$X = \operatorname{diag}(x) := \begin{pmatrix} x_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & x_n \end{pmatrix}. \tag{13}$$

The transformation $A \mapsto X^{-1}AX$ is called a *diagonal similarity scaling* of A. Such transformations do not change $\lambda(A)$ and $\mathcal{C}(A)$ [17]. They commute with max-algebraic multiplication of matrices and hence with the operation of taking the Kleene star. Geometrically, they correspond to automorphisms of $\mathbb{R}^n_+$, both in the case of max algebra and in the case of nonnegative linear algebra. The importance of such scalings is emphasized in [12, Chapter 28]. Further we define scalings which lead to particularly convenient forms of matrices.

A definite matrix $A \in \mathbb{R}^{n \times n}_+$ is called *visualized*, if

$$a_{ij} \leqslant 1, \quad \forall i, j = 1, \ldots, n, \tag{14}$$
$$a_{ij} = 1, \quad \forall (i, j) \in E_c(A). \tag{15}$$

A visualized matrix $A \in \mathbb{R}^{n \times n}_+$ is called *strictly visualized* if

$$a_{ij} = 1 \Leftrightarrow (i, j) \in E_c(A). \tag{16}$$

Visualization scalings were known already to Afriat [1] and Fiedler–Pták [19], and motivated extensive study of matrix scalings in nonnegative linear algebra, see e.g. [17,18,32,34]. We remark that some constructions and facts related to application of visualization scaling in max algebra have been observed in connection with max algebraic power method [15,16], behaviour of matrix powers [7] and max-balancing [32,34].

Visualization scalings are described in [39] in terms of the subeigencone $V^*(A)$ and its relative interior. For the convenience of the reader, we show their existence for any definite $A \in \mathbb{R}^{n \times n}_+$. In the proposition stated below, the summation in part 2 is *conventional*.

Proposition 3.1. *Let $A \in \mathbb{R}^{n \times n}_+$ be definite and $X = \operatorname{diag}(x)$.*

1. *If $x = \bigoplus_{i=1}^n A^*_{\cdot i}$ then $X^{-1}AX$ is visualized.*
2. *If $x = \sum_{i=1}^n A^*_{\cdot i}$ then $X^{-1}AX$ is strictly visualized.*

Proof. 1. Observe that $x \in V^*(A)$ and x is positive. Then $a_{ij}x_j \leqslant x_i$ for all i, j implies $x_i^{-1}a_{ij}x_j \leqslant 1$, and by (10) $x_i^{-1}a_{ij}x_j = 1$ for all $(i, j) \in E_c(A)$.
2. Observe that x is positive, and that $x \in V^*(A)$ since $V^*(A)$ is convex. Hence $X^{-1}AX$ is visualized. It remains to check that $(i, j) \notin E_c(A)$ implies $a_{ij}x_j < x_i$. We need to find k such that $a_{ij}a^*_{jk} < a^*_{ik}$. But this is true for $k = i$, since $a^*_{ii} = 1$ and $a_{ij}a^*_{ji} < 1$ by (7). This completes the proof. $\square$

More precisely [39], $A \in \mathbb{R}_+^{n \times n}$ can be visualized by any positive vector in $V^*(A)$, and it can be strictly visualized by any vector in the relative interior of $V^*(A)$.

3.2. Max algebra and Boolean matrices

Max algebra is related to the algebra of Boolean matrices. The latter algebra is defined over the Boolean semiring $\mathbb{S}$ which is the set $\{0, 1\}$ equipped with logical operations "OR" $a \oplus b := a \vee b$ and "AND" $a \otimes b := a \wedge b$. Clearly, Boolean matrices can be treated as objects of max algebra, as a very special but crucial case.

For a strongly connected graph, its cyclicity is defined as the g.c.d. of the lengths of all cycles (or equivalently, all simple cycles). If the cyclicity is 1 then the graph is called *primitive*, otherwise it is called *imprimitive*. We will not distinguish between cyclicity (or primitivity) of a Boolean matrix A and the associated digraph D_A. Further we recall an important result of Boolean matrix theory.

Proposition 3.2 [5, Theorem 3.4.5]. *Let $S \in \mathbb{S}^{n \times n}$ be irreducible, and let γ_A be the cyclicity of D_A (which is strongly connected). Then for each $k \geqslant 1$, there exists a permutation matrix P such that $P^{-1}A^kP$ has r irreducible diagonal blocks, where $r = \gcd(k, \gamma_A)$, and all elements outside these blocks are zero. The cyclicity of all these blocks is γ_A/r.*

In max algebra, let $A \in \mathbb{R}_+^{n \times n}$. Define the *critical matrix* $A^{[C]} = (a_{ij}^{[C]}) \in \mathbb{S}^{c \times c}$ and the matrix $A^{[1]} = (a_{ij}^{[1]}) \in \mathbb{S}^{c \times c}$ associated with A by

$$a_{ij}^{[C]} = \begin{cases} 1, & (i,j) \in E_c(A), \\ 0, & (i,j) \notin E_c(A), \end{cases} \quad i, j \leqslant c, \tag{17}$$

$$a_{ij}^{[1]} = \begin{cases} 1, & a_{ij} = 1, \\ 0, & a_{ij} \neq 1, \end{cases} \quad i, j \leqslant c. \tag{18}$$

Let $A, B \in \mathbb{R}_+^{n \times n}$. Assume that $\mathcal{C}(A)$ has n_c s.c.c. $\mathcal{C}_\mu$ for $\mu = 1, \ldots, n_c$, with cyclicities γ_μ. Denote by $B_{\mu\nu}$ the block of B extracted from the rows with indices in $\mathcal{C}_\mu$ and columns with indices in $\mathcal{C}_\nu$.

