Journal of Mathematical Sciences, Vol. 155, No. 6, 2008

CYCLIC PROJECTORS AND SEPARATION THEOREMS IN IDEMPOTENT CONVEX GEOMETRY

S. Gaubert and S. Sergeev

UDC 512.643+512.558

Abstract. Semimodules over idempotent semirings like the max-plus or tropical semiring have much in common with convex cones. This analogy is particularly apparent in the case of subsemimodules of the n-fold Cartesian product of the max-plus semiring: It is known that one can separate a vector from a closed subsemimodule that does not contain it. Here we establish a more general separation theorem, which applies to any finite collection of closed subsemimodules with a trivial intersection. The proof of this theorem involves specific nonlinear operators, called here cyclic projectors on idempotent semimodules. These are analogues of the cyclic nearest-point projections known in convex analysis. We obtain a theorem that characterizes the spectrum of cyclic projectors on idempotent semimodules in terms of a suitable extension of Hilbert's projective metric. We also deduce as a corollary of our main results the idempotent analogue of Helly's theorem.

1. Introduction

Some nonlinear problems in optimization theory and mathematical physics turn out to be linear over semirings with an idempotent addition $\oplus$ [1, 9, 16]. We recall that the idempotency of $\oplus$ means $a \oplus a = a$ for all a, and that the role of this addition is most often played by the operations of taking maxima or minima. The search for idempotent analogues of classical results has motivated the development of idempotent mathematics (see the recent collection of articles [18] and also [17] for more background).

One of the most studied idempotent semirings is the max-plus semiring often denoted by $\mathbf{R}_{\max}$. It is the set $\mathbf{R} \cup \{-\infty\}$ equipped with the operations of addition $a \oplus b := \max(a, b)$ and multiplication $a \odot b := a + b$. The zero element $\mathbf{0}$ of this semiring is equal to $-\infty$, and the semiring unity $\mathbf{1}$ is equal to 0. Some algebraic structures that coincide with the max-plus semiring (up to isomorphism) have appeared under other names. In particular, the min-plus or tropical semiring is obtained by replacing $-\infty$ by $+\infty$ and $\max(a, b)$ by $\min(a, b)$ above. Applying $x \mapsto \exp(x)$ to the max-plus semiring (assume that $\exp(-\infty) = 0$), we obtain the max-times semiring, further denoted by $\mathbf{R}_{\max,\times}$. It is the set of nonnegative numbers ($\mathbf{R}_+$), equipped with the operations $a \oplus b = \max(a, b)$ and $a \odot b = a \times b$. The zero and unit elements of $\mathbf{R}_{\max,\times}$ coincide with the usual 0 and 1. Our main results (see Sec. 4) will be stated over this semiring, as it makes clearer some analogies with classical convex analysis.

We shall consider here subsemimodules of the n-fold Cartesian product $\mathcal{K}^n$ of a semiring $\mathcal{K}$ and, more generally, of the set $\mathcal{K}^I$ of functions from a set I to $\mathcal{K}$. Further examples can be found, e.g., in [1, 16, 19].

In an idempotent semiring, there is a canonical order relation, for which every element is "nonnegative." Therefore, idempotent semimodules have much in common with the semimodules over the semiring of nonnegative numbers, i.e., with *convex cones* [22]. One of the first results based on this idea is the separation theorem for convex sets over "extremal algebras" proved in [24]. This theorem implies that a point in $\mathbf{R}^n_{\max,\times}$, which does not belong to a semimodule that is closed in the Euclidean topology, can be separated from it by an idempotent analogue of a closed halfspace. Generalizations of this result were obtained in [6, 7, 23]. In the special case of finitely generated semimodules, a separation theorem has also been obtained in [11], with a strong emphasis on some combinatorial aspects of the result.

The main result of this paper, Theorem 20, shows that *several* closed semimodules that do not have common nonzero points can be separated from each other. This means that for each of these semimodules,

Translated from Fundamentalnaya i Prikladnaya Matematika, Vol. 13, No. 4, pp. 31–52, 2007.

we can select an idempotent halfspace containing it, in such a way that these halfspaces also do not have common nonzero points.

Even in the case of two semimodules, this statement has not been proved in the idempotent literature. Indeed, the earlier separation theorems deal with the separation of a point from an (idempotent) convex set or semimodule, rather than with the separation of two convex sets or semimodules. Note that unlike in the classical case, separating two convex sets cannot be reduced to separating a point from a convex set. More precisely, it is easily shown that two convex sets A and B can be separated if and only if the point 0 can be separated from their Minkowski difference $A - B$, in classical convex geometry. In idempotent geometry, an analogue of Minkowski difference can still be defined; consider $A \ominus B = \{x \mid \exists b \in B \colon x \oplus b \in A\}$. However, due to the idempotency of the addition, we cannot recover a halfspace separating A and B from a halfspace separating 0 from $A \ominus B$.

In order to prove the main result, Theorem 20, we use cyclic projectors on idempotent semimodules. By these we mean finite compositions of certain nonlinear projectors on idempotent semimodules. The continuity and homogeneity of these nonlinear projectors enables us to apply to their compositions, i.e., to the cyclic projectors, some results from nonlinear Perron–Frobenius theory. The main idea is to prove the equivalence of the following three statements:

(1) that the semimodules have trivial intersection;
(2) that the separating halfspaces exist;
(3) that the spectral radius of the associated cyclic projector is strictly less than 1.

This equivalence is established in Theorems 11 and 18, which deal with the special case of Archimedean semimodules, i.e., semimodules containing at least one positive vector. As an ingredient of the proof, we use a nonlinear extension of Collatz–Wielandt's theorem obtained in [21]. To derive the main separation result, Theorem 20, we show that for any collection of trivially intersecting semimodules, there is a collection of trivially intersecting *Archimedean* semimodules such that every semimodule from the first collection is contained in an Archimedean semimodule from the second collection.

We also show in Theorems 14 and 16 that the orbit of an eigenvector of a cyclic projector maximizes a certain objective function. We call this maximum the Hilbert value of semimodules, as it is a natural generalization of Hilbert's projective metric, and characterize the spectrum of cyclic projectors in terms of these Hilbert values (Theorem 25).

The projectors on idempotent semimodules, which constitute the cyclic projectors considered here, have been studied in [8] and [9, Chap. 8], where they appear as AA^*-products. The geometrical properties of these projectors have been used in [6,7] to establish separation theorems. The same operators have also been studied in [19], where idempotent analogues of several results from functional analysis, including the analytic form of the Hahn–Banach theorem, were obtained.

The idempotent cyclic projectors have been introduced, in the case of two semimodules, in [10], where these operators give rise to an efficient (pseudo-polynomial) algorithm for finding a point in the intersection of two finitely generated subsemimodules of $\mathbf{R}^n_{\max,\times}$. In convex analysis and optimization theory, an analogous role is played by the cyclic nearest-point projections on convex sets [2].

As a corollary of Theorems 18 and 20, we deduce a max-plus analogue of Helly's theorem. This result has also been obtained, with a different proof, in [14].

Our main results apply to subsemimodules of $\mathbf{R}^n_{\max,\times}$. Some of our results still hold in a more general setting (see Sec. 3). However, the separation of several semimodules in such a generality remains an open question.

The results of this paper are presented as follows. Section 2 describes the main assumptions that are satisfied by the semimodules of the paper. In addition, it is occupied by some basic notions and facts that will be used further. Section 3 is devoted to the results obtained in the most general setting, with respect to the assumptions of Sec. 2. The main results for the case $\mathbf{R}^n_{\max,\times}$ are obtained in Sec. 4. These results include separation of several semimodules and characterization of the spectrum of cyclic projectors. Section 5 contains two illustrations of the separation theorems obtained in Secs. 3 and 4.

2. Preliminary Results on Projectors and Separation

We start this section with some details concerning the role of partial order in idempotent algebraic structures. For more background, we refer the reader to, e.g., [1, 9, 19].

