Discrete Applied Mathematics 10 (1985) 19-26
North-Holland 

# NECESSARY SOLVABILITY CONDITIONS OF SYSTEMS OF LINEAR EXTREMAL EQUATIONS

Peter BUTKOVIČ

*Department of Geometry and Algebra, Šafárik's University Košice, Jesenná 5, 041 54 Košice, Czechoslovakia*

Received 4 November 1982
Revised 23 March 1984

Systems of linear equations of the form $A\otimes X=B\otimes X$ and of the form $A\otimes X=B\otimes Y$ over the structure based on linearly ordered commutative group $(G,\otimes,\leq)$ where the role of $\oplus$ plays the maximum are treated. Necessary solvability conditions are derived using known results concerning eigenvectors of matrices in such structures. In the special case of idempotent, increasing matrices $A$ and $B$ a condition is given which is necessary and sufficient for the existence of a nontrivial solution.

## 1. Definitions and basic properties

Let $(G,\otimes,\leq)$ be a nontrivial linearly ordered commutative group (its neutral element will be denoted by 1). Denote

$$G^0=G\cup\{0\}\cup\{\infty\}$$

(0 and $\infty$ being the adjoined elements) and extend $\leq$ and $\otimes$ on $G^0$ in such a way that

$$0\leq a\leq\infty \quad \text{for all } a\in G^0$$

and

$$a\otimes 0=0\otimes a=0 \quad \text{for all } a\in G^0,$$

$$a\otimes\infty=\infty\otimes a=\infty \quad \text{for all } a\in G^0-\{0\}.$$

Let $\otimes'$ be the binary operation on $G^0$ defined by the formula

$$\begin{aligned} a\otimes' b&=\infty, && \text{if } \infty\in\{a,b\},\\ &=a\otimes b, && \text{otherwise.}\end{aligned}$$

Denote $a\oplus b=\max\{a,b\}$, $a\oplus' b=\min\{a,b\}$ for all $a,b\in G^0$. It is known (cf. [1], [3], [7]) that $S=(G^0,\oplus,\otimes)$ as well as $S'=(G^0,\oplus',\otimes')$ are commutative semirings. In the terminology used in [3] the system $(G^0,\oplus,\otimes,\oplus',\otimes')$ is a linear commutative blog. The most important interpretations seem to be those based on ordered groups

0166-218X/85/\$3.30 

$$\mathcal{G}_1=(R^+,\cdot,\le),$$
$$\mathcal{G}_2=(R^+,\cdot,\ge),$$
$$\mathcal{G}_3=(R,+,\le),$$
$$\mathcal{G}^4=(R,+,\ge),$$

where $R^+$, resp. $R$ is the set of positive reals and reals, respectively and $\le$, resp. $\ge$ is the obvious order and the inverse order of reals, respectively. The works [3] and [5] are recommended to the reader for practical motivations.

We extend by the obvious way the order $\le$ on matrices over $G^0$ as well as the operations $\oplus$, $\otimes$, resp. $\oplus'$, $\otimes'$ on matrices over $S$ and $S'$, respectively. These extensions will be denoted by the same symbols. Many interesting properties of matrices over blogs have been proved in [3]. Propositions 1–6 are corollaries of some of them.

If $a\in G^0$ we denote by $a^*$ the element of $G^0$ defined as follows:

$$a^*=a^{-1},\quad \text{if } a\in G,$$
$$0^*=\infty,$$
$$\infty^*=0.$$

For an arbitrary set $M$ we denote by $M(m,n)$ the set of all $(m,n)$ matrices over $M$; $m,n\ge 1$ being integers. If $A$ is a matrix, then the symbol $(A)_{ij}$ denotes the element of $A$ in its $i$th row and $j$th column. If $A\in G^0(m,n)$, then $A^*\in G^0(n,m)$ is defined by the relations

$$(A^*)_{ji}=(A_{ij})^*\quad \text{for all } i\in\{1,2,\dots,m\},\ j\in\{1,2,\dots,n\}.$$

The symbol $A^{\mathrm{T}}$ means the transposition of the matrix $A$. For all integers $m\ge 1$ we shall denote $M(m,1)$ by $M_m$ and its elements will be called vectors.

**Proposition 1.** *Let $A\in G^0(m,n)$, $B\in G_m^0$. If the system of equations $A\otimes X=B$ is solvable, then $A^*\otimes' B$ is its (greatest) solution.*

**Remark.** For the theory of linear systems mentioned in Proposition 1 see e.g. [3], [6]. Numerical methods for solving linear programs with constraints of this type are to be found in [6].