The following proposition can be seen as a corollary of Proposition 3.2. The idea of the proof given below is due to Schneider [33]. See also [7, Theorem 2.3], [22, Section 3.1].

Proposition 3.3. *Let $A \in \mathbb{R}_+^{n \times n}$ and $\lambda(A) \neq 0$.*

1. $\lambda(A^k) = \lambda^k(A)$.
2. $(A^{[C]})^k = (A^k)^{[C]}$.
3. *For each $k \geqslant 1$, there exists a permutation matrix P such that $(P^{-1}A^kP)_{\mu\mu}^{[C]}$, for each $\mu = 1, \ldots, n_c$, has $r_\mu := \gcd(k, \gamma_\mu)$ irreducible blocks and all elements outside these blocks are zero. The cyclicity of all blocks in $(P^{-1}A^kP)_{\mu\mu}^{[C]}$ is equal to γ_μ/r_μ.*

Proof. We can assume that A is definite. Further, the diagonal similarity scaling commutes with max algebraic matrix multiplication and changes neither $\lambda(A)$ nor $\mathcal{C}(A)$ [17], and by Proposition 3.1, part 2, there exists a strict visualization scaling. Hence we can assume that A is strictly visualized. In this case $A^{[C]} = A^{[1]}$ and it is easily seen that $(A^{[1]})^k = (A^k)^{[1]}$. As $A^{[1]} = A^{[C]}$, all entries of $A^{[1]}$ outside the blocks $A_{\mu\mu}^{[1]}$ are zero, which assures that $(A^{[1]})_{\mu\mu}^k = (A_{\mu\mu}^{[1]})^k$.

Proposition 3.2 implies that part 3 is true for $(A^{[1]})^k = (A^k)^{[1]}$. This implies that $P^{-1}(A^k)^{[1]}P$ has irreducible blocks and $\lambda(A^k) = 1$, which shows part 1. Also, $P^{-1}(A^k)^{[1]}P$ has block structure where all diagonal blocks are irreducible and all off-diagonal blocks are zero. This implies $(A^k)^{[C]} = (A^k)^{[1]}$, and parts 2 and 3 follow immediately. □
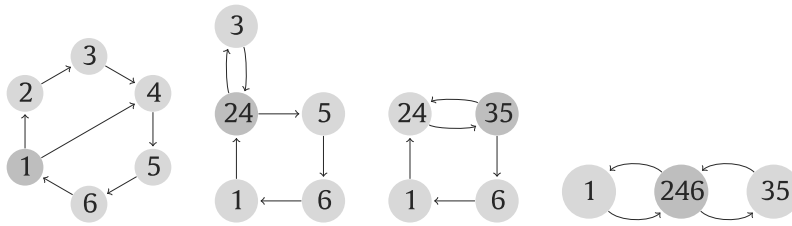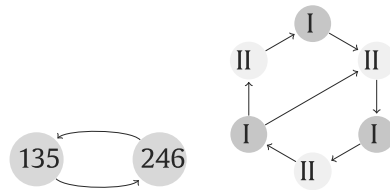
Fig. 1. Balcer–Veinott algorithm.

Fig. 2. Result of the algorithm (left) and cyclic classes (right).

3.3. Cyclic classes

For a path P in a digraph $G = (N, E)$, where $N = \{1, \ldots, n\}$, denote by $l(P)$ the length of P, i.e., the number of edges traversed by P.

Proposition 3.4 [5]. *Let $G = (N, E)$ be a strongly connected digraph with cyclicity γ_G. Then the lengths of any two paths connecting $i \in N$ to $j \in N$ (with i, j fixed) are congruent modulo γ_G.*

Proposition 3.4 implies that the following *equivalence relation* can be defined: $i \sim j$ if there exists a path P from i to j such that $l(P) \equiv 0 (\mathrm{mod}\ \gamma_G)$. The equivalence classes of G with respect to this relation are called *cyclic classes* [4,35,36]. The cyclic class of i will be denoted by $[i]$.

Consider the following *access relations* between cyclic classes: $[i] \rightarrow_t [j]$ if there exists a path P from a node in $[i]$ to a node in $[j]$ such that $l(P) \equiv t (\mathrm{mod}\ \gamma_G)$. In this case, a path P with $l(P) \equiv t (\mathrm{mod}\ \gamma_G)$ exists between any node in $[i]$ and any node in $[j]$. Further, by Proposition 3.4 the length of any path between a node in $[i]$ and a node in $[j]$ is congruent to t, so the relation $[i] \rightarrow_t [j]$ is well-defined. Classes $[i]$ and $[j]$ will be called *adjacent* if $[i] \rightarrow_1 [j]$.

Cyclic classes can be computed in $O(|E|)$ time by Balcer–Veinott digraph condensation, where $|E|$ denotes the number of edges in G. At each step of this algorithm, we look for all edges which issue from a certain node i, and condense all end nodes of these edges into a single node. A precise description of this method can be found in [4,5]. We give an example of its work, see Figs. 1 and 2.