The idempotent addition $\oplus$ defines the canonical order relation $\leq_\oplus$ on the semiring $\mathcal{K}$ by the rule $\lambda \oplus \mu = \mu \iff \lambda \leq_\oplus \mu$ for $\lambda, \mu \in \mathcal{K}$. The idempotent sum $\lambda \oplus \mu$ is equal to the least upper bound $\sup(\lambda, \mu)$ with respect to the order $\leq_\oplus$. The idempotent sum of an arbitrary subset is defined as the least upper bound of this subset, if this least upper bound exists. In a semimodule $\mathcal{V}$, we define the order relation $\leq_{\oplus,\mathcal{V}}$ in the same way. The relation $\lambda \leq_\oplus \mu$ between $\lambda, \mu \in \mathcal{K}$ implies $\lambda x \leq_{\oplus,\mathcal{V}} \mu x$ for all $x \in \mathcal{V}$. When $\mathcal{V} = \mathcal{K}^n$ and $\mathcal{K} = \mathbf{R}_{\max,\times}$, the order $\leq_\oplus$ coincides with the usual linear order on $\mathbf{R}_+$, and the order $\leq_{\oplus,\mathcal{V}}$ coincides with the standard pointwise order on $\mathbf{R}^n$. For this reason, we will write $\leq$ instead of $\leq_\oplus$ and $\leq_{\oplus,\mathcal{V}}$, in the sequel.

A semiring or a semimodule will be called *b-complete* (see [19]), if it is closed under the sum (i.e., the supremum) of any subset bounded from above, and the multiplication distributes over such sums. If the least upper bound $\oplus$ exists for all subsets bounded from above, then the greatest lower bound $\wedge$ exists for all subsets bounded from below. Consequently, the greatest lower bound exists for any subset of a b-complete semiring or a semimodule, since such a subset is bounded from below by $\mathbf{0}$.

Also note that if $\mathcal{K}$ is a b-complete semiring, and the set $\mathcal{K} \setminus \{\mathbf{0}\}$ is a multiplicative group, then this group is Abelian by Iwasawa's theorem [3]. A semiring $\mathcal{K}$ such that the set $\mathcal{K} \setminus \{\mathbf{0}\}$ is an Abelian multiplicative group is called an *idempotent semifield*.

We shall consider semirings $\mathcal{K}$ and semimodules $\mathcal{V}$ over $\mathcal{K}$ that satisfy the following assumptions:

(A0) the semiring $\mathcal{K}$ is a b-complete idempotent semifield, and the semimodule $\mathcal{V}$ is a b-complete semimodule over $\mathcal{K}$;

(A1) for all elements x and $y \neq \mathbf{0}$ from $\mathcal{V}$, the set $\{\lambda \in \mathcal{K} \mid \lambda y \leq x\}$ is bounded from above.

Assumptions (A0) and (A1) imply that the operation

$$x/y = \max\{\lambda \in \mathcal{K} \mid \lambda y \leq x\} \tag{1}$$

is defined for all elements x and $y \neq \mathbf{0}$ from $\mathcal{V}$. Operations of this kind belong to residuation theory (see [1, 4, 15] for a general reference). The following can be viewed as another definition of the operation / equivalent to (1):

$$\lambda y \leq x \iff \lambda \leq x/y. \tag{2}$$

In the case $\mathcal{V} = \mathcal{K}^I$, we have

$$x/y = \bigwedge_{i\colon y_i \neq \mathbf{0}} x_i (y_i)^{-1}, \tag{3}$$

where $^{-1}$ denotes the multiplicative inverse.

The operation / has the following properties (see [1, 4, 15]):

$$\Big(\bigwedge_\alpha x_\alpha\Big)\Big/ y = \bigwedge_\alpha (x_\alpha / y), \quad \Big(x \Big/ \bigoplus_\alpha y_\alpha\Big) = \bigwedge_\alpha (x/y_\alpha), \tag{4}$$

$$(\lambda x)/y = \lambda(x/y) \text{ for all } \lambda, \quad y/(\lambda x) = \lambda^{-1}(y/x) \text{ for all } \lambda \neq \mathbf{0}. \tag{5}$$

We also need the following lemma.

Lemma 1. *Under* (A0) *and* (A1), *$x/x = \mathbf{1}$ for all nonzero vectors $x \in \mathcal{V}$. If $\lambda x = x$ for a nonzero vector $x \in \mathcal{V}$, then $\lambda = \mathbf{1}$.*

Proof. The inequality $x \leq x$ implies that $x/x \geq \mathbf{1}$ (see (1)). On the other hand, we have that $(x/x)x \leq x$. Multiplying this by x/x, we obtain that $(x/x)^2 x \leq (x/x)x \leq x$, whence $(x/x)^2 \leq x/x$ and $x/x \leq \mathbf{1}$. Thus, $x/x = \mathbf{1}$. If $\lambda x = x$ for some $x \neq \mathbf{0}$, then $\lambda(x/x) = (\lambda x)/x = x/x$ and so $\lambda = \mathbf{1}$. □

Definition 2. A subsemimodule V of $\mathcal{V}$ is a *b-(sub)semimodule*, if V is closed under the sum of any of its subsets bounded from above in $\mathcal{V}$.

Let V be a b-subsemimodule of the semimodule $\mathcal{V}$. Consider the operator P_V defined by

$$P_V(x) = \max\{u \in V \mid u \leq x\} \tag{6}$$

for every element $x \in \mathcal{V}$. Here we use max to indicate that the least upper bound belongs to the set. The operator P_V is a *projector* onto the subsemimodule V, since $P_V(x) \in V$ for any $x \in \mathcal{V}$ and $P_V(v) = v$ for any $v \in V$. In principle, P_V can be defined for all subsets of $\mathcal{V}$, if we write sup instead of max in (6), but then P_V need not be a projector on V.

Definition 3. A subsemimodule V of $\mathcal{V}$ is called *elementary*, if $V = \{\lambda y \mid \lambda \in \mathcal{K}\}$ for some $y \in \mathcal{V}$. The projector onto such a semimodule is also called *elementary*.

Assumptions (A0) and (A1) imply that elementary semimodules are b-semimodules. For the elementary semimodule $V = \{\lambda y \mid \lambda \in \mathcal{K}\}$, the projector P_V is given by $P_V(x) = (x/y)y$, and this fact can be generalized as follows.

Proposition 4. *If V is a b-subsemimodule of $\mathcal{V}$ and $P_V(x) = \lambda y$ for some $\lambda \in \mathcal{K}$ and $x, y \in \mathcal{V}$, then $P_V(x) = (x/y)y$.*

Proof. If V is a b-semimodule, then $y \in V$, and $(x/y)y \leq x$ implies that $(x/y)y \leq P_V(x) = \lambda y$. On the other hand, $\lambda y \leq x$ implies $\lambda \leq x/y$ and hence $\lambda y \leq (x/y)y$. □

Note that P_V is isotone with respect to inclusion

$$U \subset V \implies P_U(x) \leq P_V(x) \text{ for all } x. \tag{7}$$

It is also homogeneous and isotone:

$$P_V(\lambda x) = \lambda P_V(x), \quad x \leq y \implies P_V(x) \leq P_V(y). \tag{8}$$

We remark that the operator P_V is, in general, *not* linear with respect to $\oplus$ or $\wedge$ operations, even in the case $\mathcal{V} = \mathbf{R}^n_{\max,\times}$.

In idempotent geometry, the role of halfspace is played by the following object.

Definition 5. A set H given by

$$H = \{x \mid u/x \geq v/x\} \cup \{\mathbf{0}\} \tag{9}$$

with $u, v \in \mathbf{R}^n_{\max,\times}$, $u \leq v$, will be called an (*idempotent*) *halfspace*.

Here we impose the condition $u \leq v$, since only halfspaces of this kind are really important for separation (see Theorem 6).

Properties (4) and (5) of the operation / imply that any halfspace is a semimodule. If $\mathcal{V} = \mathcal{K}^I$, then we can use (3) and then

$$H = \left\{x \;\middle|\; \bigwedge_{i\colon x_i \neq \mathbf{0}} u_i x_i^{-1} \geq \bigwedge_{i\colon x_i \neq \mathbf{0}} v_i x_i^{-1}\right\} \cup \{\mathbf{0}\}. \tag{10}$$

If $\mathcal{V} = \mathcal{K}^n$ and all coordinates of u and v are nonzero, then we have that

$$H = \left\{x \;\middle|\; \bigoplus_{1,\dots,n} x_i u_i^{-1} \leq \bigoplus_{1,\dots,n} x_i v_i^{-1}\right\}. \tag{11}$$

Such idempotent halfspaces formally resemble the closed homogeneous halfspaces of the finite-dimensional convex geometry [22].

