Discrete Applied Mathematics 11 (1985) 209–222
North-Holland

A CONDITION FOR THE STRONG REGULARITY OF MATRICES IN THE MINIMAX ALGEBRA

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Received 24 April 1984
Revised 16 November 1984

Columns of a matrix A in the minimax algebra are called strongly linearly independent if for some b the system of equations $A\otimes x=b$ is uniquely solvable (cf. [3]). This paper presents a condition which is necessary and sufficient for the strong linear independence of columns of a given matrix in the minimax algebra based on a dense linearly ordered commutative group. In the case of square matrices an $O(n^3)$ method for checking this property as well as for finding at least one b such that $A\otimes x=b$ is uniquely solvable is derived. A connection with the classical assignment problem is formulated.

1. Introduction

In the whole paper we suppose that $\mathscr{G}=(G,\otimes,\le)$ is a commutative, linearly ordered group. Its neutral element will be denoted by 1. Let $G^0=G\cup\{0\}$ where 0 is an adjoined element and extend $\otimes$ and $\le$ on G^0 by the rules:

$$0\otimes a=a\otimes 0=0 \quad \text{for all } a\in G^0, \tag{1}$$

$$0\le a \quad \text{for all } a\in G^0. \tag{2}$$

Obviously, $\le$ is a linear ordering on G^0. For $a,\ b\in G^0$ the symbol $a\oplus b$ will denote $\max\{a,b\}$. Many properties of $\oplus$, $\otimes$, $\le$ are derived in [3] and it will be useful to mention here two of them ($a<b$ means $a\le b$ and $a\ne b$):

$$a\le b \ \Rightarrow\ c\otimes a\le c\otimes b \quad \text{for all } a,b,c\in G^0; \tag{3}$$

$$a<b \ \Rightarrow\ c\otimes a<c\otimes b \quad \text{for all } a,b\in G^0, c\in G. \tag{4}$$

For $a,b\in G^0$ we denote by (a,b) the interval

$$\{x\in G^0 \mid a<x<b\}.$$

The order $\le$ on G^0 is called dense if $(a,b)\ne\emptyset$ for all $a,b\in G^0$, $a<b$. The following assertion can easily be verified using definitions.

Proposition 1. *Let $\le$ be dense on G^0 and a,b,c,d be elements of G^0 such that $a<b$, $c<d$. Then*

$$(a,b)\cap(c,d)\ne\emptyset \quad \textit{if and only if} \quad a<d \textit{ and } c<b.$$

We recall another simple fact which will be used later:

$$\text{An arbitrary finite system of intervals in a linearly ordered set has a nonempty intersection whenever each pair of intervals of this system has a nonempty intersection.} \tag{5}$$

The set of all (m, n) matrices over G and G^0 will be denoted by $G(m, n)$ and $G^0(m, n)$, respectively. If $n = 1$, then these sets will be written shortly G_m and G^0_m, respectively and their elements will be called vectors. Properties of matrices have also been investigated in [3] and we recall the associative and distribute laws for the operations $\oplus$, $\otimes$ extended in the natural way on these matrices.

In what follows we always suppose that $m, n \geq 1$ are given integers and we denote by S and N the sets $\{1, 2, \ldots, m\}$ and $\{1, 2, \ldots, n\}$, respectively; $P(n)$ will mean the set of all permutations of the set N.

If $\sigma \in P(n), d_1, \ldots, d_n \in G^0$ then $P_\sigma(d_1, \ldots, d_n)$ denotes the matrix $(p_{ij}) \in G^0(n, n)$ the elements of which satisfy the conditions

$$p_{ij} = d_i, \quad \text{if } j = \sigma(i),$$

$$p_{ij} = 0, \quad \text{if } j \neq \sigma(i)$$

for all $i, j \in N$. If σ is, moreover, an identity then instead of $P_\sigma(d_1, \ldots, d_n)$ we write as usual $\operatorname{diag}(d_1, \ldots, d_n)$. The matrix $\operatorname{diag}(1, 1, \ldots, 1)$ is called unity matrix.

For $A = (a_{ij}) \in G^0(m, n)$ and $b \in G_m$ we denote for all $j \in N$ by $S_j(A, b)$ the set

$$\left\{ i \in S \mid b_i^{-1} \otimes a_{ij} = \bigoplus_{s \in S} (b_s^{-1} \otimes a_{sj}) \right\}.$$

Systems of linear equations of the form

$$A \otimes x = b \tag{6}$$

have been treated in [3] and in the case of some special groups in [6].

Let us mention that the relation between dual variables of the classical transportation problem can be expressed as the system of equations of the form (6) where G is the additive group of reals with the inverse ordering. A more detailed explanation of this fact can be found in [3, pp. 7–8].

Recall two results concerning the solution of (6) on which the main results of this paper are based.