In this example, see Fig. 1 at the left, we start by condensing nodes 2 and 4, which are "next to" node 1, into the node 24. Further we proceed with condensing nodes 3 and 5 into the node 35. In the end, see Fig. 2 at the left, there are just two nodes 135 and 246. They correspond to two cyclic classes $\{1,3,5\}$ and $\{2,4,6\}$ of the initial graph, see Fig. 2 at the right.

The notion of cyclic classes and access relations can be generalized to the case when G has n_c disjoint components G_μ with cyclicities γ_μ, for $\mu = 1, \ldots, n_c$ (just like the critical graph in max algebra). In this case we write $i \sim j$ if i, j belong to the same component and there exists a path P from i to j such that $l(P) \equiv 0 (\mathrm{mod}\ \gamma_\mu)$. If $l(P) \equiv t (\mathrm{mod}\ \gamma_\mu)$, then we write $[i] \rightarrow_t [j]$. In this case the *cyclicity* of G is $\gamma := \mathrm{lcm}\ \gamma_\mu, \mu = 1, \ldots, n_c$.

We will be interested in the cyclic classes of critical graphs, and below we interpret these in terms of the Boolean matrix $A^{[C]}$. Let $A \in \mathbb{R}_+^{n \times n}$. Following Brualdi and Ryser [5] we can find such ordering of the indices that any submatrix $A_{\mu\mu}^{[C]}$, which corresponds to an imprimitive component $\mathcal{C}_\mu$ of $\mathcal{C}(A)$, will be of the form

$$
\begin{pmatrix}
\mathbf{0} & A^{[C]}_{s_1 s_2} & \mathbf{0} & \cdots & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & A^{[C]}_{s_2 s_3} & \cdots & \mathbf{0} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & A^{[C]}_{s_{k-1} s_k} \\
A^{[C]}_{s_k s_1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0}
\end{pmatrix}, \tag{19}
$$

where k is the number of cyclic classes in $\mathcal{C}_\mu$. Indices s_i and s_{i+1} for $i = 1, \ldots, k-1$, and s_k and s_1 correspond to adjacent cyclic classes. By Proposition 3.3, part 2, when A is raised to power k, $A^{[C]}$ is also raised to the same power over the Boolean algebra. Any power of $A^{[C]}$ has a similar block-permutation form. In particular, $(A^{\gamma_\mu})^{[C]}_{\mu\mu}$ looks like

$$
\begin{pmatrix}
(A^{\gamma_\mu})^{[C]}_{s_1 s_1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\mathbf{0} & (A^{\gamma_\mu})^{[C]}_{s_2 s_2} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & (A^{\gamma_\mu})^{[C]}_{s_k s_k}
\end{pmatrix} \tag{20}
$$

Theorem 5.4.11 of [25] implies that the sequence $(A^k)^{[C]} = (A^{[C]})^k$ becomes periodic after $k = (n-1)^2 + 1$, with period $\gamma = \mathrm{lcm}(\gamma_\mu)$, $\mu = 1, \ldots, n_c$. This period, which is equal to the cyclicity of $\mathcal{C}(A)$, will be denoted by γ everywhere in the sequel.

In the periodic regime, all entries of nonzero blocks of $(A^{[C]})^k$ are equal to 1. In particular, it means the following.

Proposition 3.5. *Let* $A \in \mathbb{R}_+^{n \times n}$ *be definite and let* $t \geqslant 0$ *be such that* $t\gamma \geqslant n^2$. *Let* $i, j \leqslant c$ *be such that* $[i] \rightarrow_l [j], l \leqslant \gamma$, *hence also* $[j] \rightarrow_s [i]$ *where* $l + s = \gamma$.

1. *If A is visualized, then* $a_{ij}^{(t\gamma+l)} = a_{ji}^{(t\gamma+s)} = 1$.
2. *In the general case,* $a_{ij}^{(t\gamma+l)} a_{ji}^{(t\gamma+s)} = 1$.

Proof. Part 1. follows from the theory of Boolean matrices [5,25]. To obtain part 2, we apply to A a visualization scaling, which does not change the value of $a_{ij}^{(t\gamma+l)} a_{ji}^{(t\gamma+s)}$. $\square$

4. Periodicity and complexity

4.1. Spectral projector and matrix periodicity

For a definite and irreducible $A \in \mathbb{R}_+^{n \times n}$, consider the matrix $Q(A)$ with entries

$$
q_{ij} = \bigoplus_{k=1}^{c} a_{ik}^* a_{kj}^*, \quad i, j = 1, \ldots, n. \tag{21}
$$

The max-linear operator whose matrix is $Q(A)$, is a max-linear *spectral projector* associated with A, in the sense that it projects $\mathbb{R}_+^n$ on the eigencone $V(A)$ [3].

This operator is closely related to the periodicity questions, as the following fact suggests. Under more restrictive conditions, it appears in [12, Section 27.3], where $Q(A)$ is called the *orbital matrix*.

Proposition 4.1 [3, Theorem 3.109]. *Let* $A \in \mathbb{R}_+^{n \times n}$ *be irreducible and definite, and let all s.c.c. of* $\mathcal{C}(A)$ *be primitive. Then there is an integer* $T(A)$ *such that* $A^r = Q(A)$ *for all* $r \geqslant T(A)$.