Since the operation / is isotone with respect to the first argument, we can replace the inequalities in (9), (10), and (11) by the equalities. For instance, (9) can be rewritten as

$$H = \{x \mid u/x = v/x\} \cup \{\mathbf{0}\}, \tag{12}$$

where $u \leq v$. A semimodule defined by (12), with general u and v, is also called a max-plus hyperplane [6]. The structure of max-plus hyperplanes, for the case of $\mathbf{R}^n_{\max}$, has been studied in [20].

The present paper is concerned with the separation of several b-semimodules, whereas the separation theorems that have been established previously, like the ones of [6, 7], deal with the separation of one point from a semimodule. For the convenience of the reader, we next state a theorem, which is a variant of a separation theorem of [6]. The difference is in that we deal with b-complete semimodules rather than with complete semimodules. Both results are closely related with the idempotent Hahn–Banach theorem of [19].

Theorem 6 (cp. [6, Theorem 8])**.** *Let V be a b-subsemimodule of $\mathcal{V}$ and let $u \notin V$. Then the halfspace*

$$H = \{x \mid P_V(u)/x \geq u/x\} \cup \{\mathbf{0}\} \tag{13}$$

contains V but not u.

Proof. Take a nonzero vector $x \in V$ (the case $x = \mathbf{0}$ is trivial). Since $(u/x)x \leq u$, we have $(u/x)x \leq P_V(u)$, which is, by (2), equivalent to $u/x \leq P_V(u)/x$. Hence $V \subseteq H$.

Take $x = u$ and assume that $P_V(u)/u \geq u/u = \mathbf{1}$. This is equivalent to $u \leq P_V(u)$ and hence to $u = P_V(u)$. Since V is a b-semimodule, we have that $u \in V$, which is a contradiction. Hence $u \notin H$. $\square$

Definition 7. Consider the preorder relation $\preceq$ defined by

$$x \preceq y \iff y/x > \mathbf{0}. \tag{14}$$

We say that x and y are *comparable*, and we write $x \sim y$, if $x \preceq y$ and $y \preceq x$. Equivalently,

$$x \sim y \iff (x/y)(y/x) > \mathbf{0}. \tag{15}$$

Note that if $y = \lambda x$ with $\lambda \neq \mathbf{0}$, then $y \sim x$, and that the inequality $x \leq y$, if $x \neq \mathbf{0}$, implies that $x \preceq y$. In particular, $P_V(x) \preceq x$ for any nonzero $x \in \mathcal{V}$ and any semimodule V, provided that $P_V(x)$ is nonzero.

When $\mathcal{V} = \mathcal{K}^n$, comparability can be characterized in terms of supports. Recall that the *support* of a vector x in $\mathcal{K}^n$ is defined by $\operatorname{supp}(x) = \{i \mid x_i \neq 0\}$. It can be checked that for all $x, y \in \mathcal{K}^n$, we have $x \preceq y$ if and only if $\operatorname{supp}(x) \subset \operatorname{supp}(y)$, and so, $x \sim y$ if and only if $\operatorname{supp}(x) = \operatorname{supp}(y)$.

Proposition 8. *Let $x \in \mathcal{V}$ be a nonzero vector and let $V \subseteq \mathcal{V}$ be a b-semimodule containing a nonzero vector y. If $y \preceq x$, then $P_V(x)$ is nonzero, and $y \preceq P_V(x) \preceq x$. If $y \sim x$, then $P_V(x) \sim x$.*

Proof. By the definition of $/$ and by (14), there exists α such that $\alpha y \leq x$. Then $\alpha y \leq P_V(x)$, whence $P_V(x)$ is nonzero and $y \preceq P_V(x)$. $\square$

Proposition 9. *Let F be an isotone and homogeneous operator, let λ, μ be arbitrary scalars from $\mathcal{K}$, and let v and u be nonzero vectors such that $v \prec u$. Suppose that one of the following is true*:

(1) *$Fv \geq \mu v$ and $Fu = \lambda u$;*
(2) *$Fv = \mu v$ and $Fu \leq \lambda u$.*

Then $\mu \leq \lambda$.

Proof. Applying F to the inequality $(u/v)v \leq u$ and using any of the given conditions, we obtain that $(u/v)\mu v \leq \lambda u$. If $\lambda = \mathbf{0}$, then $\mu = \mathbf{0}$. If λ is invertible, then by (2) $(u/v)\mu\lambda^{-1} \leq u/v$. Cancelling u/v, we get $\mu \leq \lambda$. $\square$

Properties (4) and (5) imply that the sets $\{x \mid x \preceq y\}$, $\{x \mid x \succeq y\}$, and hence $\{x \mid x \sim y\}$ are subsemimodules of $\mathcal{V}$. For any semimodule $V \subset \mathcal{V}$ and any vector $y \in \mathcal{V}$, we define

$$V^y = \{x \in V \mid x \preceq y\}, \tag{16}$$

which is a subsemimodule of V. When $\mathcal{V} = \mathcal{K}^n$, V^y is uniquely determined by the support M of y. For this reason, for all $M \subseteq \{1, \ldots, n\}$, we set

$$V^M = \{x \in V \mid \operatorname{supp}(x) \subset M\}. \tag{17}$$

Definition 10. A vector $x \in \mathcal{V}$ is called *Archimedean*, if $y \preceq x$ for all $y \in \mathcal{V}$. A semimodule $V \subseteq \mathcal{V}$ is called *Archimedean*, if it contains an Archimedean vector. A halfspace will be called *Archimedean* if both vectors defining it (e.g., u and v in (9)) are Archimedean.

Thus, a semimodule is Archimedean if it contains an Archimedean vector y, i.e., a vector with the property that for any other vector $x \in \mathcal{V}$ there exists $\lambda > \mathbf{0}$ such that $\lambda x \leq y$.

Of course, Definition 10 makes sense only in the case where $\mathcal{V}$ satisfies the following assumption:

(A2) the semimodule $\mathcal{V}$ has an Archimedean vector.

This assumption is satisfied by the semimodules $\mathcal{V} = \mathcal{K}^n$ (we are also assuming (A0), (A1)). In this case, Archimedean halfspaces have been written explicitly in (11).

3. Cyclic Projectors and Separation Theorems: General Results

In this section, we study cyclic projectors, i.e., compositions of projectors

$$P_{V_k} \cdots P_{V_1}, \tag{18}$$

where $V_1, \ldots, V_k$ are always assumed to be b-subsemimodules of $\mathcal{V}$. We assume (A0) and (A1), which means, in particular, that $\mathcal{K}$ is an idempotent semifield, and state general results concerning cyclic projectors and separation properties. For the notational convenience, we will write P_t instead of P_{V_t}. We will also adopt a convention of cyclic numbering of indices of projectors and semimodules, so that $P_{l+k} = P_l$ and $V_{l+k} = V_l$ for all l.

First we prove the following separation theorem, where the main role is played by cyclic projectors. Here we assume the existence of Archimedean vectors (A2). See Sec. 5 for an illustration.

Theorem 11. *Suppose that $P_k \cdots P_1$ has an Archimedean eigenvector y with nonzero eigenvalue λ. The following are equivalent*:

(1) *there exist an Archimedean vector x and a scalar $\mu < \mathbf{1}$ such that $P_k \cdots P_1 x \leq \mu x$;*

(2) *for all $i = 1, \ldots, k$, there exist Archimedean halfspaces H_i such that $V_i \subseteq H_i$ and $H_1 \cap \cdots \cap H_k = \{\mathbf{0}\}$;*

(3) $V_1 \cap \cdots \cap V_k = \{\mathbf{0}\}$;

(4) $\lambda < \mathbf{1}$.