Proposition 2. *Let* $A \in G(m, n)$, $b \in G_m$. *Then:*

(a) *The system* (6) *is solvable if and only if*

$$\bigcup_{j \in N} S_j(A, b) = S, \tag{7}$$

i.e. the system $\{S_j(A, b) \mid j \in N\}$ *is a covering of* S.

(b) *The system* (6) *is uniquely solvable if and only if* (7) *and the implication*

$$N' \subseteq N,\ N' \neq N \ \Rightarrow\ \bigcup_{j \in N'} S_j(A, b) \neq S$$

hold, i.e. the system $\{S_j(A, b) \mid j \in N\}$ *is a minimal covering of* S.

Recall that if $\mathcal{S} = \{S_j(A, b) \mid j \in N\}$ is a minimal covering of N, then $S_1(A, b), \ldots, S_n(A, b)$ are pairwise disjoint one-element sets. To see this realize that for every $k \in N$ there must exist $i_k \in S_k(A, b) - \bigcup_{j \in N - \{k\}} S_j(A, b)$ for, otherwise $\mathcal{S} - \{S_k(A, b)\}$ would be a covering of N, too. As a consequence, for $m = n$ (6) is uniquely solvable if and only if $S_1(A, b), \ldots, S_n(A, b)$ are pairwise disjoint one-element sets.

It has been shown in [3] that certain job-scheduling problems can be formulated as problems of solving the system (6) for $m = n$ in the additive group of reals. Here b plays the role of prescribed termination times of the work on m machines after a certain finite number of cycles and x_i are starting times we want to know. In many cases the components of x can move in an interval without any change of the fact that x is a solution of (6) but a natural question arises: can it happen for some b that (6) will have exactly one solution? It turns out that there exists a class of matrices for which the answer is positive.

Let $A \in G(m, n)$. The set

$$\mathrm{ir}(A) = \{b \in G_m \mid \text{there exists a unique } x \in G_n \text{ such that } A \otimes x = b\}$$

is called irreducibility set of the matrix A. We say that the columns of A are strongly linearly independent (cf. [3]) or, shortly SLI, if $\mathrm{ir}(A) \neq \emptyset$. In the case $m = n$ we say that A is strongly regular.

In the light of what has been said above matrix A is strongly regular if and only if there exists a vector b such that $S_1(A, b), \ldots, S_n(A, b)$ are pairwise disjoint one-element sets.

Example 1. Let $\mathcal{G}$ be the multiplicative group of positive reals with the obvious ordering. The columns of A are SLI where

$$A = \begin{pmatrix} 4 & 4 & 6 \\ 1 & 4 & 1 \\ 2 & 4 & 3 \\ 1 & 1 & 1 \\ 2 & 3 & 6 \end{pmatrix}$$

because for $b = (2, 1, 1, \frac{1}{2}, 2)^{\mathrm{T}}$ we get $S_1(A, b) = \{1, 3, 4\}$, $S_2(A, b) = \{2, 3\}$, $S_3(A, b) = \{1, 3, 5\}$ yielding that these sets form a minimal covering of the set $S = \{1, 2, 3, 4, 5\}$. Hence, the system $A \otimes x = b$ has a unique solution and $b \in \mathrm{ir}(A)$.

Example 2. For the same $\mathcal{G}$ consider

$$A = \begin{pmatrix} 3 & 1 & 3 \\ 1 & 1 & 1 \\ 2 & 4 & 3 \end{pmatrix}.$$

We will not be succesful in finding some $b \in R_3^{\dagger}$ such that the system $A \otimes x = b$ would be uniquely solvable. Theorem 3 will show that such a b in fact does not exist.

Let $A = (a_{ij}) \in G^0(n, n)$, $\sigma \in P(n)$. The product

$$a_{1,\sigma(1)} \otimes a_{2,\sigma(2)} \otimes \ldots \otimes a_{n,\sigma(n)}$$

will be denoted by $w(A, \sigma)$ and the sum $\oplus_{\sigma \in P(n)} w(A, \sigma)$ by per(A) (and called permanent of A). If, moreover, $j_1, \ldots, j_t \in N$, $t \geq 2$, then the product

$$a_{j_1 j_2} \otimes a_{j_2 j_3} \otimes \ldots \otimes a_{j_{t-1} j_t}$$

will be denoted by $A(j_1, \ldots, j_t)$. A permutation $\sigma \in P(n)$ is called maximal with respect to A if per$(A) = w(A, \sigma)$. We say that A has a strong permanent if there exists just one permutation maximal with respect to A.

The aim of this paper is

(i) to show that strongly regular matrices are exactly those with strong permanent whenever $\leq$ is dense, and

(ii) to derive a method for checking this property as well as for finding at least one $b \in \mathrm{ir}(A)$.

2. Auxiliary results

The following two assertions show that permutations of the rows and columns of a matrix A as well as multiplying them by non-zero constants do not influence the strong regularity of A as well as the fact that A has a strong permanent.