We will also need the following property of $Q(A)$ which follows directly from (21).

Proposition 4.2. *For $A \in \mathbb{R}_+^{n \times n}$ irreducible and definite, any critical column (or row) of $Q(A)$ is equal to the corresponding column (or row) of A^*.*

We also note that $Q(A)$ is important for the policy iteration algorithm of [10].

When $\mathcal{C}(A)$ has imprimitive components, it follows from Proposition 3.3, part 3 that all components of $C(A^\gamma)$ are primitive, where γ is the cyclicity of $\mathcal{C}(A)$. Hence, for any r great enough which is a multiple of γ, A^r is the matrix of the spectral projector onto the eigencone of A^γ. This also implies that for large enough r we have $A^r = A^{r+\gamma}$. The number r, after which this starts, is called the *transient* of $\{A^r\}$. It will be denoted by $T(A)$. Also, it is well known that γ is the *ultimate period* of $\{A^r\}$, i.e., it is the least integer α such that $A^{r+\alpha} = A^r$ for all $r \geqslant T(A)$.

It is also important that the entries $a_{ij}^{(r)}$, where i or j are critical, become periodic much faster than the non-critical part of A. The following proposition is a known result, which is proved here for convenience of the reader.

Proposition 4.3 [30]. *Let $A \in \mathbb{R}_+^{n \times n}$ be a definite irreducible matrix. Critical rows and columns of A^r become periodic for $r \geqslant n^2$.*

Proof. We prove the claim for rows, and for columns everything is analogous. Let $i \leqslant c$, Then there is a critical cycle of length l_i to which i belongs. Hence $a_{ii}^{(kl_i)} = 1$ for $k \geqslant 1$. Since for all $m < k$ and any $t = 1, \ldots, n$ we have

$$a_{is}^{(ml_i)} = a_{ii}^{((k-m)l_i)} a_{is}^{(ml_i)} \leqslant a_{is}^{(kl_i)},$$

it follows that

$$a_{is}^{(kl_i)} = \bigoplus_{m=1}^{k} a_{is}^{(ml_i)}. \tag{22}$$

Entries $a_{is}^{(kl_i)}$ are maximal weights of paths of length k with respect to the matrix A^{l_i}. Since the weights of all cycles are less than or equal to 1 and all paths of length n are not simple, the maximum is achieved at $k \leqslant n$. Using (22) we obtain that $a_{is}^{((t+1)l_i)} = a_{is}^{(tl_i)}$ for all $t \geqslant n$. Further,

$$a_{is}^{(tl_i+d)} = \bigoplus_p a_{ip}^{(tl_i)} a_{ps}^{(d)}$$

and it follows that $a_{is}^{((t+1)l_i+d)} = a_{is}^{(tl_i+d)}$ for all $t \geqslant n$ and $0 \leqslant d \leqslant l_i - 1$. Hence $a_{is}^{(k)}$ is periodic for $k \geqslant nl_i$, and all these sequences, for any $i \leqslant c$ and any s, become periodic for $k \geqslant n^2$. $\square$

4.2. Properties of periodic powers

Periodic powers of definite and irreducible matrices are described by the following propositions.

Proposition 4.4. *Let $A \in \mathbb{R}_+^{n \times n}$ be a definite and irreducible matrix, and let $t \geqslant 0$ be such that $t\gamma \geqslant T(A)$. Then for every integer $l \geqslant 0$*

$$A_{k\cdot}^{t\gamma+l} = \bigoplus_{i=1}^{c} a_{ki}^{(t\gamma)} A_{i\cdot}^{t\gamma+l}, \quad A_{\cdot k}^{t\gamma+l} = \bigoplus_{i=1}^{c} a_{ik}^{(t\gamma)} A_{\cdot i}^{t\gamma+l}, \quad 1 \leqslant k \leqslant n. \tag{23}$$

Proof. Due to Proposition 4.1, for $B = A^\gamma$ and any $t \geqslant T(B)$ we have

$$b_{kj}^{(r)} = \bigoplus_{i=1}^{c} b_{ki}^* b_{ij}^*, \quad 1 \leqslant k, j \leqslant n. \tag{24}$$

By Propositions 4.1 and 4.2, we have $b_{ki}^* = b_{ki}^{(t)} = a_{ki}^{(t\gamma)}$ and $b_{ij}^* = b_{ij}^{(t)} = a_{ij}^{(t\gamma)}$ for all $t \geqslant T(B)$ or equivalently $t\gamma \geqslant T(A)$, and any $i \leqslant c$. Hence

$$a_{kj}^{(t\gamma)} = \bigoplus_{i=1}^{c} a_{ki}^{(t\gamma)} a_{ij}^{(t\gamma)}, \quad 1 \leqslant k, j \leqslant n. \tag{25}$$

In the matrix notation, this is equivalent to:

$$A_{k\cdot}^{t\gamma} = \bigoplus_{i=1}^{c} a_{ki}^{(t\gamma)} A_{i\cdot}^{t\gamma}, \quad A_{\cdot k}^{t\gamma} = \bigoplus_{i=1}^{c} a_{ik}^{(t\gamma)} A_{\cdot i}^{t\gamma}, \quad 1 \leqslant k \leqslant n. \tag{26}$$

Multiplying (26) by any power A^l, we obtain (23). $\quad\square$

In the proof of the next proposition we will use the following simple principle

$$a_{ij}^{(r)} a_{jk}^{(s)} \leqslant a_{ik}^{(r+s)}, \quad \forall i, j, k, r, s, \tag{27}$$

which holds for the matrix powers in max algebra.