Proof. (1) $\Longrightarrow$ (2). Denote $x^0 = x$ and $x^i = P_i \cdots P_1 x^0$. Note that all the x^i are also Archimedean by Proposition 8. For all $i = 1, \ldots, k$, we have that

$$V_i \subseteq \{u\colon x^{i-1}/u = x^i/u\} = H_i. \tag{19}$$

Indeed, if $x^{i-1} = x^i$, then H_i coincides with the whole $\mathcal{V}$. If $x^{i-1} \neq x^i$, which means that $x^i \notin V_{i-1}$, then the inclusion in (19) follows from Theorem 6. Assume that there exists a nonzero vector u that belongs to every H_i. Then $x^k/u = x/u$. But $x^k/u \leq (\mu x)/u \leq x/u$, whence $\mu(x/u) = (\mu x)/u = x/u$. Cancelling x/u, we get $\mu = \mathbf{1}$, which contradicts (1). Thus, the halfspaces H_i can intersect only trivially. Also note that, as all x^i are Archimedean, the halfspaces H_i are Archimedean. The implication is proved.

(2) $\Longrightarrow$ (3). Immediate.

(3) $\Longrightarrow$ (4). By the conditions of this theorem, $P_k \cdots P_1$ has an eigenvector y with eigenvalue λ. As any vector is greater than or equal to its image by the projector P_i, we have that $\lambda \leq \mathbf{1}$. Assume that $\lambda = \mathbf{1}$. Then the inequalities

$$P_k \cdots P_1 y \leq P_{k-1} \cdots P_1 y \leq \cdots \leq y$$

turn into equalities, and hence y is a common vector of $V_1, \ldots, V_k$, which contradicts (3).

(4) $\Longrightarrow$ (1). Take $x = y$. $\square$

Theorem 11 has the condition that $P_k \cdots P_1$ has an Archimedean eigenvector with nonzero eigenvalue. This implies that the semimodules $V_1, \ldots, V_k$ are Archimedean, but the converse implication is not true. To see this, consider two subsemimodules of $\mathbf{R}^4_{\max,\times}$: V_1 generated by $a^1 = (1, 1, 0, 0)$ and $a^2 = (0, 0, 1, 1)$

and V_2 generated by $b^1 = (1, 2, 0, 0)$ and $b^2 = (0, 0, 1, 3)$. Any eigenvector of P_1P_2 belongs to V_1, whence it has the form $(\lambda, \lambda, \mu, \mu)$. The action of P_1P_2 takes this vector to $\left(\frac{1}{2}\lambda, \frac{1}{2}\lambda, \frac{1}{3}\mu, \frac{1}{3}\mu\right)$, which shows that P_1P_2 does not have Archimedean eigenvectors. The condition of the theorem can be relaxed in the case $\mathcal{V} = \mathbf{R}^n_{\max,\times}$ as will be shown in the next section (see Theorems 18 and 20).

Theorem 11 invokes some interest in spectral properties of cyclic projectors. Below we will prove some results on these properties. First, let us give the following definition.

Definition 12. Let $x^1, \dots, x^k$ be nonzero elements of $\mathcal{V}$. The value

$$d_{\mathrm{H}}(x^1, \dots, x^k) = (x^1/x^2)(x^2/x^3)\cdots(x^k/x^1) \tag{20}$$

will be called the *Hilbert value* of $x^1, \dots, x^k$.

It follows from Definition 7 that $d_{\mathrm{H}}(x^1, \dots, x^k) \neq \mathbf{0}$ if and only if all vectors $x^1, \dots, x^k$ are comparable. One can show that $d_{\mathrm{H}}(x^1, \dots, x^k) \leq \mathbf{1}$. This inequality is an equality if and only if $x^1, \dots, x^k$ differ from each other only by scalar multiples. The Hilbert value is invariant under multiplication of any of its arguments by an invertible scalar, and under cyclic permutation of its arguments.

The Hilbert value of two vectors x^1 and x^2 was studied in [6]. For two comparable vectors in $\mathbf{R}^n_{\max,\times}$, i.e., for two vectors with common support M it is given by

$$d_{\mathrm{H}}(x^1, x^2) = \min_{i,j\in M}(x^1_i(x^2_i)^{-1}x^2_j(x^1_j)^{-1}), \tag{21}$$

so that $-\log\bigl(d_{\mathrm{H}}(x^1, x^2)\bigr)$ coincides with Hilbert's projective metric

$$\delta_{\mathrm{H}}(x^1, x^2) = \log\bigl(\max_{i,j\in M}(x^1_i(x^2_i)^{-1}x^2_j(x^1_j)^{-1})\bigr) = -\log\bigl(d_{\mathrm{H}}(x^1, x^2)\bigr). \tag{22}$$

Definition 13. The *Hilbert value* of k subsemimodules $V_1, \dots, V_k$ of $\mathcal{V}$ is defined by

$$d_{\mathrm{H}}(V_1, \dots, V_k) = \sup_{x^1\in V_1,\dots,x^k\in V_k} d_{\mathrm{H}}(x^1, \dots, x^k). \tag{23}$$

Theorem 14. *Suppose that the operator $P_k \cdots P_1$ has an eigenvector y with eigenvalue λ. Then*

$$\lambda = \max_{x^1\in V^y_1,\dots,x^k\in V^y_k} d_{\mathrm{H}}(x^1, \dots, x^k) = d_{\mathrm{H}}(\bar{x}^1, \dots, \bar{x}^k), \tag{24}$$

where $\bar{x}^i = P_i \cdots P_1 y$.

Proof. Note that $\bar{x}^i$, for any i, is an eigenvector of $P_{i+k} \cdots P_{i+1}$ and that all these vectors are comparable with y. Further, let $x^1, \dots, x^k$ be arbitrary elements of $V^y_1, \dots, V^y_k$, respectively, and let $\alpha_1, \dots, \alpha_k$ be scalars such that

$$\begin{aligned} \alpha_1 x^2 &\leq P_2 x^1, \\ &\vdots \\ \alpha_{k-1} x^k &\leq P_k x^{k-1}, \\ \alpha_k x^1 &\leq P_1 x^k. \end{aligned} \tag{25}$$

Take the last inequality. Applying P_2 to both sides and using the first inequality, we have that $\alpha_1\alpha_k x^2 \leq P_2P_1x^k$. Further, we apply P_3 to this inequality and use the inequality $\alpha_2 x^3 \leq P_3 x^2$. Proceeding in the same manner, we finally obtain

$$\alpha_1 \cdots \alpha_k x^k \leq P_k \cdots P_1 x^k. \tag{26}$$

It follows from Proposition 9 that $\alpha_1 \cdots \alpha_k \leq \lambda$. We take $\alpha_i = x^i/x^{i+1}$ for $i = 1, \dots, k-1$, and $\alpha_k = x^k/x^1$. This leads to

$$d_{\mathrm{H}}(V^y_1, \dots, V^k_y) \leq \lambda. \tag{27}$$

Note that this inequality is true if $V_1, \dots, V_k$ are not b-semimodules. Applying Proposition 4 we have that $\lambda y = d_{\mathrm{H}}(\bar{x}^1, \dots, \bar{x}^k)y$. By Lemma 1, we can cancel y, and the observation that $\bar{x}^i \in V^y_i$ for all i yields the desired equality. $\square$

The situation where $P_k \cdots P_1$ has an eigenvector with nonzero eigenvalue occurs, if at least one of the semimodules $V_1, \dots, V_k$ is elementary, i.e., generated by a single vector x^i, and if all other semimodules have vectors comparable with x^i. In this case, $P_k \cdots P_{i+1} x^i$ is the only eigenvector of $P_k \cdots P_1$ with nonzero eigenvalue.

To obtain the following lemma, we use Proposition 8.