Proposition 3. *Let $\sigma, \tau \in P(n)$, $s_1, \ldots, s_n$, $t_1, \ldots, t_n \in G$.*

(a) *$A \in G^0(n, n)$ has a strong permanent if and only if the matrix $B = P_\sigma(s_1, \ldots, s_n) \otimes A \otimes P_\tau(t_1, \ldots, t_n)$ has a strong permanent, and*

(b) per$(B) = s_1 \otimes \ldots \otimes s_n \otimes t_1 \otimes \ldots \otimes t_n \otimes$ per(A).

Proof. One can easily verify that the product

$$P_\sigma(s_1, \ldots, s_n) \otimes P_{\sigma^{-1}}(s_{\sigma^{-1}(1)}^{-1}, \ldots, s_{\sigma^{-1}(n)}^{-1})$$

gives as result the unity matrix. Because of this fact it suffices to prove only the necessity in (a) since

$$A = P_{\sigma^{-1}}(s_{\sigma^{-1}(1)}^{-1}, \ldots, s_{\sigma^{-1}(n)}^{-1}) \otimes B \otimes P_{\tau^{-1}}(t_{\tau^{-1}(1)}^{-1}, \ldots, t_{\tau^{-1}(n)}^{-1}).$$

Let $B' = (b'_{ij}) = P_\sigma(s_1, \ldots, s_n) \otimes A$. Since

$$b'_{ij} = (\underbrace{0, \ldots, 0}_{\sigma(i)}, s_i, 0, \ldots, 0) \otimes (a_{1j}, \ldots, a_{nj})^{\mathrm{T}} = s_i \otimes a_{\sigma(i),j},$$

for all $\varrho \in P(n)$ we have

$$\begin{aligned} w(B', \varrho) &= s_1 \otimes a_{\sigma(1),\varrho(1)} \otimes s_2 \otimes a_{\sigma(2),\varrho(2)} \otimes \cdots \otimes s_n \otimes a_{\sigma(n),\varrho(n)} \\ &= s_1 \otimes \ldots \otimes s_n \otimes a_{1,\omega(1)} \otimes \ldots \otimes a_{n,\omega(n)} \quad \text{where } \omega = \varrho\sigma^{-1}. \end{aligned}$$

Hence, denoting $s_1 \otimes \ldots \otimes s_n \in G$ by s we get

$$w(B', \varrho) = s \otimes w(A, \varrho\sigma^{-1})$$

for all $\varrho \in P(n)$ and thus $\mathrm{per}(B') = s \otimes \mathrm{per}(A)$. Moreover, $\varrho' \neq \varrho$ implies $\varrho'\sigma^{-1} \neq \varrho\sigma^{-1}$. From this fact we get that $w(B', \varrho) = \mathrm{per}(B')$ and $\varrho' \in P(n) - \{\varrho\}$ imply $w(B', \varrho') < w(B', \varrho)$ for, otherwise we would have

$$w(A, \varrho'\sigma^{-1}) = s^{-1} \otimes w(B', \varrho') = s^{-1} \otimes w(B', \varrho) = w(A, \varrho\sigma^{-1}) = \mathrm{per}(A),$$

a contradiction.

The product $B' \otimes P_\tau(t_1, \ldots, t_n)$ can be treated similarly. □

Proposition 4. *Let* $\sigma \in P(m)$, $\tau \in P(n)$, $s_1, \ldots, s_m$, $t_1, \ldots, t_n \in G$. *Columns of* $A \in G(m, n)$ *are* SLI *if and only if columns of*

$$P_\sigma(s_1, \ldots, s_m) \otimes A \otimes P_\tau(t_1, \ldots, t_n)$$

are SLI.

Proof. We prove only the necessity of the condition for the same reasons as in the foregoing proof.

Let $B' = P_\sigma(s_1, \ldots, s_m) \otimes A$. If x is the unique solution of

$$A \otimes x = b, \tag{8}$$

then x is also the unique solution of

$$B' \otimes x = b' \tag{8'}$$

where $b' = P_\sigma(s_1, \ldots, s_m) \otimes b \in G_m$ because

$$A = P_{\sigma^{-1}}(s^{-1}_{\sigma^{-1}(1)}, \ldots, s^{-1}_{\sigma^{-1}(n)}) \otimes B'$$

(cf. the foregoing proof) and the existence of another solution of (8′), say y, would yield that y solves (8), too.

Let $B'' = A \otimes P_\tau(t_1, \ldots, t_n)$, $\tau \in P(n)$ and (8) be uniquely solvable. Clearly, $B'' \in G(m, n)$ and $S_{\tau(j)}(B'', b) = S_j(A, b)$ for all $j \in N$ and thus the system $\{S_1(B'', b), \ldots, S_n(B'', b)\}$ is the same covering of S as $\{S_1(A, b), \ldots, S_n(A, b)\}$ which is minimal. □

A square matrix A over G is said to be normal if all its diagonal elements as well as $\operatorname{per}(A)$ are 1.