Proposition 4.5. *Let $A \in \mathbb{R}_+^{n \times n}$ be a definite and irreducible matrix, and let $i, j \in N_c(A)$ be such that $[i] \rightarrow_l [j]$, for some $0 \leqslant l < \gamma$.*

1. There exists t such that for any $r \geqslant n^2$

$$a_{ij}^{(t\gamma+l)} A_{\cdot i}^r = A_{\cdot j}^{r+l}, \quad a_{ij}^{(t\gamma+l)} A_{j\cdot}^r = A_{i\cdot}^{r+l}. \tag{28}$$

2. If A is visualized, then for all $r \geqslant n^2$

$$A_{\cdot i}^r = A_{\cdot j}^{r+l}, \quad A_{j\cdot}^r = A_{i\cdot}^{r+l}. \tag{29}$$

Proof. Let $s = \gamma - l$. By Proposition 3.5 there exists t such that $a_{ij}^{(t\gamma+l)} a_{ji}^{(t\gamma+s)} = 1$. Combining this with (27) we obtain

$$\begin{aligned} A_{\cdot i}^r &= A_{\cdot i}^r a_{ij}^{(t\gamma+l)} a_{ji}^{(t\gamma+s)} \leqslant A_{\cdot j}^{r+t\gamma+l} a_{ji}^{(t\gamma+s)} \leqslant A_{\cdot i}^{r+(2t+1)\gamma}, \\ A_{j\cdot}^r &= A_{j\cdot}^r a_{ij}^{(t\gamma+l)} a_{ji}^{(t\gamma+s)} \leqslant A_{i\cdot}^{r+t\gamma+l} a_{ji}^{(t\gamma+s)} \leqslant A_{j\cdot}^{r+(2t+1)\gamma}. \end{aligned} \tag{30}$$

By Proposition 4.3, $A_{\cdot i}^r = A_{\cdot i}^{r+t\gamma}$ and $A_{j\cdot}^r = A_{j\cdot}^{r+t\gamma}$ for all $t \geqslant 0$ and $r \geqslant n^2$, hence all inequalities in (30) are equalities. Multiplying them by $a_{ij}^{(t\gamma+l)}$ we obtain (28), from the equalities between the first and the third terms. In the visualized case $a_{ij}^{(t\gamma+l)} = a_{ji}^{(t\gamma+s)} = 1$, hence (29). $\quad\square$

Letting $l = 0$ in Proposition 4.5 we obtain the following.

Corollary 4.6. *Let $A \in \mathbb{R}_+^{n \times n}$ and $r \geqslant n^2$. All rows of A^r with indices in the same cyclic class are equal to each other, and the same statement holds for the columns.*

Proposition 4.5 says that in any power A^r for $r \geqslant n^2$, the critical columns (or rows) can be obtained from the critical columns (or rows) of the spectral projector $Q(A^\gamma)$ by permuting the sets of columns (or rows) which correspond to the cyclic classes of $\mathcal{C}(A)$. Proposition 4.4 adds to this that all non-critical columns (or rows) of any periodic power are in the max cone spanned by the critical columns (or rows). From this we conclude the following.

Proposition 4.7. *All powers A^r for $r \geqslant T(A)$ have the same column span, which is the eigencone $V(A^\gamma)$.*

Proposition 4.7 enables us to say that $V(A^\gamma)$ is the *ultimate column span* of A. Similarly, we have the *ultimate row span* which is $V((A^T)^\gamma)$. These cones are generated by critical columns (or rows) of the Kleene star $(A^\gamma)^*$. For a basis of this cone, we can take any set of columns $(A^\gamma)^*$ (equivalently $Q(A^\gamma)$ or $A^{t\gamma}$ for $t\gamma \geqslant T(A)$), whose indices form a minimal set of representatives of all cyclic classes of $\mathcal{C}(A)$. This basis is tropically independent in the sense of [2,23,24].

Propositions 4.5 and 4.4 also admit the following nice matrix formulation which follows an idea of [33]. Take $t\gamma \geqslant T(A)$. For any $l \geqslant 0$, let $C^{(l)} \in \mathbb{R}_+^{n \times c}$, or resp. $R^{(l)} \in \mathbb{R}_+^{c \times n}$, be the matrices extracted from the critical columns of $A^{t\gamma + l}$, or resp. from the critical rows of $A^{t\gamma + l}$. Let $C := C^{(0)}, R := R^{(0)}$. With this notation and from Propositions 4.4 and 4.5 respectively, we conclude that for an irreducible visualized $A \in \mathbb{R}_+^{n \times n}$

$$A^{t\gamma + l} = C^{(l)} \otimes R = C \otimes R^{(l)}, \tag{31}$$

$$C^{(l)} = C \otimes (A^{[C]})^l, \quad R^{(l)} = (A^{[C]})^l \otimes R. \tag{32}$$

Indeed, (31) is evidently equivalent to (23). Further observe that $[i] \rightarrow_l [j]$ if and only if there exist indices $p \in [i]$ and $s \in [j]$ such that the (p, s)-entry of $(A^{[C]})^l$ is 1, and that all rows of R (or columns of C) with indices in the same cyclic class are equal to each other, by Corollary 4.6. Hence (32) is equivalent to (29). Note that the role of $A^{[C]}$ in (32) can be played by any other Boolean matrix that corresponds to a graph with the same cyclic classes as $C(A)$. In particular, we can take the block-permutation matrix which has all entries equal to 1 in all nonzero blocks of (19).