Lemma 15. *Let $x^1 \in V_1$ and $x^i = P_i x^{i-1}$ for $i = 2, \dots, k$. Then, the Hilbert value $d_{\mathrm{H}}(x^1, \dots, x^k)$ is not equal to $\mathbf{0}$ if and only if $V_2, \dots, V_k$ have vectors comparable with x^1.*

Theorem 16. *Suppose that the vectors x^i, $i = 1, \dots,$ are such that $x^1 \in V_1$ and $x^i = P_i x^{i-1}$ for $i = 2, \dots$. Then $d_{\mathrm{H}}(x^{l+1}, \dots, x^{l+k})$ is nondecreasing with l so that the following inequalities hold for all l:*

$$d_{\mathrm{H}}(x^1, \dots, x^k) \le d_{\mathrm{H}}(x^2, \dots, x^{k+1}) \le \cdots \le \mathbf{1}. \tag{28}$$

Proof. As V_i are b-semimodules, $x^i \in V_i$ for all i. If the Hilbert value is $\mathbf{0}$ for all l, then there is nothing to prove. Thus, we assume that there exists a least $l = l_{\min}$ for which the Hilbert value $d_{\mathrm{H}}(x^l, \dots, x^{l+k-1})$ is nonzero. As it is nonzero, by Lemma 15, all $x^l, \dots, x^{l+k-1}$ are comparable. By Proposition 8, x^{l+k} is also comparable with them, and the same is true about the rest of the sequence, whence $d_{\mathrm{H}}(x^l, \dots, x^{l+k-1})$ is nonzero for all $l \ge l_{\min}$. Now we take any $l \ge l_{\min}$ and consider the composition

$$P_{l+k} P'_{l+k-1} \cdots P'_{l+1}, \tag{29}$$

where P'_i, for $i = l+1, \dots, l+k-1$, are elementary projectors onto the semimodules generated by x^i. The operator (29) has an eigenvector x^{l+k}. By Theorem 14,

$$d_{\mathrm{H}}(x^l, \dots, x^{l+k-1}) \le \max_{y \in V_l,\ y \preceq x^{l+k}} d_{\mathrm{H}}(x^{l+1}, \dots, x^{l+k-1}, y) = d_H(x^{l+1}, \dots, x^{l+k}) \tag{30}$$

for all $l = 1, \dots$. $\square$

4. Cyclic Projectors and Separation Theorems in $\mathbf{R}^n_{\max,\times}$

In $\mathbf{R}^n_{\max,\times}$, it is natural to consider semimodules that are closed in the Euclidean topology. One can easily show that such semimodules are b-semimodules. From [7, Theorem 3.11] it follows that projectors onto closed subsemimodules of $\mathbf{R}^n_{\max,\times}$ are continuous.

In order to relax the assumption concerning Archimedean vectors in Theorem 11, we shall use some results from nonlinear spectral theory, that we next recall. By Brouwer's fixed point theorem, a continuous homogeneous operator $x \mapsto Fx$ that maps $\mathbf{R}^n_+$ to itself has a nonzero eigenvector. This allows us to define the nonlinear spectral radius of F:

$$\rho(F) = \max\{\lambda \in \mathbf{R}_+ \mid \exists x \in (\mathbf{R}^n_+) \setminus 0,\ Fx = \lambda x\}. \tag{31}$$

Suppose, in addition, that F is isotone. Then it can be shown (or deduced from Proposition 9) that if $Fx = \lambda x$, $Fy = \mu y$, and x and y are comparable, then $\lambda = \mu$. It follows that the number of eigenvalues of F is bounded by the number of nonempty supports of vectors of $\mathbf{R}^n_+$, i.e., by $2^n - 1$. This implies, in particular, that the maximum is attained in (31). We shall need the following nonlinear generalization of the Collatz–Wielandt formula for the spectral radius of a nonnegative matrix.

Theorem 17 (R. D. Nussbaum, [21, Theorem 3.1])**.** *For any isotone, homogeneous, and continuous map F from $\mathbf{R}^n_+$ to itself, we have*

$$\rho(F) = \inf_{x \in (\mathbf{R}_+ \setminus \{0\})^n} \max_{1 \le i \le n} [F(x)]_i x_i^{-1}. \tag{32}$$

As the projectors on subsemimodules of $\mathbf{R}^n_{\max,\times}$ are isotone, homogeneous, and continuous, so are their compositions, i.e., cyclic projectors. Consequently, we can apply Theorem 17 to them. The following results refine Theorem 11, which is a separation result obtained under assumptions (A0)–(A2).

Theorem 18. *Suppose that $V_1, \dots, V_k$ are closed Archimedean subsemimodules of $\mathbf{R}^n_{\max,\times}$. The following are equivalent*:

(1) *there exist a positive vector x and a number $\lambda < 1$ such that $P_k \cdots P_1 x \leq \lambda x$;*
(2) *there exist Archimedean halfspaces H_i that contain V_i and are such that $H_1 \cap \cdots \cap H_k = \{\mathbf{0}\}$;*
(3) *$V_1 \cap \cdots \cap V_k = \{\mathbf{0}\}$;*
(4) *$\rho(P_k \cdots P_1) < \mathbf{1}$.*

Proof. The implications (1) $\Longrightarrow$ (2), (2) $\Longrightarrow$ (3), and (3) $\Longrightarrow$ (4) are proved in Theorem 11. The implication (4) $\Longrightarrow$ (1) follows from Eq. (32). $\square$

Proposition 19. *Suppose that V_i, $i = 1, \dots, k$, are closed semimodules in $\mathbf{R}^n_{\max,\times}$ with zero intersection. Then there exist closed Archimedean semimodules V_i', $i = 1, \dots, k$, with zero intersection and such that each V_i' contains V_i.*

Proof. In every semimodule V_i, we find a vector y^i such that $\|y^i\| = \max(y^i_1, \dots, y^i_n) = 1$. For all scalars $\delta > 0$, define

$$z^i(\delta) = y^i \oplus \delta \bigoplus_{j \notin \operatorname{supp}(y^i)} e^j \tag{33}$$

and the semimodules

$$V_i(\delta) = \{x \mid x = v \oplus \lambda z^i(\delta),\ v \in V_i,\ \lambda \in \mathbf{R}_+\}. \tag{34}$$

These semimodules are closed, as all arithmetical operations are continuous (see also [5, Proposition 24]). We show that for $\delta > 0$ small enough these semimodules have zero intersection. Assume by contradiction that for all $\delta > 0$, there exists a nonzero vector $u(\delta)$ in the intersection $V_1(\delta) \cap \cdots \cap V_k(\delta)$. After normalizing $u(\delta)$, we may assume that $\|u(\delta)\| = 1$. For any $i = 1, \dots, k$ and any δ, we have that

$$u(\delta) = v^i(\delta) \oplus \lambda_i(\delta) y^i \oplus \lambda_i(\delta)\delta \bigoplus_{j \notin \operatorname{supp}(y^i)} e^j, \tag{35}$$

where $v^i(\delta)$ is a vector from V_i and $\lambda_i(\delta)$ is a scalar. As $\|u(\delta)\| = 1$ and $\|y^i\| = 1$, we have that $\lambda_i(\delta) \leq 1$. Thus, there exists a sequence $(\delta_m)_{m \geq 1}$ converging to 0 such that for all $1 \leq i \leq k$, both $\lambda_i(\delta_m)$ and $v^i(\delta_m)$ converge to some limits as m tends to infinity. Then

$$w := \lim_{m \to \infty} u(\delta_m) = \lim_{m \to \infty} v^i(\delta_m) \oplus \lambda_i(\delta_m) y^i$$

for all i. As V_i are closed, w belongs to V_i at all i. Since $\|w\| = 1$, w is not equal to $\mathbf{0}$, which is a contradiction. Thus, there exists δ such that $V_1(\delta) \cap \cdots \cap V_k(\delta) = \{0\}$. The semimodules $V_1(\delta), \dots, V_k(\delta)$ have all desired properties, since they are Archimedean and contain $V_1, \dots, V_k$. $\square$

The following is an immediate corollary of Theorem 18 and Proposition 19 (see Sec. 5 for an illustration of this theorem).

Theorem 20 (separation theorem)**.** *If V_i, $i = 1, \dots, k$, are closed semimodules with zero intersection, then there exist Archimedean halfspaces H_i, $i = 1, \dots, k$, that contain the corresponding semimodules V_i and have zero intersection.*

The following separation theorem for two closed semimodules is a corollary of Theorem 20.

Theorem 21. *If U and V are two closed semimodules with zero intersection, then there exists a closed halfspace H_U that contains U and has zero intersection with V, and there exists a closed halfspace H_V that contains V and has zero intersection with U.*

As a consequence of Theorem 20, we further deduce a separation theorem for convex subsets of $\mathbf{R}^n_{\max,\times}$. We recall here some definitions from idempotent convex geometry (see, e.g., [13]). A subset $C \subset \mathbf{R}^n_{\max,\times}$ is (*idempotently*) *convex* if $\lambda u \oplus \mu v \in C$ for all $u, v \in C$ and $\lambda, \mu \in \mathbf{R}_{\max,\times}$ such that $\lambda \oplus \mu = 1$. In what follows, we call such subsets convex, as the traditional convexity is not used in this paper.