Proposition 5. *Let $A=(a_{ij})\in G(n,n)$ and $\sigma\in P(n)$ be maximal with respect to A. Then the matrix*

$$B=(b_{ij})=P_{\sigma^{-1}}(a^{-1}_{\sigma^{-1}(1),1},\dots,a^{-1}_{\sigma^{-1}(n),n})\otimes A$$

is normal.

Proof. Denote $a^{-1}_{\sigma^{-1}(i),i}$ by d_i for all $i\in N$ and $d_1\otimes\dots\otimes d_n$ by d. It follows from Proposition 3(b) that

$$\operatorname{per}(B)=d\otimes w(A,\sigma)=1.$$

Moreover, taking an arbitrary $i\in N$ we obtain

$$b_{ii}=(\underbrace{0,\dots,0,d_i}_{\sigma^{-1}(i)},0,\dots,0)\otimes(a_{1i},\dots,a_{ni})^{T}=d_i\otimes a_{\sigma^{-1}(i),i}=1. \qquad \square$$

3. Every strongly regular matrix has a strong permanent

Theorem 1. *Let $A\in G(n,n)$ be strongly regular. Then A has a strong permanent.*

Proof. Suppose that $b=(b_1,\dots,b_n)^T\in G_n$ is such that the system $\{S_j(A,b)\mid j\in N\}$ is a minimal covering of N and let $B=(b_{ij})=\operatorname{diag}(b_1^{-1},\dots,b_n^{-1})\otimes A$. Then $S_1(A,b),\dots,S_n(A,b)$ are disjoint one-element sets. According to Proposition 4 we may assume without any loss of generality that

$$S_j(A,b)=\{j\}\quad\text{for all } j\in N \tag{9}$$

and due to Proposition 3 it is sufficient to prove that B has a strong permanent. But (9) yields that $b_{jj}>b_{sj}$ for all $j\in N$ and $s\in N-\{j\}$. That's why we get (using (4)) $\operatorname{per}(B)=b_{11}\otimes\dots\otimes b_{nn}>w(B,\sigma)$ for every $\sigma\in P(n)$ different from the identity. $\square$

4. Every matrix with strong permanent is strongly regular

Theorem 2. *Let the ordering $\le$ be dense and $A\in G(n,n)$ be normal. If A has a strong permanent, then A is strongly regular.*

Theorem 2 will be proved in Section 6.

Theorem 3. *Let the ordering $\le$ be dense. If $A\in G(n,n)$ has a strong permanent, then A is strongly regular.*

Proof. Suppose that $A = (a_{ij}) \in G(n, n)$ has a strong permanent. According to Proposition 5 there exist $d_1, \ldots, d_n \in G$ and $\sigma \in P(n)$ such that

$$P_\sigma(d_1, d_2, \ldots, d_n) \otimes A$$

is normal. This fact ensures using Propositions 3, 4 and Theorem 2 that A is strongly regular. □

Theorems 1 and 3 give a condition being necessary and sufficient for the strong regularity of an arbitrary square matrix over a dense linearly ordered commutative group.

Example 2 (continued). For the matrix A we can now easily check the strong regularity. Its permanent is

$$9 \oplus 2 \oplus 12 \oplus 6 \oplus 12 \oplus 3 = 12$$

and thus equals $w(A, \sigma)$ for two permutations $\sigma \in P(3)$.

Hence we conclude that A has no strong permanent and according to Theorem 1 it is not strongly regular.

Example 3. For the same $\mathcal{G}$ and matrix

$$A = \begin{pmatrix} 3 & 4 & 2 \\ 5 & 3 & 1 \\ 5 & 3 & 2 \end{pmatrix}$$

$\operatorname{per}(A) = 18 \oplus 20 \oplus 30 \oplus 30 \oplus 9 \oplus 40$, i.e. A has a strong permanent and thus (Theorem 3) it is strongly regular.

The problem of finding some $b \in \operatorname{ir}(A)$ will be solved at the end of the paper and the method will be illustrated at this matrix.

Remark. Theorem 3 does not hold, in general, without the assumption that $\leq$ is dense. To demonstrate this fact consider matrix $A = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right)$ over the additive group of integers. In this case A has a strong permanent $(3 \oplus 2)$ but obviously A is not strongly regular.

5. The case of rectangular matrices

We say that a matrix $A \in G(m, n)$ has rank k (written $r(A) = k$) if k is the greatest natural number for which there exists a strongly regular submatrix $B \in G(k, k)$ of A.

Theorem 4. *Let $A \in G(m, n)$. Then the columns of A are* SLI *if and only if $r(A) = n$.*

Proof. We note at first that according to Proposition 2, $m<n$ would yield $\text{ir}(A)=\emptyset$.