Combining (31) with (32) we obtain

$$A^{t\gamma + l} = C \otimes (A^{[C]})^l \otimes R, \tag{33}$$

a concise matrix expression of Propositions 4.4 and 4.5.

Note that if A is not necessarily visualized but just definite irreducible, then equations (32) and (33) can be also written. Indeed, let $B = X^{-1}AX$ be visualised. Applying the inverse scaling XBX^{-1} to (33) (where B is substituted for A), we obtain equation of the same form, where C and R are the matrices extracted from the critical columns, resp. rows, of $A^{t\gamma}$, and $A^{[C]}$ is replaced by the matrix $A^{(C)} = (a_{ij}^{(C)}) \in \mathbb{R}_+^{c \times c}$ defined by

$$a_{ij}^{(C)} = \begin{cases} a_{ij}, & \text{if } (i, j) \in E_c(A), \\ 0, & \text{if } (i, j) \notin E_c(A), \end{cases} \quad i, j \leqslant c.$$

4.3. Solving periodicity problems by square multiplication

Let $A \in \mathbb{R}_+^{n \times n}$ and $\lambda(A) = 1$. The t-attraction cone $\text{Attr}(A, t)$ is the max cone which consists of all vectors x, for which there exists an integer r such that $A^r \otimes x = A^{r+t} \otimes x$, and hence this is also true for all integers greater than or equal to r. Actually we may speak of any $r \geqslant T(A)$, due to the following observation.

Proposition 4.8. *Let A be irreducible and definite. The systems $A^r \otimes x = A^{r+t} \otimes x$ are equivalent for all $r \geqslant T(A)$.*

Proof. Let x satisfy $A^s \otimes x = A^{s+t} \otimes x$ for some $s \geqslant T(A)$, then it also satisfies this system for all greater s. Due to the periodicity, for all k from $T(A) \leqslant k \leqslant s$ there exists $l > s$ such that $A^k = A^l$. Hence $A^k \otimes x = A^{k+t} \otimes x$ also hold for $T(A) \leqslant k \leqslant s$. □

Corollary 4.9. $\text{Attr}(A, t) = \text{Attr}(A^t, 1)$.

Proof. By Proposition 4.8, $\text{Attr}(A, t)$ is solution set to the system $A^r \otimes x = A^{r+t} \otimes x$ for any $r \geqslant T(A)$ which is a multiple of t, hence the statement. □

A component (that is, equation) of $A^r \otimes x = A^{r+t} \otimes x$ with index in $N_c(A) = \{1, \dots, c\}$ will be called *critical*, and the subsystem of components with these indices will be called the *critical subsystem*.

Lemma 4.10. *Let A be irreducible and definite and let $r \geqslant T(A)$. Then $A^r \otimes x = A^{r+t} \otimes x$ is equivalent to its critical subsystem*.

Proof. Consider a non-critical component $A_{k.}^r \otimes x = A_{k.}^{r+t} \otimes x$. Using (23) it can be written as

$$\bigoplus_{i=1}^{c} a_{ki}^{(r)} A_{i.}^r \otimes x = \bigoplus_{i=1}^{c} a_{ki}^{(r)} A_{i.}^{r+t} \otimes x, \tag{34}$$

hence it is a max combination of equations in the critical subsystem. $\square$

Next we give a bound on the computational complexity of deciding whether $x \in \mathrm{Attr}(A, t)$, as well as other related problems which we formulate below.

P1. For a given x, decide whether $x \in \mathrm{Attr}(A, t)$.

P2. For a given $k: \ 0 \leqslant k < \gamma$, compute periodic power A^r where $r \equiv k \pmod{\gamma}$.

P3. For a given x compute the ultimate period of $\{A^r \otimes x, \ r \geqslant 0\}$, meaning the least integer α such that $A^{r+\alpha} \otimes x = A^r \otimes x$ for all $r \geqslant T(A)$.

Theorem 4.11. *For any irreducible matrix $A \in \mathbb{R}_+^{n \times n}$, the problems P1–P3 can be solved in $O(n^3 \log n)$ time.*

Proof. First note that we can compute both $\lambda(A)$ and a subeigenvector, and identify all critical nodes in no more than $O(n^3)$ operations, which is done essentially by Karp and Floyd–Warshall algorithms [31]. Further we can identify all cyclic classes of $\mathcal{C}(A)$ by Balcer–Veinott condensation in $O(n^2)$ operations.

By Proposition 4.3 the critical rows and columns become periodic for $r \geqslant n^2$. To know the critical rows and columns of a given power $r\prime \geqslant T(A)$, it suffices to compute A^r for arbitrary $r \geqslant n^2$ which can be done in $O(\log n)$ matrix squaring ($A, A^2, A^4, \ldots$) and takes $O(n^3 \log n)$ time, and following (29), to apply the corresponding permutation on cyclic classes which takes $O(n^2)$ overrides. By Lemma 4.10 we readily solve P1 by the verification of the critical subsystem of $A^{r'} \otimes x = A^{r'+t} \otimes x$ which takes $O(n^2)$ operations. Using linear dependence (23) the remaining non-critical submatrix of A^r, for any $r \geqslant T(A)$ such that $r \equiv k \pmod{\gamma}$, can be computed in $O(n^3)$ time. This solves P2.