The *recession cone* of a convex set C, $\operatorname{rec}(C)$, is the set of vectors u such that $v \oplus \lambda u \in C$ for all $\lambda \in \mathbf{R}_{\max,\times}$, where v is an arbitrary vector of C. As shown in [13, Proposition 2.6], if C is closed, then the recession cone is independent of the choice of v. Observe that when C is compact, its recession cone is zero.

A set H^{aff} given by

$$H^{\mathrm{aff}} = \{x \mid u/x \wedge \alpha \geq v/x \wedge \gamma\} \tag{36}$$

with $u, v \in \mathbf{R}^n_{\max,\times}$, $u \leq v$, $\alpha, \gamma \in \mathbf{R}_{\max,\times}$, and $\alpha \leq \gamma$, will be called (*idempotent*) *affine halfspace.* It is called *Archimedean*, if u, v, α, and γ are positive.

For a convex set $C \subset \mathbf{R}^n_{\max,\times}$, define $V(C) \subset \mathbf{R}^{n+1}_{\max,\times}$ as the semimodule of vectors of the form $(x_1\lambda, \ldots, x_n\lambda, \lambda)$ with $x = (x_1, \ldots, x_n) \in C$ and $\lambda \in \mathbf{R}_{\max,\times}$.

Theorem 22 (separation of convex sets)**.** *Let $C_1, \ldots, C_k$ be closed convex subsets of $\mathbf{R}^n_{\max,\times}$ with empty intersection, and assume that the intersection of the recession cones of $C_1, \ldots, C_k$ is zero. Then there exist affine Archimedean halfspaces $H^{\mathrm{aff}}_1, \ldots, H^{\mathrm{aff}}_k$ that contain the corresponding convex sets C_i, $i = 1, \ldots, k$, and have empty intersection.*

Proof. From [13, Proposition 2.16] we know that the closure of $V(C_i)$, $\overline{V(C_i)}$, is equal to $V(C_i) \cup (\mathrm{rec}(C_i) \times \{0\})$. Hence, the assumptions imply that the intersection of $\overline{V(C_i)}$, $i = 1, \ldots, k$, is zero. By Theorem 20, we can find Archimedean halfspaces $H_i \supset \overline{V(C_i)}$ with zero intersection. Every H_i can be written as

$$H_i = \{(x_1, \ldots, x_n, \mu) \mid u^i/x \wedge \alpha^i \mu^{-1} \geq v^i/x \wedge \gamma^i \mu^{-1}\} \cup \{0\} \tag{37}$$

with $u^i \leq v^i$ and $\alpha^i \leq \gamma^i$, understanding that $x := (x_1, \ldots, x_n)$ and that the terms with μ^{-1} disappear if $\mu = \mathbf{0}$. Observe that for all $x \in C_i$, $(x, 1) \in V(C_i) \subset H_i$. We deduce that the affine Archimedean halfspace

$$H^{\mathrm{aff}}_i = \{x \mid u^i/x \wedge \alpha^i \geq v^i/x \wedge \gamma^i\}$$

contains C_i. Since the intersection of the halfspaces H_i is zero, the intersection of the affine halfspaces H^{aff}_i must be empty. $\square$

In convex analysis, there is an analogous separation theorem for several compact convex sets (see [12, pp. 39–40].

We now deduce an idempotent analogue of the classical Helly's theorem. As observed in [14], there is another proof of this theorem, which is based on the direct idempotent analogue of Radon's argument (see [12]).

Theorem 23 (Helly's theorem)**.** *Suppose that V_i, $i = 1, \ldots, m$, is a collection of $m \geq n$ semimodules in $\mathbf{R}^n_{\max,\times}$. If any n semimodules intersect nontrivially, then the whole collection has a nontrivial intersection.*

Proof. It suffices to consider the case where the semimodules V_i are all closed. Indeed, the assumption implies that for all $j := (j_1, \ldots, j_n) \in \{1, \ldots, m\}^n$, we can choose a nonzero element z_j in the intersection $V_{j_1} \cap \cdots \cap V_{j_n}$. Let V'_i denote the semimodule generated by the elements z_j that belong to V_i. The collection of semimodules V'_i, $i = 1, \ldots, m$, still has the property that any n semimodules intersect nontrivially. Moreover, V'_i is closed, because it is finitely generated (see, e.g., [13, Lemma 2.20] or [5, Corollary 27]). Hence, if the conclusion of the theorem holds for closed semimodules, then we deduce that the whole collection V'_i, $i = 1, \ldots, m$, has a nontrivial intersection, and, since $V_i \supset V'_i$, the conclusion of the theorem also holds without any closure assumption.

In the discussions that follow, the semimodules V_i are all closed. We argue by contradiction, assuming that the whole collection has zero intersection. By Theorem 19, we can also assume that the semimodules V_i are Archimedean. For some number $k < m$, any k semimodules intersect nontrivially, but there are $k + 1$ semimodules, say $V_1, \ldots, V_{k+1}$, that have zero intersection. By Theorem 18, there exist a positive vector $y = y^0$ and a scalar $\lambda < 1$ such that

$$P_{k+1} \cdots P_1 y \leq \lambda y.$$

For all i, we denote $y^i = P_i \cdots P_1 y^0$, where projectors are indexed modulo $(k+1)$. By the homogeneity and isotonicity of projectors, we have that

$$P_{l+k+1} \cdots P_{l+1} y^l \leq \lambda y^l \tag{38}$$

for all $l = 1, \dots$. Consider the vectors

$$z^l = P_{l+k} \cdots P_{l+1} y^l$$

for $l = 1, \dots, k+1$. Since any k semimodules intersect nontrivially, the vector z^l must have at least one coordinate equal to that of y^l, for otherwise y^l would satisfy the first condition of Theorem 18, giving a contradiction. As $k \geq n$, there are at least two numbers l and at least one number i such that z^l has the same ith coordinate as y^l. If we take the smallest of these two l numbers, then

$$(P_{l+k+1} \cdots P_{l+1} y^l)_i = y^l_i.$$

But this contradicts (38). Hence any $k+1$ semimodules intersect nontrivially, which is again a contradiction. The theorem is proved. ◻

There is also an affine version of this theorem.

Theorem 24. *Suppose that C_i, $i = 1, \dots, m$, is a collection of $m \geq n+1$ convex subsets of $\mathbf{R}^n_{\max,\times}$. If any $n+1$ of these convex sets have a nonempty intersection, then the whole collection has a nonempty intersection.*

Proof. Consider the semimodules $V(C_1), \dots, V(C_m)$ defined above and apply Theorem 23 to them. ◻

Now we come to the study of spectral properties of cyclic projectors. As a corollary to the Collatz–Wielandt formula for spectral radius (32), we have that the spectral radius of cyclic projectors is isotone: if F and G are two cyclic projectors and $F(x) \leq G(x)$ for any $x \in \mathbf{R}^n_+$, then $\rho(F) \leq \rho(G)$. This implies that if V_i', $i = 1, \dots, k$, and V_i, $i = 1, \dots, k$, are closed semimodules in $\mathbf{R}^n_{\max,\times}$ and such that $V_i' \subseteq V_i$, $i = 1, \dots, k$, then

$$\rho(P_k' \cdots P_1') \leq \rho(P_k \cdots P_1), \tag{39}$$

since the projectors are isotone with respect to inclusion (7). In the following theorem, this observation helps us to characterize the spectrum of cyclic projectors in terms of Hilbert values.

