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ON THE DEPENDENCE OF THE MAXIMUM CYCLE MEAN OF A MATRIX ON PERMUTATIONS OF THE ROWS AND COLUMNS

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Let $(G,\cdot,\leq)$ be a radicable, linearly ordered, commutative group. Given a square matrix $A=(a_{ij})$ of order n with entries from G and a cyclic permutation $\sigma=(i_1,\dots,i_l)$ of a subset of $N=\{1,2,\dots,n\}$ we define $\mu(\sigma)$, the mean weight of σ, as $(a_{i_1i_2}\cdot a_{i_2i_3}\cdot\cdots\cdot a_{i_{l-1}i_l}\cdot a_{i_li_1})^{1/l}$ and $\lambda(A)$, the maximum cycle mean (MCM) of A, as $\max_\sigma\mu(\sigma)$, where σ ranges over all cyclic permutations of subsets of N. We study the dependence of the MCM of a matrix on the permutations of its rows and columns and particularly we prove an $O(n^2)$ algorithm for checking whether $\lambda(A)=\lambda(A')$ holds for any matrix A' which can be obtained from A by permuting its rows and columns.

1. Introduction

Let $(G,\cdot,\leq)$ be a linearly ordered, commutative group. For positive integer n we denote by G_n the set of all square matrices of order n with elements from G. The letter N stands for the set $\{1,2,\dots,n\}$. Given any $A=(a_{ij})\in G_n$ and a cyclic permutation $\sigma=(i_1,\dots,i_l)$ of a subset of N (shortly, cycle in N), define $w(\sigma)$, the weight of σ, as

$$a_{i_1i_2}\cdot a_{i_2i_3}\cdot\cdots\cdot a_{i_{l-1}i_l}\cdot a_{i_li_1}.$$

The number l, the length of σ, will be denoted by $l(\sigma)$.

Everywhere we suppose that G is radicable, i.e. for any $a\in G$ and positive integer t there exists $b\in G$ such that $b^t=a$. Such an element b is unique and will be denoted by $\sqrt[t]{a}$. Given any cycle σ, define $\mu(\sigma)$, the mean weight of σ, as $\sqrt[l(\sigma)]{w(\sigma)}$ and define $\lambda(A)$, the maximum cycle mean (MCM) of A, as $\max_\sigma\mu(\sigma)$, where σ ranges over all cycles in N. The task of finding $\lambda(A)$ can be formulated using the graph theoretic concepts as follows: given an arc-weighted digraph G find a cycle in G for which the sum of its arc weights divided by their number is the maximum possible. This is exactly the MCM problem in the additive group of reals for the associated matrix $A=(a_{ij})$ where a_{ij} is the weight of the arc(i,j) and $a_{ij}=-\infty$ if the arc(i,j) (including loops) does not exist.

Probably the most efficient method for computing the maximum cycle mean was presented in [3] (for the additive group of reals). An exhaustive investigation of this concept (among many other topics) was carried out in [1]. It was shown in this monograph, that $\lambda(A)$ is the unique eigenvalue of the matrix A in the structure derived from $(G,\cdot)$ by setting $+$ for the maximum and extending the operations $+$ and $\cdot$ to matrices in the same way as in conventional linear algebra. Besides, an application to scheduling in industrial processes was described. Another application devoted to ship routing problems was introduced in [2].

The aim of this paper is to study the dependence of the MCM of a matrix with respect to the permutation of its rows and columns. Note that the exchange of two columns corresponds, in the interpretation of [1], to the exchange of the role of two machines in an industrial process. To investigate this dependence we denote by P_n the set of all permutations of N. Given any

$$\pi,\varrho\in P_n \quad \text{and} \quad A=(a_{ij})\in G_n,$$

$A(\pi,\varrho)$ stands for the matrix $B=(b_{ij})\in G_n$ such that $b_{ij}=a_{\pi(i),\varrho(j)}$ for all $i,j\in N$. $L(A)$ denotes the set $\{\lambda(A(\pi,\varrho)):\pi,\varrho\in P_n\}$. As obvious,

$$A\begin{pmatrix} i_1 i_2\cdots i_k \\ j_1 j_2\cdots j_l\end{pmatrix}=\begin{bmatrix} a_{i_1j_1} & \cdots & a_{i_1j_l} \\ \vdots & & \vdots \\ a_{i_kj_1} & \cdots & a_{i_kj_l}\end{bmatrix} \quad \text{for } i_1,\dots,i_k,j_1,\dots,j_l\in N$$

and $A(i_1,\dots,i_k;j_1,\dots,j_l)$ denotes the matrix arising from A by deleting the rows with indices $i_1,\dots,i_k$ and columns with indices $j_1,\dots,j_l$.