As the non-critical rows of A are generated by the critical rows, the ultimate period of $\{A^r \otimes x\}$ is determined by the critical components. For visualized matrix we know that $A_{i.}^{r+t} = A_{j.}^r$ for all i, j such that $[i] \to_t [j]$. This implies $(A^{r+t} \otimes x)_i = (A^r \otimes x)_j$ for $[i] \to_t [j]$, meaning that, to determine the period we need only the critical subvector of $A^r \otimes x$ for any fixed $r \geqslant n^2$. Indeed, for any $i \leqslant c$ and $r \geqslant n^2$ the sequence $\{(A^{r+t} \otimes x)_i, \ t \geqslant 0\}$ can be represented as a sequence of critical coordinates of $A^r \otimes x$ determined by a permutation on γ_μ cyclic classes of the s.c.c. to which i belongs. To compute the period, we take a sample of γ_μ numbers appearing consecutively in the sequence, and check all possible periods, which takes no more than γ_μ^2 operations. The period of $A^r \otimes x$ appears as the l.c.m. of these periods. It remains to note that all operations above do not require more than $O(n^3)$ time. This solves P3. $\square$

5. Example

In this section we will examine the problems that can be solved by square multiplication on 9×9 real matrix over the *max-plus semiring*:

$$A = \begin{pmatrix}
-1 & 0 & -1 & -1 & -9 & -7 & -10 & -4 & -8 \\
0 & -1 & 0 & -1 & -10 & -1 & -10 & -9 & -4 \\
-1 & -1 & -1 & 0 & -2 & -3 & -2 & -6 & -6 \\
0 & -1 & -1 & -1 & -10 & -6 & -10 & -6 & -1 \\
-10 & -2 & -8 & -1 & -1 & 0 & -1 & -10 & -1 \\
-5 & -5 & -10 & -9 & -1 & -1 & 0 & -3 & -6 \\
-9 & -10 & -7 & -10 & 0 & -1 & -1 & -8 & -8 \\
-75 & -80 & -77 & -83 & -80 & -77 & -82 & -2 & -0.5 \\
-84 & -81 & -77 & -80 & -78 & -77 & -78 & -0.5 & -2
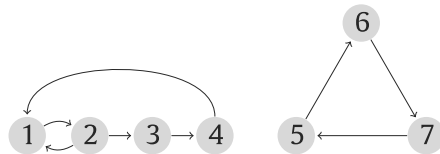\end{pmatrix}.$$

Fig. 3. The critical graph of A.

The corresponding max-times example is obtained by, e.g., taking *exponents* of the entries.

The critical graph of A, see Fig. 3, has two s.c.c.: $\mathcal{C}_1$ with nodes $N_1 = \{1, 2, 3, 4\}$ and $\mathcal{C}_2$ with nodes $N_2 = \{5, 6, 7\}$. The cyclicity of $\mathcal{C}_1$ is $\gamma_1 = 2$ and the cyclicity of $\mathcal{C}_2$ is $\gamma_2 = 3$, so the cyclicity of $\mathcal{C}(A)$ is $\gamma = \mathrm{lcm}(2, 3) = 2 \times 3 = 6$.

The matrix can be decomposed into blocks

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{1M} \\ A_{21} & A_{22} & A_{2M} \\ A_{M1} & A_{M1} & A_{MM} \end{pmatrix},$$

where the submatrices A_{11} and A_{22} correspond to two s.c.c. $\mathcal{C}_1$ and $\mathcal{C}_2$ of $\mathcal{C}(A)$, see Fig. 3. They equal

$$A_{11} = \begin{pmatrix} -1 & 0 & -1 & -1 \\ 0 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 \\ 0 & -1 & -1 & -1 \end{pmatrix}, \quad A_{22} = \begin{pmatrix} -1 & 0 & -1 \\ -1 & -1 & 0 \\ 0 & -1 & -1 \end{pmatrix},$$

and A_{MM} is the non-critical principal submatrix

$$A_{MM} = \begin{pmatrix} -2 & -0.5 \\ -0.5 & -2 \end{pmatrix}.$$

The submatrices A_{12}, A_{21}, A_{1M} and A_{2M} are composed of randomly taken numbers from -1 to -10, and A_{M1} and A_{M2} are composed of randomly taken numbers from -75 to -85.

It can be checked that the powers of A become periodic after $T(A) = 154$.

We will consider the following instances of problems P2 and P3.

P2. Compute A^r for $r \geqslant T(A)$ and $r \equiv 2 (\mathrm{mod}\ 6)$.

P3. For given $x \in \mathbb{R}_+^9$, find ultimate orbit period of $A^k \otimes x$.