**Proposition 2.** *If $A\in G^0(m,n)$, $X\in G_m^0$, then*

$$A\otimes(A^*\otimes' X)\le(A\otimes A^*)\otimes' X\le X\le(A\otimes' A^*)\otimes X\le A\otimes'(A^*\otimes X).$$

**Proposition 3.** *If $A\in G^0(n,n)$, then*

$$(A\otimes' A^*)\otimes(A\otimes' A^*)=A\otimes' A^*$$

*and*

$$(A\otimes A^*)\otimes'(A\otimes A^*)=A\otimes A^*.$$

Let an integer $n \geq 1$ and a matrix $A \in G^0(n,n)$ be given. The elements of the set $V(A) = \{X \in G_n \mid A \otimes X = X\}$, resp. $V'(A) = \{X \in G_n \mid A \otimes' X = X\}$ will be called eigenvectors and dual eigenvectors of the matrix $A$, respectively. Denote by $\Delta(A)$ the oriented, strongly complete and weighted graph with the set of nodes $\{1, 2, \dots, n\}$ and the weight function $w$ satisfying

$$w(i,j) = (A)_{ij} \quad \text{for all } i, j \in \{1, 2, \dots, n\}.$$

To each path $p = (i_0, i_1, \dots, i_t)$ of $\Delta(A)$ we associate a path product (dual path product)

$$w(p) = w(i_0, i_1) \otimes w(i_1, i_2) \otimes \cdots \otimes w(i_{t-1}, i_t)$$

$$(w'(p) = w(i_0, i_1) \otimes' w(i_1, i_2) \otimes' \cdots \otimes' w(i_{t-1}, i_t)).$$

The matrix $A$ is called definite (dually definite) if $\oplus_c w(c) = 1$ $(\oplus'_c w'(c) = 1)$ where the summation is taken over all elementary circuits in $\Delta(A)$. The matrix $\Gamma(A) = A \oplus A^2 \oplus \cdots \oplus A^n$ is called metric matrix generated by $A$.

**Proposition 4.** (a) *Let $A \in G^0(n,n)$. If $V(A) \neq \emptyset$ $(V'(A) \neq \emptyset)$, then $A$ is (dually) definite.*

(*b*) *Let $A \in G(n,n)$. Then $V(A) \neq \emptyset$ $(V'(A) \neq \emptyset)$ if and only if $A$ is (dually) definite.*

**Proposition 5.** *If $A \in G(n,n)$ is definite, then $\Gamma(A)$ has at least one diagonal element equal to* 1. *Moreover, every linear combination over $S$ with the coefficients from $G$ of the columns of $\Gamma(A)$ having the diagonal element equal* 1 *is an element of $V(A)$.*

In correspondence with [3] we say that the matrix $A \in G^0(n,n)$ is
- increasing, if $A \otimes X \geq X$ for all $X \in G_n^0$,
- decreasing, if $A \otimes' X \leq X$ for all $X \in G_n^0$,
- idempotent, if $A \otimes A = A$,
- dually idempotent, if $A \otimes' A = A$,
- projection matrix, if there exists $B \in G^0(n,m)$ such that $A = B \otimes' B^*$,
- dual projection matrix, if there exists $B \in G^0(n,m)$ such that $A = B \otimes B^*$.

Hence, Proposition 3 may be formulated as follows: every (dual) projection matrix is (dually) idempotent.

**Proposition 6.** *The matrix $A \in G^0(n,n)$ is increasing (decreasing) if and only if $(A)_{ii} \geq 1$ $(\leq 1)$ for all $i = 1, 2, \dots, n$ or, equivalently $A \oplus I = A$ $(A \oplus' I = A)$.*

## 2. Common eigenvectors and eigen-bivectors

**Proposition 7.** *Let $A, B \in G^0(n,n)$. Then*

(i) *$V(A) \cap V(B) \subseteq V(A \oplus B)$ and*

(ii) *if, moreover, $A, B$ are increasing, then $V(A \oplus B) \subseteq V(A) \cap V(B)$.*

**Proof.** (i) Let $A \otimes X = X = B \otimes X$. Then

$$(A \oplus B) \otimes X = A \otimes X \oplus B \otimes X = X \oplus X = X.$$

(ii) Let $(A \oplus B) \otimes X = X$. Then $A \otimes X \oplus B \otimes X = X$ and thus $A \otimes X \leq X$, $B \otimes X \leq X$. But since, $A, B$ are increasing, the last inequalities are in fact equalities. □

By the same way the following proposition can easily be verified.