Clearly, if $\pi\in P_n$ and $(i_1,\dots,i_l)$ is a cycle in N then $(\pi(i_1),\dots,\pi(i_l))$ is a cycle in N too, yielding that the weight of a cycle with respect to $A(\pi,\pi)$ is equal to the weight of a (possibly different) cycle with respect to A. It follows then from the monotonicity of $\sqrt[l]{\ }$ (cf. [1]) that

$$\lambda(A(\pi,\pi))\le\lambda(A),$$

where due to the symmetry, in fact, equality holds. It is clear that $A(\pi,\varrho)=B(\pi,\pi)$ where $\pi,\varrho\in P_n$ and $B=A(\mathrm{id},\pi^{-1}\circ\varrho)$. This indicates that every element of $L(A)$ is equal to an MCM of a matrix arising from A by permuting its columns (say) only. Hence $n!$ is an upper bound for the cardinality of $L(A)$. This can be improved to $n!-(n-1)!+1$ since there exist at least $(n-1)!$ permutations for which the maximal element of A will be diagonal.

We intend to present some more information concerning $L(A)$. The first deals with bounds for elements of $L(A)$. Further we characterize those matrices A for which $L(A)$ is a one element set and moreover, an $O(n^2)$ algorithm for checking this property will be proved. Matrices possessing this property, i.e. $\lambda(A(\pi,\varrho))=\lambda(A)$ for all $\pi,\varrho\in P_n$ will be called stationary.

Any cycle in a subset of N is also a cycle in N. This simple fact and the monotonicity of $\sqrt[l]{\ }$ leads to

Proposition 1.1. *MCM of a principal submatrix of A is less than or equal $\lambda(A)$.*

2. Bounds for $L(A)$

For $A=(a_{ij})\in G_n$ the symbol $\Delta(A)$ will denote $\max_{i,j\in N} a_{ij}$. The case of the least upper bound is easy: since

$$\sqrt[k]{x_1\cdot x_2\cdot\cdots\cdot x_k}\leq \max(x_1,\dots,x_k)$$

for $x_1,\dots,x_k\in G$ and positive integer k, we have that every cycle mean and hence also $\lambda(A)$ is less than or equal to $\Delta(A)$; moreover using any permutation of columns (say) after which the maximal element will belong to the diagonal, we achieve the equality. Thus we have proved the following proposition:

Proposition 2.1. $\max L(A)=\Delta(A)$.

On the other hand the case of the greatest lower bound is more complicated and can be shown to be NP-complete (see [4]). Nevertheless we derive a lower bound which is the best in some cases and will be useful later. For this purpose denote for $A=(a_{ij})\in G_n$:

$$\delta_C(A)=\min_{j\in N}\max_{i\in N} a_{ij},\qquad \delta_R(A)=\min_{i\in N}\max_{j\in N} a_{ij},$$

$$C_j(A)=\{i\in N\colon a_{ij}\geq\delta_C(A)\},$$

$C(A)$, respectively $R(A)$, denotes the set of column and row indices of A, respectively.

Proposition 2.2. $\min L(A)\geq\max(\delta_C(A),\delta_R(A))$.

Proof. Due to the symmetry it suffices to prove only $\delta_C(A)\leq\min L(A)$. Without loss of generality we suppose that $\lambda(A)=\min L(A)$, since $\delta_C(A)=\delta_C(A(\pi,\varrho))$ for all $\pi,\varrho\in P_n$. Let

$$\bigcup_{j\in C(A)} C_j(A)=\{i_1,\dots,i_k\}$$

and put

$$A_1=A\begin{pmatrix} i_1\cdots i_k\\ i_1\cdots i_k\end{pmatrix}.$$

Construct A_2 from A_1 and further $A_3,A_4,\dots$ in the same way. Put $A_0=A$. Clearly, $\delta_C(A_t)\geq\delta_C(A_{t-1})$ for $t=1,2,\dots$. Since $C_j(A_t)\neq\emptyset$ for all $t=1,2,\dots$ and $j\in C(A_t)$, an index p with property

$$\bigcup_{j \in C(A_p)} C_j(A_p) = R(A_p) \tag{1}$$

(and hence $A_p = A_{p+1} = \cdots$) exists. It is a well known combinatorial property that in a finite digraph a cycle exists whenever the outdegree of each vertex is nonzero. In other words, if each row of a matrix contains at least one element possessing a (fixed) property then a cycle containing only such elements exists. Thus, a cycle $\sigma = (i_1, \ldots, i_s)$ satisfying

$$i_q \in C_{i_{q+1}}(A_p) \quad \text{for } q = 1, \ldots, s-1$$

and

$$i_s \in C_{i_1}(A_p)$$

exists. Hence $a_{i_q i_{q+1}} \geq \delta_C(A_p)$ for all $q = 1, \ldots, s-1$ and $a_{i_s i_1} \geq \delta_C(A_p)$. Therefore

$$\lambda(A) \geq \mu(\sigma) \geq \delta_C(A_p) \geq \delta_C(A). \qquad \square$$