Solving P2. Using the idea of Proposition 4.11, we perform 7 squarings $A, A^2, A^4, \ldots$ to raise A to the power $128 > 9 \times 9$. This brings us to the matrix

$$A^{128} = \begin{pmatrix} A_{11}^{(128)} & A_{12}^{(128)} & A_{1M}^{(128)} \\ A_{21}^{(128)} & A_{22}^{(128)} & A_{2M}^{(128)} \\ A_{M1}^{(128)} & A_{M2}^{(128)} & A_{MM}^{(128)} \end{pmatrix},$$

where

$$A_{11}^{(128)} = \begin{pmatrix} 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 \end{pmatrix}, \quad A_{22}^{(128)} = \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ -1 & 0 & -1 \end{pmatrix},$$

all entries of $A_{12}^{(128)}$ and $A_{21}^{(128)}$ are -1 and

$$A_{1M}^{(128)} = \begin{pmatrix} -2.5 & -1 \\ -1.5 & -2 \\ -2.5 & -1 \\ -1.5 & -2 \end{pmatrix}, \quad A_{2M}^{(128)} = \begin{pmatrix} -1.5 & -2 \\ -2.5 & -2 \\ -2.5 & -1 \end{pmatrix}$$

$$A_{M1}^{(128)} = \begin{pmatrix} -76 & -75.5 \\ -75 & -76.5 \\ -76 & -75.5 \\ -75 & -76.5 \end{pmatrix}^T , \quad A_{M2}^{(128)} = \begin{pmatrix} -76 & -76.5 \\ -76 & -76.5 \\ -76 & -76.5 \end{pmatrix}^T .$$

We are lucky since $128 \equiv 2 \pmod 6$, as we already have true critical columns and rows of A^r. However, the non-critical principal submatrix of A^{128} is

$$A_{MM}^{(128)} = \begin{pmatrix} -64 & -65.5 \\ -65.5 & -64 \end{pmatrix} .$$

It can be checked that this is *not* the non-critical submatrix of A^r that we seek (recall that $T(A) = 154$). Hence, it remains to compute the principal non-critical submatrix $A_{MM}^{(r)}$.

We note that A^{132} has critical rows and columns of the spectral projector $Q(A)$, since 132 is a multiple of $\gamma = 6$. In A^{132}, the critical rows and columns 1–4 (in $\mathcal{C}_1$) are the same as that of A^{128}, since $\gamma_1 = 2$ and both 128 and 132 are even. The critical rows 5–7 (in $\mathcal{C}_2$) can be computed from those of A^{128} by cyclic permutation $(5, 6, 7) \to (7, 5, 6)$, and the critical columns 5–7 can be computed by the inverse permutation $(5, 6, 7) \to (6, 7, 5)$. We conclude that all blocks in A^{132} are the same as in A^{128} above (in the analogous block decomposition of A^{132}), except for

$$A_{22}^{(132)} = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix} , \quad A_{2M}^{(132)} = \begin{pmatrix} -2.5 & -2 \\ -2.5 & -1 \\ -1.5 & -2 \end{pmatrix} .$$

Now the remaining non-critical submatrix of A^r can be computed using linear dependence (23), which specifies to

$$A_{\cdot k}^{(r)} = \bigoplus_{i=1}^{7} a_{ik}^{(132)} A_{\cdot i}^{(128)}, \quad k = 8, 9.$$

This yields

$$A_{MM}^{(r)} = \begin{pmatrix} -76.5 & -77 \\ -78 & -76.5 \end{pmatrix}$$

Solving P3. We examine the orbit period of $A^k x$ for $x = x^1, x^2, x^3, x^4$, where

$$\begin{aligned}
x^1 &= [1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9], \\
x^2 &= [1 \quad 2 \quad 3 \quad 4 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0], \\
x^3 &= [0 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1 \quad 1], \\
x^4 &= [0 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0].
\end{aligned}$$

We compute $y = A^{128} x$ for $x = x^1, x^2, x^3, x^4$:

$$\begin{aligned}
y^1 &= A^{128} \otimes x^1 = [8 \quad 7 \quad 8 \quad 7 \quad 7 \quad 7 \quad 8 \quad \times \quad \times], \\
y^2 &= A^{128} \otimes x^2 = [3 \quad 4 \quad 3 \quad 4 \quad 3 \quad 3 \quad 3 \quad \times \quad \times], \\
y^3 &= A^{128} \otimes x^3 = [1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 0 \quad 0 \quad \times \quad \times], \\
y^4 &= A^{128} \otimes x^4 = [1 \quad 1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0 \quad \times \quad \times].
\end{aligned}$$

Here $\times$ correspond to non-critical entries which we do not need. The cyclic classes of $\mathcal{C}_1$ are $\{1, 3\}$, $\{2, 4\}$, and the cyclic classes of $\mathcal{C}_2$ are $\{5\}$, $\{6\}$ and $\{7\}$. From the considerations of Proposition 4.11, it follows that the coordinate sequences $\{(A^r x)_i, \ r \geqslant T(A)\}$ are

$$y_1, \ y_2, \ y_1, \ y_2, \ldots, \quad \text{for } i = 1, 2, 3, 4,$$

$$y_5, \ y_6, \ y_7, \ y_5, \ y_6, \ y_7, \ldots, \quad \text{for } i = 5, 6, 7.$$

Looking at $y^1, \ldots, y^4$ above, we conclude that the orbit of x^1 is of the largest possible period 6, the orbit of x^2 is of the period 2 (in other words, $x^2 \in \mathrm{Attr}(A, 2)$), the orbit of x^3 is of the period 3 (i.e., $x^3 \in \mathrm{Attr}(A, 3)$), and the orbit of x^4 is of the period 1 (i.e., $x^1 \in \mathrm{Attr}(A, 1)$).

Acknowledgements

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