Theorem 25. *Let V_i, $i = 1, \dots, k$, be closed semimodules in $\mathbf{R}^n_+$. Then the Hilbert value $d_{\mathrm{H}}(V_1, \dots, V_k)$ is the spectral radius of $P_k \cdots P_1$. The spectrum of $P_k \cdots P_1$ is the set of Hilbert values $d_{\mathrm{H}}(V_1^M, \dots, V_k^M)$, where M ranges over all nonempty subsets of $\{1, \dots, n\}$.*

Proof. First, we prove that the Hilbert value $d_{\mathrm{H}}(V_1, \dots, V_k)$ is the spectral radius of the cyclic projector, and hence an eigenvalue. We take k elementary subsemimodules spanned by $x^i \in V_i$, $i = 1, \dots, k$, and consider elementary projectors P_i' onto them. Observe that

$$\rho(P_k' \cdots P_1') = d_{\mathrm{H}}(x^1, \dots, x^k).$$

Denote by $\bar{x}^0$ an eigenvector of $P_k \cdots P_1$ associated with the spectral radius, and let $\bar{x}^i = P_i \cdots P_1 \bar{x}^0$. Then

$$\rho(P_k \cdots P_1) = d_{\mathrm{H}}(\bar{x}^1, \dots, \bar{x}^k).$$

By (39), it follows that $\rho(P_k \cdots P_1) \geq \rho(P_k' \cdots P_1')$, i.e.,

$$d_{\mathrm{H}}(\bar{x}^1, \dots, \bar{x}^k) \geq d_{\mathrm{H}}(x^1, \dots, x^k)$$

for any $x^1 \in V_1, \dots, x^k \in V^k$. Thus, the Hilbert value of $V_1, \dots, V_k$ is the spectral radius of $P_k \cdots P_1$.

Now consider $d_{\mathrm{H}}(V_1^M, \dots, V_k^M)$ for arbitrary $M \subseteq \{1, \dots, n\}$. Note that the semimodules $V_1^M, \dots, V_k^M$ are closed, and denote by $P_1^M, \dots, P_k^M$ the projectors onto these. It is easy to see that $P_i^M(y) = P_i(y)$ for all i and all y with $\mathrm{supp}(y) \subseteq M$. It follows that $d_{\mathrm{H}}(V_1^M, \dots, V_k^M)$ is the spectral radius of $P_k^M \cdots P_1^M$ and also an eigenvalue of $P_k \cdots P_1$.

We have proved that any Hilbert value $d_{\mathrm{H}}(V_1^M, \dots, V_k^M)$ is an eigenvalue of $P_k \cdots P_1$. The converse statement follows from Theorem 14. ◻

5. Illustrations

In this section, we give graphical illustrations of Theorems 11 and 20.

To illustrate Theorem 11, consider the matrices

$$A = \begin{pmatrix} 0 & 0 & 0 & -\infty \\ 1 & 2 & -\infty & 1 \\ 0 & -1 & 2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 2 & 2 \\ 0 & 0 & 0 \\ -\infty & 0 & -1 \end{pmatrix}.$$

Let a^i and b^i denote the ith column of A and B, respectively. For all vectors $x = (x_1, \dots, x_n)$ and $\beta > 0$, we denote by $\exp(\beta x)$ the vector of the same size with entries $\exp(\beta x_j)$. We define V_1 (V_2) as the subsemimodule of $\mathbf{R}^3_{\max,\times}$ generated by the vectors $\exp(\beta a^i)$ for $1 \le i \le 4$ (respectively, $\exp(\beta b^i)$ for $1 \le i \le 3$). The discussions that follow are independent of the choice of the scaling parameter $\beta > 0$, which is adjusted to make Fig. 1 readable (we took $\beta = 2/3$). The two semimodules V_1 and V_2 and their generators are represented as follows at the left of the figure. Here a nonzero vector $w = (w_1, w_2, w_3) \in \mathbf{R}^3_{\max,\times}$ is represented by the point of the two dimensional simplex that is the barycenter with weights w_j of the three vertices of this simplex. The generators a^i and b^i correspond to the bold dots. For instance, a^1 corresponds to the barycenter of the vertices of the simplex with weights $(1, \exp(\beta), 1)$. The semimodules V_1 and V_2 correspond to the two medium grey regions, together with the bold broken segments joining the generators to each of these regions.

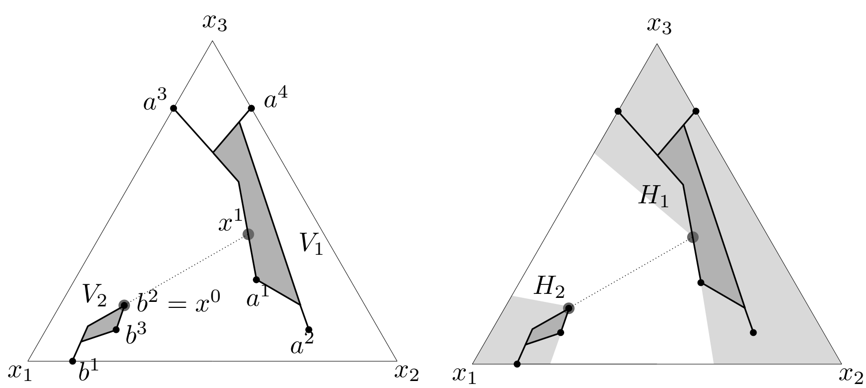

Fig. 1. Two semimodules (left) with separating halfspaces (right).

Since the entries of $x^0 := b^2 = \exp(\beta(2,0,0)) \in V_2$ are nonzero, the vector x^0 is Archimedean, and one can check, using the explicit formula of the projector (see [6, Theorem 5]), that x^0 is an eigenvector of P_2P_1. Indeed,

$$x^1 := P_1 x^0 = \exp\bigl(\beta(-1,0,0)\bigr)$$

and

$$x^2 := P_2 x^1 = \exp\bigl(\beta(-1,-3,-3)\bigr) = \exp(-3\beta)x^0.$$

The halfspaces constructed in the proof of Theorem 11 are given by

$$\begin{aligned} H_1 &= \{u \mid x^0/u = x^1/u\} \\ &= \{u \mid \min(\exp(2\beta)/u_1,\ 1/u_2,\ 1/u_3) = \min(\exp(-\beta)/u_1,\ 1/u_2,\ 1/u_3)\} \\ &= \{u \mid \max(u_2, u_3) \ge \exp(\beta) u_1\} \end{aligned}$$

and

$$\begin{aligned} H_2 &= \{u \mid x^1/u = x^2/u\} \\ &= \{u \mid \min(\exp(-\beta)/u_1,\ 1/u_2,\ 1/u_3) = \min(\exp(-\beta)/u_1,\ \exp(-3\beta)/u_2,\ \exp(-3\beta)/u_3)\} \\ &= \{u \mid u_1 \ge \exp(2\beta)\max(u_2, u_3)\}. \end{aligned}$$

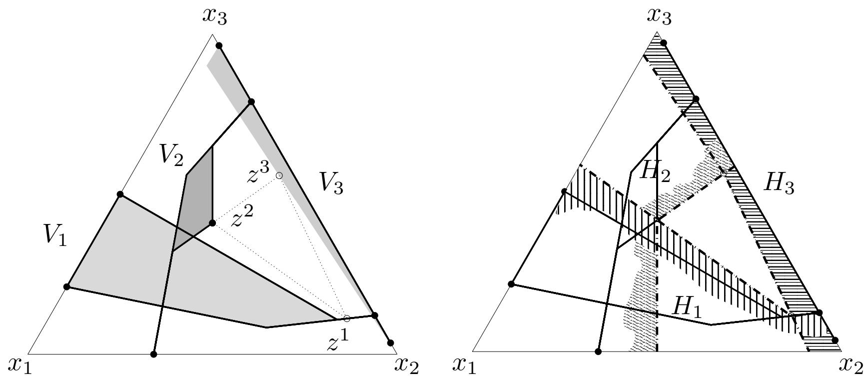

Fig. 2. Separation of three semimodules one of which is not Archimedean.

These two halfspaces are represented by the zones in light grey (right). The proof of Theorem 11 shows that their intersection is zero (meaning that it is reduced to the zero vector).