It is not difficult to find matrices for which the lower bound in Proposition 2.2 is the greatest. For instance, consider

$$A = \begin{pmatrix} 1 & 2 & 2 \cdots 2 \\ 0 & 1 & 0 \cdots 0 \\ 0 & 0 & 1 \cdots 1 \\ \vdots & & \ddots \vdots \\ 0 & 0 & 0 \cdots 1 \end{pmatrix}$$

in the additive group of reals. Evidently, the greatest diagonal element of $A(\pi, \varrho)$ is 1 or 2 for any $\pi, \varrho \in P_n$ and hence $\min L(A) = \delta_C(A) = \delta_R(A) = 1$. On the other hand, the following matrix shows that this is not the case in general:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Here $\min L(A) = \frac{2}{3}$, $\delta_C(A) = \delta_R(A) = 0$.

3. Stationary matrices

Given $A = (a_{ij}) \in G_n$, $d \in G$ we say that the kth row (the lth column) of A is d-triangular if

$$a_{kj} < d \quad \text{for all } j = 1, 2, \ldots, k$$
$$(a_{il} < d \quad \text{for all } i = l, l+1, \ldots, n).$$

Furthermore, A will be called d-triangular if all its rows (and hence also columns) are d-triangular: a row or column of A is said to be d-weak if all its elements are less than d. The prefix d will be omitted in the case when $d=\Delta(A)$. Hence the first column and the last row of any triangular matrix are weak.

Two matrices are said to be equivalent if one can be obtained from the other by permuting its rows and columns only. The set of matrices in G_n equivalent to a triangular matrix will be denoted by T_n.

It follows from Proposition 2.1 that for the stationary matrix A the equality $\lambda(A(\pi,\varrho))=\Delta(A)$ holds for all $\pi,\varrho\in P_n$. Furthermore, the condition

$$\max(\delta_C(A),\delta_R(A))=\Delta(A)$$

is sufficient for A to be stationary by Proposition 2.2. (Note that it is not necessary since e.g. the matrix

$$A=\begin{pmatrix}0 & 1 & 1\\ 0 & 1 & 1\\ 0 & 0 & 0\end{pmatrix}$$

is stationary but $\delta_C(A)=\delta_R(A)=0$ and $\Delta(A)=1$.) Clearly, for any triangular matrix A we have $\lambda(A)<\Delta(A)$. Hence we see immediately that all matrices in T_n are not stationary. Our main result shows that the converse implication is also true.

Theorem 3.1. *$A\in G_n$ is stationary if and only* if $A\notin T_n$.

Proof. It remains to prove the "if" part. Without loss of generality suppose that $\lambda(A)=\min L(A)<\Delta(A)$ and $n>2$ (note that the case $n=1$ is trivial since $T_1=\emptyset$ and the case $n=2$ can be easily verified). We find the triangular matrix equivalent to A by the construction of a finite sequence of matrices $B_0, A_1, B_1, A_2, \ldots, A_t$ possessing the following properties for all $j=0,1,2,\ldots$:

(a) B_j is equivalent to A and the first j columns and the last j rows of B_j are triangular;

(b) A_{j+1} is a principal submatrix of A and can be obtained from B_j by deleting the j first and j' ($j'\le j$) last columns and the j' first and j last rows.

For this purpose we put $B_0=A$, $A_1=A$ and suppose that $B_0, A_1, B_1, \ldots, A_s$ are already defined.

(i) If $\Delta(A_s)<\Delta(A)$ then ($s\ge 2$ and) B_{s-1} is triangular; the theorem follows. Note that this case covers all matrices A_s of order 1 since then $A_s=(a)$ for some $a\in G$ and $\Delta(A_s)=a=\lambda(A_s)\le\lambda(A)<\Delta(A)$.