Suppose that $b\in \text{ir}(A)$. Hence the system

$$\{S_j(A,b)\mid j\in N\}$$

is a minimal covering of S and thus for each $k\in N$ there exists some $i_k\in S_k(A,b)-\bigcup_{j\in N-\{k\}} S_j(A,b)$. Choosing rows with indices $i_1,\dots,i_n$ from A we get a matrix $A'\in G(n,n)$ we are looking for because denoting by b' the subvector of b corresponding to the choosen rows we obtain that $S_j(A',b')$, $j\in N$ are pairwise disjoint one-element sets.

To prove the converse implication let us suppose that the matrix A' consisting of the rows of A with indices $i_1,i_2,\dots,i_n$ is strongly regular. Then there exists $c\in G_n$ such that the system

$$A'\otimes x=c \tag{9'}$$

has unique solution, say $\hat{x}$. Denote $A\otimes\hat{x}$ by b. Then $\hat{x}$ is, naturally, a solution of $A\otimes x=b$ and the existence of another solution would yield that (9′) has more than one solution, a contradiction. □

6. The proof of Theorem 2

Before proving Theorem 2 we establish some lemmas. Everywhere we suppose that $A=(a_{ij})\in G(n,n)$.

Lemma 1. *If A is normal, then*

$$A(j_1,\dots,j_k,j_1)\le 1 \tag{10}$$

for all $j_1,\dots,j_k\in N$, $k\ge 1$ integer such that $j_r\ne j_s$ for $r\ne s$. Moreover, if A has a strong permanent, then equality in (10) *holds only for $k=1$.*

Proof. Take $\sigma\in P(n)$ defined by formulas:

$$\sigma(j_i)=j_{i+1}\quad \text{for } i=1,2,\dots,k-1;$$

$$\sigma(j_k)=j_1;$$

$$\sigma(j)=j\qquad \text{for } j\in N-\{j_1,\dots,j_k\}=N'.$$

Then

$$\begin{aligned}1&\ge w(A,\sigma)\\ &=a_{j_1j_2}\otimes a_{j_2j_3}\otimes\dots\otimes a_{j_kj_1}\otimes\bigotimes_{j\in N'}a_{jj}\\ &=A(j_1,\dots,j_k,j_1).\end{aligned}$$

If $k>1$, then σ is different from identity and thus we have $A(j_1,\dots,j_k,j_1)=w(A,\sigma)<\text{per}(A)=1$, assuming that A has a strong permanent. □

Lemma 2. *If A is normal, then*

$$A(j_1,\dots,j_k)\le A(j_1,\dots,j_r,j_{s+1},\dots,j_k) \tag{11}$$

for arbitrary $j_1,\dots,j_k\in N$, $k\ge 3$ integer and $r,s\in\{1,2,\dots,k-1\}$ such that $r<s$, $j_r=j_s$ and $j_p\ne j_q$ for $p\ne q$ and $p,\ q\in\{r,r+1,\dots,s-1\}$. If, moreover, A has a strong permanent then the equality in (11) *holds only for $s=r+1$.*

Proof. Obviously, $A(j_1,\dots,j_k)=A(j_1,\dots,j_r,j_{s+1},\dots,j_k)\otimes A(j_r,\dots,j_s)$ $\le A(j_1,\dots,j_r,j_{s+1},\dots,j_k)$ since $j_r=j_s$ and

$$A(j_r,\dots,j_s)\le 1 \tag{12}$$

according to Lemma 1. If A has a strong permanent, then Lemma 1 yields that the equality in (12) can hold only for $s=r+1$. □

In what follows we denote by Z^+ the set of nonnegative integers. For $k,l\in N$ we denote by $E(A,k,l)$ and $F(A,k,l)$ the finite sets

$$\{A(k,j_1,\dots,j_t,l)\,|\,j_1,\dots,j_t\in N-\{k,l\};\ t\in Z^+;\ j_r\ne j_s \text{ for } r\ne s\}$$

and

$$\{(A(k,j_1,\dots,j_t,l))^{-1}\,|\,j_1,\dots,j_t\in N-\{k,l\};\ t\in Z^+;\ j_r\ne j_s \text{ for } r\ne s\},$$

respectively and we put

$$m(A,k,l)=\max E(A,k,l),$$

$$M(A,k,l)=\min F(A,k,l).$$

It is obvious that $m(A,k,l)=(M(A,k,l))^{-1}$ for all $k,l\in N$.