To illustrate Theorem 20, consider the matrices

$$A_1 = \begin{pmatrix} 0 & 2 & -\infty \\ -\infty & -\infty & 3 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & -\infty & 0 \\ 0 & 0 & 0 \\ -\infty & 2 & 0.5 \end{pmatrix}, \quad A_3 = \begin{pmatrix} -\infty & -\infty \\ 5 & 0 \\ 0 & 5 \end{pmatrix}.$$

As in the previous example, we define V_i as the semimodule generated by the vectors $\exp(\beta y)$, where y is a column of the matrix A_i. The three semimodules are represented at the left of Fig. 2, with the same conventions as above. The semimodule V_3, which does not contain Archimedean vectors, is represented by the bold segment included in the edge $[x_2, x_3]$ of the simplex. The grey zone containing this segment represents the semimodule generated by the vectors $\exp(\beta y)$, where y is a column of the matrix

$$A_3' = \begin{pmatrix} -\infty & -\infty & 1 \\ 5 & 0 & 0 \\ 0 & 5 & 5 \end{pmatrix}.$$

This semimodule, further denoted by V_3', is Archimedean and such that $V_3 \subset V_3'$ and $V_1 \cap V_2 \cap V_3' = \{0\}$. Its construction follows the proof of Proposition 19: we join the generators of V_3 with a new vector of the form $y \oplus \delta e^1$, where $y = \exp\big(\beta(-\infty, 0, 5)\big)$ is a vector from V_3, and $\delta = \exp(\beta)$ is sufficiently small to make $V_1 \cap V_2 \cap V_3' = \{0\}$.

We denote by P_i the projector on V_i, for $i = 1, 2$, and by P_3' the projector on V_3'. The cyclic projector $P_3' P_2 P_1$ has an Archimedean eigenvector with eigenvalue $\lambda < 1$. Indeed, let

$$\begin{aligned} z^0 &:= \exp\big(\beta(-3.5, 0, 0.5)^{\mathrm{T}}\big), \\ z^1 &:= P_1 z^0 = \exp\big(\beta(-3.5, 0, -3)^{\mathrm{T}}\big), \\ z^2 &:= P_2 z^1 = \exp\big(\beta(-3.5, -3.5, -3)^{\mathrm{T}}\big). \end{aligned}$$

It can be checked that

$$z^3 := P_3' z^2 = P_3' P_2 P_1 z^0 = \lambda z^0$$

with $\lambda = \exp(-3.5\beta)$. The points z^0, z^1, and z^2 are the vertices of the triangle with dotted edges on the left-hand side of the figure. These dotted edges indicate the action of the projectors P_1, P_2, and P_3'. Using Theorem 11, we get the following halfspaces:

$$\begin{aligned} H_1 &= \{u \mid \exp(3\beta)u_3 \le \max(\exp(3.5\beta)u_1,\ u_2)\}, \\ H_2 &= \{u \mid \exp(3.5\beta)u_2 \le \max(\exp(3.5\beta)u_1,\ \exp(3\beta)u_3)\}, \\ H_3 &= \{u \mid \exp(7\beta)u_1 \le \max(\exp(3.5\beta)u_2,\ \exp(3\beta)u_3)\}. \end{aligned}$$

We have $H_i \supset V_i$ and $H_1 \cap H_2 \cap H_3 = \{0\}$. The boundary of the halfspaces H_1, H_2, and H_3 is represented by bold dashed broken lines on the right-hand side of the figure. Each of these lines divides the main simplex into two parts, and the corresponding halfspace is the part indicated by the hatched region.

6. Acknowledgments

The two authors thank Peter Butkovič and Hans Schneider for illuminating discussions that led to the present work. The first author also thanks Frédéric Meunier for having drawn his attention to the max-plus analogues of Helly-type theorems. The second author is grateful to Andreĭ Sobolevskiĭ for valuable ideas and discussions concerning the analogy between convex geometry and idempotent analysis. The two authors are grateful to Ivan Singer and to an unknown referee for their careful reading and numerous remarks on the initial version of this paper.

This work was supported by the RFBR grant 05-01-00824 and the joint RFBR/CNRS grant 05-01-02807.

REFERENCES

1. F. L. Baccelli, G. Cohen, G. J. Olsder, and J. P. Quadrat, *Synchronization and Linearity*, Wiley, New York (1992).
2. H. H. Bauschke, J. M. Borwein, and A. S. Lewis, "The method of cyclic projections for closed convex sets in Hilbert space," in: Y. Censor and S. Reich, eds., *Recent Developments in Optimization Theory and Nonlinear Analysis*, Contemp. Math., Vol. 204, Amer. Math. Soc., Providence (1997), pp. 1–42.
3. G. Birkhoff, *Lattice Theory*, Amer. Math. Soc., Providence (1967).
4. T. S. Blyth and M. F. Janowitz, *Residuation Theory*, Pergamon Press (1972).
5. P. Butkovič, H. Schneider, and S. Sergeev, "Generators, extremals and bases of max cones," *Linear Algebra Appl.*, **421**, 394–406 (2007), `arXiv:math.RA/0604454`.
6. G. Cohen, S. Gaubert, and J. P. Quadrat, "Duality and separation theorems in idempotent semimodules," *Linear Algebra Appl.*, **379**, 395–422 (2004), `arXiv:math.FA/0212294`.
7. G. Cohen, S. Gaubert, J. P. Quadrat, and I. Singer, "Max-plus convex sets and functions," in: G. Litvinov and V. Maslov, eds., *Idempotent Mathematics and Mathematical Physics*, Contemp. Math., Vol. 377, Amer. Math. Soc., Providence (2005), pp. 105–129, `arXiv:math.FA/0308166`.
8. R. A. Cuninghame-Green, "Projections in minimax algebra," *Math. Programming*, **10**, No. 1, 111–123 (1976).
9. R. A. Cuninghame-Green, *Minimax Algebra*, Lect. Notes Economics Math. Systems, Vol. 166, Springer, Berlin (1979).
10. R. A. Cuninghame-Green and P. Butkovič, "The equation $A \otimes x = B \otimes y$ over $(\max, +)$," *Theor. Comput. Sci.*, **293**, 3–12 (2003).
11. M. Develin and B. Sturmfels, "Tropical convexity," *Documenta Math.*, **9**, 1–27 (2004), `arXiv:math.MG/0308254`.
12. H. G. Eggleston, *Convexity*, Cambridge Univ. Press (1958).
13. S. Gaubert and R. Katz, "The Minkowski theorem for max-plus convex sets," *Linear Algebra Appl.*, **421**, 356–369 (2007), `arXiv:math.GM/0605078`.
14. S. Gaubert and F. Meunier, private communication (2006).
15. J. Golan, *Semirings and Their Applications*, Kluwer, Dordrecht (2000).
16. V. N. Kolokoltsov and V. P. Maslov, *Idempotent Analysis and Applications*, Kluwer, Dordrecht (1997).
17. G. L. Litvinov, "Maslov dequantization, idempotent and tropical mathematics: A brief introduction," *J. Math. Sci.*, **140**, No. 3, 426–444 (2007).
18. G. Litvinov and V. Maslov, eds., *Idempotent Mathematics and Mathematical Physics*, Contemp. Math., Vol. 377, Amer. Math. Soc., Providence (2005).
19. G. L. Litvinov, V. P. Maslov, and G. B. Shpiz, "Idempotent functional analysis. An algebraic approach," *Math. Notes*, **69**, No. (5), 696–729 (2001), `arXiv:math.FA/0009128`.

20. V. Nitica and I. Singer, "The structure of max-plus hyperplanes," *Linear Algebra Appl.*, **426**, 382–414 (2007).
21. R. D. Nussbaum, "Convexity and log convexity for the spectral radius," *Linear Algebra Appl.*, **73**, 59–122 (1986).
22. R. T. Rockafellar, *Convex Analysis*, Princeton Univ. Press (1970).
23. S. N. Samborskiĭ and G. B. Shpiz, "Convex sets in the semimodule of bounded functions," in: V. P. Maslov and S. N. Samborskiĭ, eds., *Idempotent Analysis*, Adv. Sov. Math., Vol. 13, Amer. Math. Soc., Providence (1992), pp. 135–137.
24. K. Zimmermann, "A general separation theorem in extremal algebras," *Ekonomicko-Matematický Obzor*, **13**, No. 2, 179–201 (1977).

Stéphane Gaubert
INRIA, Rocquencourt, B.P. 105, 78105 Le Chesnay cedex, France
E-mail: `Stephane.Gaubert@inria.fr`

Sergeĭ Sergeev
Department of Physics, Sub-Department of Quantum Statistics and Field Theory,
Moscow State University, Moscow, 119992 Leninskie Gory, Russia
E-mail: `sergiej@gmail.com`