(ii) Suppose $\Delta(A_s)=\Delta(A)$ and the order of A_s to be 2. Then by Proposition 1.1 we have $\lambda(A_s)\le\lambda(A)<\Delta(A)=\Delta(A_s)$ and A_s contains exactly one element equal to $\Delta(A)$ since it has (by Proposition 2.2) a weak (and hence also a $\Delta(A)$-weak) column and row. Thus, A_s is equivalent to a triangular matrix and hence $B_{s-1}\in T_n$.

(iii) Suppose $\Delta(A_s)=\Delta(A)$ and the order of A_s to be at least 3. Then by the same arguments as in (ii) A_s contains a $\Delta(A)$-weak column (say kth) and row (say lth). Suppose they are contained in the column k' and row l' of B_{s-1} and that A_s can be obtained from B_{s-1} by deleting the first $s'-1$ ($s'\le s$) and last $s-1$ rows and the first $s-1$ and last $s'-1$ columns. Clearly,

$$s'\le l'\le n-s+1$$

and

$$s\le k'\le n-s'+1.$$

In the rest of this proof we denote for $i,j\in N$, $i<j$ by $\sigma(i,j)$, respectively $\tau(i,j)$ the permutations

$$(1)(2)\dots(i-1)(i,i+1,\dots,j)(j+1)\dots(n)$$

and

$$(1)(2)\dots(i-1)(j,j-1,\dots,i)(j+1)\dots(n),$$

respectively. If $k=l$ then we put $B_s=B_{s-1}(\pi,\varrho)$ where $\pi=\sigma(l',n-s+1)$ and $\varrho=\tau(s,k')$ and $A_{s+1}=A_s(k;k)$. For treating the case $k\ne l$ we denote by k'' the index of that row of B_{s-1} which contains the kth row of A_s and let l'' be the index of the column of B_{s-1} containing the lth column of A_s. If $k<l$ then obviously

$$s'\le k''<l'\le n-s+1$$

and

$$s\le k'<l''\le n-s'+1.$$

We put

$$\pi=\sigma(l',n-s+1)\circ\tau(s',k'')$$

and

$$\varrho=\tau(s,k')\circ\sigma(l'',n-s'+1).$$

If $k>l$ then

$$s\le l''<k'\le n-s'+1 \quad\text{and}\quad s'\le l'<k''\le n-s+1.$$

We put

$$\pi=\sigma(l',n-s+1)\circ\tau(s',k''-1)$$

and

$$\varrho=\tau(s,k')\circ\sigma(l'',n-s'+1).$$

In both cases we take $B_s=B_{s-1}(\pi,\varrho)$ and $A_{s+1}=A_s(k,l;k,l)$. Hence A_{s+1} as a principal submatrix of A_s is also a principal submatrix of A. The order of A_s decreases in (iii) always by 1 or 2 and thus the procedure will stop after a finite number of steps by (i) or (ii). □

Theorem 3.1 turns in fact the problem of stationarity to the question of whether the given matrix is equivalent to a triangular one. We show now that this question can be answered by a simple $O(n^2)$ algorithm. It is based on the fact that the second column of the triangular matrix $A=(a_{ij})$ is either weak or contains at most one element equal to $\Delta(A)$. Its element a_{12} will be called the leader.

If we look for the permutations of rows and columns of a matrix transforming it to the triangular form then the following elements of A can become the leader only:

(a) any element equal to $\Delta(A)$ belonging to the column all other elements of which are less than $\Delta(A)$;

(b) any element of a weak column, if in A at least two weak columns exist.

We shall refer to any element of A satisfying (a) or (b) as to the candidate for the leader (or shortly, candidate). If $A=(a_{ij})\in G_n$ is equivalent to a triangular matrix in which a_{kl} is the leader then obviously $B=A(k;l)$ is equivalent to a $\Delta(A)$-triangular matrix. Conversely, if a_{kl} is a candidate and B is equivalent to a $\Delta(A)$-triangular matrix then A has this property too. Hence, if in A a unique candidate would exist then our question could be reduced to the $\Delta(A)$-triangularity of B. In the first insight it is probably not clear whether in the case of more candidates it is possible to take anyone to carry out this reduction. The last assertion shows that this is true.

Proposition 3.2. *If $A=(a_{ij})\in T_n$ and a_{kl} is an arbitrary candidate, then the columns and rows can be permuted in such a way that we get as the result a triangular matrix in which a_{kl} will become the leader.*

Proof. Without loss of generality suppose A to be triangular. Take a candidate, say a_{kl}. We construct a triangular matrix equivalent to A in which a_{kl} will become the leader.