Lemma 3. *If A is normal, then for all $k,l\in N$*

$$\max\{A(k,j_1,\dots,j_t,l)\,|\,j_1,\dots,j_t\in N,t\in Z^+\}\ \textit{(denoted by } m'(A,k,l)\textit{)},$$

$$\min\{(A(k,j_1,\dots,j_t,l))^{-1}\,|\,j_1,\dots,j_t\in N,\ t\in Z^+\}\ \textit{(denoted by } M'(A,k,l)\textit{)}$$

exist and the following equalities hold:

$$m'(A,k,l)=m(A,k,l),$$

$$M'(A,k,l)=M(A,k,l).$$

Proof. It is sufficient to prove the inequalities

$$A(k,j_1,\dots,j_t,l)\le m(A,k,l) \tag{13}$$

and

$$(A(k,j_1,\dots,j_t,l))^{-1}\ge M(A,k,l) \tag{14}$$

for all $k, l, j_1, \dots, j_t \in N$. According to Lemma 2 the subsequences of the sequence $k, j_1, \dots, j_t, l$ the equal members of which are only the first and the last ones may be omitted successively (with the exception of the first members) without decreasing the value of the corresponding product. Obviously, after finite number of such deletions we obtain a product which is an element of $E(A, k, l)$. This yields (13) and (14) can be proved similarly. □

Denote by $\Delta(A)$ the strongly complete, arc-weighted digraph associated with A. We notice that the quantities $m(A, k, l)$, $M(A, k, l)$, resp. $m'(A, k, l)$, $M'(A, k, l)$ are just the lengths of the shortest and the longest paths and elementary paths in $\Delta(A)$, respectively. Thus, Lemma 3 describes the following property: If A is normal, then the lengths of the longest and the shortest paths between arbitrary two (not necessarily distinct) nodes in $\Delta(A)$ are lengths of elementary paths.

For $k, l \in N$, $k<l$ and a normal matrix A we define intervals

$$I(A, k, l) = (m'(A, k, l),\ M'(A, l, k)). \tag{15}$$

It follows from Lemma 3 that

$$I(A, k, l) = (m(A, k, l),\ M(A, l, k)) \quad \text{for all } k, l \in N.$$

Lemma 4. *Suppose that $\leq$ is dense, A is normal and has a strong permanent. Then $I(A, k, l) \neq \emptyset$ for all $k, l \in N, k<l$.*

Proof. It is sufficient to prove that

$$A(k, j_1, \dots, j_t, l) < (A(l, i_1, \dots, i_q, k))^{-1} \tag{16}$$

for arbitrary $j_1, \dots, j_t$, $i_1, \dots, i_q \in N - \{k, l\}$, t, $q \in Z^+$, $i_r \neq i_s$ and $j_r \neq j_s$ for $r \neq s$ because the sets $E(A, k, l)$ and $F(A, k, l)$ are finite and $\leq$ is dense. Inequality (16) is, however, equivalent to

$$1 > A(l, i_1, \dots, i_q, k) \otimes A(k, j_1, \dots, j_t, l) =$$
$$= A(l, i_1, \dots, i_q, k, j_1, \dots, j_t, l).$$

Using Lemma 2 several times we get

$$A(l, i_1, \dots, i_q, k, j_1, \dots, j_t, l) \leq A(l, h_1, \dots, h_p, l)$$

where

$$\{h_1, \dots, h_p\} \subseteq \{i_1, \dots, i_q, j_1, \dots, j_t, k\}$$

and $h_r \neq h_s$ for $r \neq s$. Thus by Lemma 1 we have that

$$A(l, h_1, \dots, h_p, l) \leq 1$$

and the equality would hold only if $\{h_1, \dots, h_p\} = \emptyset$ which is impossible because $l \notin \{i_1, \dots, i_q, j_1, \dots, j_t, k\}$. □

If $I=(a,b)\subseteq G$ and $c\in G$, then the interval $(c\otimes a,\ c\otimes b)$ will be denoted by cI. For $l=2,3,\dots,n$ and $w_1,\dots,w_{l-1}\in G$ the symbol $J(w_1,\dots,w_{l-1};l)$ will denote the set

$$w_1 I(A,1,l)\cap w_2 I(A,2,l)\cap\dots\cap w_{l-1}I(A,l-1,l)$$

and $J(1)$ will mean $\{1\}$.

Lemma 5. *Suppose that $\leq$ is dense, A is normal, $I(A,k,l)\neq\emptyset$ for all $k,l\in N$, $k<l$. Let $l\in N$ and $w_1,\dots,w_{l-1}$ be arbitrary elements of G satisfying the condition*

$$w_{l'}\in J(w_1,\dots,w_{l'-1};l')$$

for all $l'\in\{1,2,\dots,l-1\}$. Then

$$J(w_1,\dots,w_{l-1};l)\neq\emptyset.$$

Proof. Fact (5) ensures that it is sufficient to prove

$$w_k I(A,k,l)\cap w_m I(A,m,l)\neq\emptyset \tag{17}$$

for $1\leq k<m<l$ (case $l=2$ is trivial) or equivalently (cf. Proposition 1), to prove the inequalities

$$w_k\otimes A(k,j_1,\dots,j_t,l)<w_m\otimes(A(l,i_1,\dots,i_q,m))^{-1} \tag{18}$$

and

$$w_m\otimes A(m,j_1,\dots,j_t,l)<w_k\otimes(A(l,i_1,\dots,i_q,k))^{-1}. \tag{19}$$