**Proposition 8.** *Let $A, B \in G^0(n, n)$. Then*
(i) $V'(A) \cap V'(B) \subseteq V'(A \oplus' B)$ *and*
(ii) *if, moreover, $A, B$ are decreasing, then $V'(A \oplus' B) \subseteq V'(A) \oplus' V'(B)$.*

**Corollary 9.** *If $A, B \in G^0(n, n)$ are increasing, then*

$$V(A) \cap V(B) = V(A \oplus B)$$

*and if $A, B$ are decreasing, then*

$$V'(A) \cap V'(B) = V'(A \oplus' B).$$

Let $A \in G^0(2n, 2n)$. The elements of the set

$$V_2(A) = \{X = (x_1, \ldots, x_{2n})^{\mathrm{T}} \in V(A) \mid x_i = x_{n+i};\ i = 1, 2, \ldots, n\},$$

resp.

$$V_2'(A) = \{X = (x_1, \ldots, x_{2n})^{\mathrm{T}} \in V'(A) \mid x_i = x_{n+i};\ i = 1, 2, \ldots, n\}$$

are called eigen-bivectors and dual eigen-bivectors of the matrix $A$, respectively. We shall use the following notations:

$$A^+ = A \oplus \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \qquad A^- = A \oplus' \begin{pmatrix} 0^* & I^* \\ I^* & 0^* \end{pmatrix}$$

where 0, resp. $I$ are the zero and the unity matrix from $G^0(n, n)$, respectively.

**Proposition 10.** *If $A \in G^0(2n, 2n)$ is*
(i) *increasing, then $V_2(A) = V(A^+)$,*
(ii) *decreasing, then $V_2'(A) = V'(A^-)$.*

**Proof.** (i) Let $Z \in V_2(A)$. Then $Z = \binom{X}{X}$ where $X \in G_n$ and

$$A \otimes \begin{pmatrix} X \\ X \end{pmatrix} = \begin{pmatrix} X \\ X \end{pmatrix}.$$

Hence,

$$A^+ \otimes Z = \left(A \oplus \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\right) \otimes \begin{pmatrix} X \\ X \end{pmatrix} = A \otimes \begin{pmatrix} X \\ X \end{pmatrix} \oplus \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \otimes \begin{pmatrix} X \\ X \end{pmatrix}$$

$$=\begin{pmatrix}X\\X\end{pmatrix}\oplus\begin{pmatrix}X\\X\end{pmatrix}=\begin{pmatrix}X\\X\end{pmatrix}.$$

Let $Z\in V(A^+)$. Then $Z=\binom{U}{W}$ where $U,W\in G_n$ and

$$\begin{pmatrix}U\\W\end{pmatrix}=A^+\otimes\begin{pmatrix}U\\W\end{pmatrix}=\left(A\oplus\begin{pmatrix}0&I\\I&0\end{pmatrix}\right)\otimes\begin{pmatrix}U\\W\end{pmatrix}=A\otimes\begin{pmatrix}U\\W\end{pmatrix}\oplus\begin{pmatrix}W\\U\end{pmatrix}.$$

Thus,

$$A\otimes\begin{pmatrix}U\\W\end{pmatrix}\le\begin{pmatrix}U\\W\end{pmatrix}\quad\text{and}\quad\begin{pmatrix}W\\U\end{pmatrix}\le\begin{pmatrix}U\\W\end{pmatrix}.$$

Since $A$ is increasing and $\le$ is antisymmetric, the last inequalities are in fact equalities. Hence, $U=W$ and $Z=\binom{U}{W}\in V_2(A)$.

Part (ii) can be verified similarly and dually. $\square$

## 3. Necessary conditions of the solvability of linear extremal systems

Let $A,B\in G(m,n)$. Consider the system of equations

(I) $\qquad A\otimes X=B\otimes X.$

Its solutions are always the zero vector from $G_n^0$ and every vector from $G_n^0$ at least one component of which is equal to $\infty$. These solutions will be called trivial and all other nontrivial. A general procedure for solving such (and slightly more general) systems, as well as for minimization of isotone function over their solution sets, has been derived in [2]. But this procedure works too long in the case when the solution set is empty. To avoid this we shall now look for some (at least necessary) solvability conditions of (I).