At first suppose that a_{kl} is of the type (a) (thus $k<l$). Exchange the lth and $(k+1)$st column. The obtained matrix, say $B=(b_{ij})$, is evidently triangular too. Continue by exchanging the columns k and $k+1$ and the rows $k-1$ and k. The triangularity is again not touched since $b_{k-1,k+1}<\Delta(A)$. Our candidate is now in the row $k-1$ and column k. Using the analogue of the last exchange we can translate the candidate along the main diagonal until it becomes the leader.

Suppose now the lth column to be weak $(l>1)$. One can easily verify that the matrix $A(\pi,\varrho)$ where

$$\pi=(k,k-1,\dots,1)(k+1)\dots(n),$$
$$\varrho=(1)(l,l-1,\dots,2)(l+1)\dots(n)$$

is triangular and $a_{\pi(1),\varrho(2)}=a_{kl}$. □

The last proposition enables to immediately compile an $O(n^3)$ algorithm for checking the equivalence to a triangular matrix consisting of the repeated choice of any candidate a_{kl} (in $O(n^2)$ steps) and reduction (at most $n-2$ times) of the problem to the submatrix obtained by deleting the row k and column l. A little more careful approach will, however, lead to an essential decrease of the computational complexity. For this purpose, given any $A=(a_{ij})\in G_n$ we denote for all $j\in N$ by m_j the number of elements in the jth column equal to $\Delta(A)$ and define for all $i,j\in N$:

$$p_{ij}=\begin{cases}1, & \text{if } a_{ij}=\Delta(A),\\ 0, & \text{otherwise.}\end{cases}$$

In what follows we suppose without loss of generality that the first column and last row of A are weak. Hence to find a candidate means to look for an index $j\in N, j\neq 1$ satisfying $m_j\leq 1$. If it does not exist, we stop with the negative answer. Otherwise we take any j with this property (say the first). If $m_j=0$ then we take for the candidate any element of the jth column (say the first); if $m_j=1$, then the candidate will be its (unique) maximal element. The algorithm then proceeds in the same way for the matrix arising by deleting the candidate's row and column, until the matrix of order 2 is obtained. The new values of m_j, say m_j', can be recomputed easily by the formula

$$m_j'=m_j-p_{kj} \quad \text{for all } j\in N,$$

where k is the index of the deleted row. Thus in each iteration the number of operations for finding the candidate is $O(n)$ (for finding j with $m_j\leq 1)+O(n)$ (for finding the candidate in the jth column) $=O(n)$, instead of $O(n^2)$ in the original version.

At last we offer a more formal description of the algorithm written in PIDGIN-ALGOL:

Algorithm TRIANGULARITY

Input. $A=(a_{ij})\in G_n$, $n\geq 2$,

$$\max\left(\max_{j\in N} a_{nj}, \max_{i\in N} a_{i1}\right)<\Delta(A).$$

Output. $\pi,\varrho\in P_n$ such that $A(\pi,\varrho)$ is triangular and 'yes' in variable tr if $A\in T_n$; 'no' in tr otherwise.

begin
$\Delta(A):=\max_{i,j\in N} a_{ij}$, tr := 'no', $r:=2$;
$\pi(n):=n$, $\varrho(1):=1$;
fr(n) := 'no', fc(1) := 'no';
for all $i=1,2,\ldots,n-1$ **do** fr(i) := 'yes';
for all $j=2,3,\ldots,n$ **do** fc(j) := 'yes', $m_j:=0$;
for all $i,j\in N$ **do**
  if $a_{ij}=\Delta(A)$ **then** $m_j:=m_j+1$, $p_{ij}:=1$ **else** $p_{ij}:=0$;

stage:
  if $r = n$ **then goto** last;
  find $\min\{m_j\colon \mathrm{fc}(j) = \text{'yes'}\}$, say m_l;
  if $m_l > 1$ **then** stop;
  if $m_l = 1$ **then** $k :=$ (unique) $i \in N$ satisfying $\mathrm{fr}(i) = $ 'yes' and $a_{il} = \Delta(A)$;
  if $m_l = 0$ **then** $k := \min\{i \in N\colon \mathrm{fr}(i) = \text{'yes'}\}$;
  $\pi(r-1) := k$, $\varrho(r) := l$;
  $\mathrm{fr}(k) :=$ 'no', $\mathrm{fc}(l) :=$ 'no';
  $r := r + 1$, **goto** stage;
last:
  $\pi(n-1) :=$ (unique) i with $\mathrm{fr}(i) =$ 'yes';
  $\varrho(n) :=$ (unique) j with $\mathrm{fr}(j) =$ 'yes';
  tr := 'yes'
end

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