However (18) is equivalent to the inequality

$$w_k\otimes A(k,j_1,\dots,j_t,l,i_1,\dots,i_q,m)<w_m$$

which follows from the assumption $w_m\in w_k I(A,k,m)$. The inequality (19) can be proved similarly. □

Lemma 6. *Let $A\in G(n,n)$ and all diagonal elements of A be 1. If $d=(d_1,\dots,d_n)^{\mathrm{T}}\in G_n$ is a solution of the system of inequalities*

$$a_{ij}\otimes d_i<d_j,\qquad i,j\in N,\quad i\neq j \tag{20}$$

then $\bar d=(d_1^{-1},\dots,d_n^{-1})^{\mathrm{T}}\in\mathrm{ir}(A)$ (and hence A is strongly regular).

Proof. From (20) it follows that $S_j(A,\bar d)=\{j\}$ for all $j\in N$. Thus $\{S_j(A,\bar d)\mid j\in N\}$ is a minimal covering of $\{1,2,\dots,n\}$. □

Theorem 2 follows immediately from Lemma 4 and from the following assertion.

Lemma 7. *Let A be normal and $\leq$ be dense. Then the condition*

$$I(A,k,l)\neq\emptyset\quad\textit{for all } k,l\in N,\ k<l \tag{21}$$

is necessary and sufficient for A to be strongly regular.

Moreover, every vector $(w_1^{-1}, \ldots, w_n^{-1})^{\mathrm{T}}$ *such that*

$$w_l \in J(w_1, \ldots, w_{l-1}; l) \quad \textit{for all } l \in N \tag{22}$$

is an element of $\mathrm{ir}(A)$.

Proof. If A is strongly regular, then by Theorem 1 it has a strong permanent and thus Lemma 4 implies the necessity of (21).

If (21) is fulfilled, then by Lemma 5 there exists $w = (w_1, \ldots, w_n)^{\mathrm{T}}$ satisfying (22). Due to Lemma 6 the proof will be completed by showing that w is a solution of (20).

If $i<j$, then $w_j \in w_i I(A, i, j)$ and thus

$$w_j > w_i \otimes m(A, i, j) \geq w_i \otimes A(i, j) = w_i \otimes a_{ij}.$$

If $i>j$, then $w_i \in w_j I(A, j, i)$ and thus

$$w_i < w_j \otimes M(A, j, i) \leq w_j \otimes (A(i, j))^{-1} = w_j \otimes a_{ij}^{-1}. \qquad \square$$

7. A method for checking the strong regularity

Checking the strong regularity of a given square matrix A by the results of Sections 4 and 6 would not be effective in general because one would have to compute $w(A, \sigma)$ for all $\sigma \in P(n)$, i.e. for $n!$ permutations. Besides, it is not clear enough how to find at least one $b \in \mathrm{ir}(A)$ (if such b exists). We try now to make these aspects clear.

If a maximal permutation with respect to A is known then Propositions 4 and 5 reduce the problem of checking the strong regularity of A to the same problem for a normal matrix. Note that in the case when G is the additive group of reals the problem of finding the maximal permutation is in fact the classical assignment problem the updated algorithm for which can be implemented in $O(n^3)$ time (cf. [4]).

Proposition 6. *Let $A \in G(n, n)$ be normal and $A^{n-1} = (g_{ij})$. Then for all $i, j \in N$, $i<j$*

$$I(A, i, j) = (g_{ij}, g_{ji}^{-1})$$

where $I(A, i, j)$ are intervals defined by (15).

Proof. It is not difficult to verify (cf. [3]) that $a_{ii} = 1$ for all $i \in N$ implies the equality

$$A \oplus A^2 \oplus \ldots \oplus A^{n-1} = A^{n-1}$$

Thus, taking $i, j \in N$, $i \neq j$ we get

$$g_{ij} = \bigoplus_{t=0}^{n-2} \bigoplus_{j_1, \ldots, j_t = 1}^{n} a_{ij_1} \otimes a_{j_1 j_2} \otimes \ldots \otimes a_{j_t j} = \max H$$

where we have denoted by H the set

$$\{A(i, j_1, \ldots, j_t, j) \mid j_1, \ldots, j_t \in N,\ t \in \{0, 1, \ldots, n-2\}\}.$$

Since $E(A, i, j) \subseteq H$, we have the inequality

$$m(A, i, j) \leq \max H.$$

The reverse inequality follows from the fact that

$$\max H \leq m'(A, i, j)$$

and from Lemma 3. Thus, $g_{ij} = m(A, i, j)$. $\square$

We summarize obtained results in a method for checking the strong regularity of a given $A = (a_{ij}) \in G(n, n)$, assuming that $\leq$ is dense:

(i) Find $\sigma \in P(n)$ being maximal with respect to A.
(ii) Set $B = P_{\sigma^{-1}}(a^{-1}_{\sigma^{-1}(1), 1}, \ldots, A^{-1}_{\sigma^{-1}(n), n}) \otimes A$ and compute $B^{n-1} = (g_{ij})$.
(iii) Check whether

$$g_{ij} < g_{ji}^{-1} \text{ for all } i, j \in N,\ i < j.$$

In the negative case stop (A is not strongly regular by Propositions 5, 6 and Lemma 7).
(iv) Find $\bar{w} = (w_1^{-1}, \ldots, w_n^{-1})^{\mathrm{T}}$ by the formula

$$w_l \in J(w_1, \ldots, w_{l-1}; l) \quad \text{for } l = 1, 2, \ldots, n$$

(such $\bar{w}$ exists according to Proposition 6 and Lemma 5).
(v) Compute $b = P_\sigma(a_{1,\sigma(1)}, \ldots, a_{n,\sigma(n)}) \otimes \bar{w}$ which is an element of $\mathrm{ir}(A)$ (cf. the beginning of the proof of Proposition 4).

Note that it remains an open question how to describe the whole set $\mathrm{ir}(A)$.

Example 3 (continued). We check alternatively the strong regularity of A by the just described method. One can easily verify that here (in the algebraic notation) $\sigma = (12)(3) = \sigma^{-1}$ and

$$P_{\sigma^{-1}}(a^{-1}_{\sigma^{-1}(1), 1}, a^{-1}_{\sigma^{-1}(2), 2}, a^{-1}_{\sigma^{-1}(3), 3}) = \begin{pmatrix} 0 & \frac{1}{5} & 0 \\ \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

Thus

$$B = \begin{pmatrix} 1 & \frac{3}{5} & \frac{1}{5} \\ \frac{3}{4} & 1 & \frac{1}{2} \\ \frac{5}{2} & \frac{3}{2} & 1 \end{pmatrix} \quad \text{and} \quad B^2 = \begin{pmatrix} 1 & \frac{3}{5} & \frac{3}{10} \\ \frac{5}{4} & 1 & \frac{1}{2} \\ \frac{5}{2} & \frac{3}{2} & 1 \end{pmatrix}.$$

Hence,

$$I(B, 1, 2) = (\tfrac{3}{5}, \tfrac{4}{5}); \qquad I(B, 1, 3) = (\tfrac{3}{10}, \tfrac{2}{5}); \qquad I(B, 2, 3) = (\tfrac{1}{2}, \tfrac{2}{3}).$$

We find successively $w_1 = 1$, $w_2 = \frac{7}{10}$,

$$w_3 = \tfrac{3}{8} \in (\tfrac{3}{10}, \tfrac{2}{5}) \cap \tfrac{7}{10}(\tfrac{1}{2}, \tfrac{2}{3}) = (\tfrac{7}{20}, \tfrac{2}{5}) \quad \text{and} \quad \bar{w} = (1, \tfrac{10}{7}, \tfrac{8}{3})^{\mathrm{T}}.$$

The element of ir(A) we wanted to find is thus

$$b = P_\sigma(a_{1,\sigma(1)},\ a_{2,\sigma(2)},\ a_{3,\sigma(3)}) \otimes \bar{w}$$

$$= \begin{pmatrix} 0 & 4 & 0 \\ 5 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ \frac{10}{7} \\ \frac{8}{3} \end{pmatrix} = \begin{pmatrix} \frac{40}{7} \\ 5 \\ \frac{16}{3} \end{pmatrix}$$

In conclusion two remarks.

Remark 1. One can easily verify (see e.g. [3]) that $A^n = A^{n-1}$ for a normal matrix $A \in G(n, n)$. In order to compute A^n the generalized Warshall algorithm can be used, i.e. defining the matrices $A^{(k)} = (a_{ij}^{(k)})$; $k = 0, 1, \dots, n$ by the rules

$$A^{(0)} = A,$$

$$a_{ij}^{(k)} = a_{ij}^{(k-1)} \oplus (a_{ik}^{(k-1)} \otimes a_{kj}^{(k-1)}) \quad \text{for } k \geq 1,$$

we get $A^{(n)} = A^n$. By Lemma 3 this assertion can be proved in the same way as in [5]. Thus, step (ii) can be carried out in $2\ n^2 n = \mathrm{O}(n^3)$ steps.

Step (iii) of the presented method does not require more time. Furthermore, it follows from [1] that (even in more general structures) the assignment problem (step (i)) can be solved in $\mathrm{O}(n^3)$ steps. Thus, the problem of the strong regularity can be solved in $\mathrm{O}(n^3)$ steps, too.

Remark 2. Theorems 1 and 3 yield that the strong regularity of a square matrix is in fact equivalent to the uniqueness of the assignment problem solution (APS) whenever $\leq$ is dense. Hence as an immediate corollary we get that the uniqueness of the APS can be decided in $\mathrm{O}(n^3)$ steps whenever $\leq$ is dense. We want to emphasize at this place that the last statement is true also without the assumption of density.

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