**Theorem 1.** *If* (I) *has a nontrivial solution, then $C^-$ is dually definite where*

$$C=\begin{pmatrix}A\\B\end{pmatrix}\otimes\begin{pmatrix}A\\B\end{pmatrix}^*.$$

**Proof.** Let $X=(x_1,\dots,x_n)^{\mathrm{T}}$ be the nontrivial solution of (I) and let $x_t\ne 0$. Then there exists $Z=(z_1,\dots,z_m)^{\mathrm{T}}\in G_m^0$ satisfying

$$A\otimes X=Z=B\otimes X. \tag{1}$$

At first we show that $Z\in G_m$. Take an arbitrary $i\in\{1,2,\dots,m\}$. Clearly $z_i\ne\infty$ and

$$z_i=\max\{(A)_{ij}\otimes x_j \mid j=1,2,\dots,n\}\ge(A)_{it}\otimes x_t\in G$$

because $x_t$ is an element of $G$ and all $(A)_{it}$ are in $G$. That's why $z_i\in G$. The equations (1) can be rewritten blockwise as

$$\begin{pmatrix}A\\B\end{pmatrix}\otimes X=\begin{pmatrix}Z\\Z\end{pmatrix}.$$

Hence, using Propositions 1 and 2 we get

$$\binom{Z}{Z}=\binom{A}{B}\otimes\left(\binom{A}{B}^*\otimes'\binom{Z}{Z}\right)\leq\left(\binom{A}{B}\otimes\binom{A}{B}^*\right)\otimes'\binom{Z}{Z}\leq\binom{Z}{Z}$$

Thus $\binom{Z}{Z}\in V_2'(C)$. But $V_2'(C)\subseteq V'(C^-)$ which implies together with Proposition 4 that $C^-$ is dually definite. □

The following example will illustrate the applicability of Theorem 1.

**Example 1.** Take the interpretation based on $\mathscr{G}_1$ and consider the system (I) with

$$A=\begin{pmatrix}2 & 2\\ 1 & 2\end{pmatrix},\qquad B=\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}.$$

We compute that

$$C^-=\binom{A}{B}\otimes\binom{A}{B}^*\oplus'\begin{pmatrix}0^* & I^*\\ I^* & 0^*\end{pmatrix}=\begin{pmatrix}1 & 2 & 1 & 2\\ 1 & 1 & 2 & 1\\ \frac{1}{2} & 1 & 1 & 1\\ \frac{1}{2} & 1 & 1 & 1\end{pmatrix}.$$

Since $w'(1,3,1)=1\cdot\frac{1}{2}=\frac{1}{2}<1$, we conclude that the considered system has only trivial solutions.

**Remark.** The condition in Theorem 1 is not sufficient because for

$$A=\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix},\qquad B=\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$$

the system (I) has only trivial solutions but $C^-$ is dually definite.

Consider now the system

(II) $\qquad A\otimes X=B\otimes Y$

where $A\in G(m,n)$, $B\in G(m,k)$. The vector $\binom{X}{Y}\in G_{n+k}^0$ is its solution whenever $\binom{X}{Y}$ is a zero vector or at least one component of both $X$ and $Y$ is equal to $\infty$. These solutions will be called trivial and all other nontrivial. Note that (II) cannot be regarded as a special case of (I).

**Theorem 2.** *If* (II) *has a nontrivial solution, then the matrix* $(A\otimes A^*)\oplus'(B\otimes B^*)$ *is dually definite.*

**Proof.** Let $\binom{X}{Y}\in G_{n+k}^0$ be a nontrivial solution of (II). One can verify that $Z=A\otimes X=B\otimes Y\in G_m$ by the same way as in the proof of Theorem 1. Proposition 1 yields that

$$A\otimes(A^*\otimes' Z)=Z\quad\text{and}\quad B\otimes(B^*\otimes' Z)=Z.$$

According to Proposition 2 we have

$$Z = A \otimes (A^* \otimes' Z) \leq (A \otimes A^*) \otimes' Z \leq Z$$

and

$$Z = B \otimes (B^* \otimes' Z) \leq (B \otimes B^*) \otimes' Z \leq Z.$$

Hence

$$Z \in V'(A \otimes A^*) \cap V'(B \otimes B^*).$$

The last set is a subset of $V'((A \otimes A^*) \oplus' (B \otimes B^*))$ due to Proposition 8. Thus, according to Proposition 4 the matrix $(A \otimes A^*) \oplus' (B \otimes B^*)$ is dually definite. □

**Example 2.** Take the same interpretation as in Example 1 and

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}, \qquad B = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}.$$

Hence,

$$C = (A \otimes A^*) \oplus' (B \otimes B^*) = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{2}{3} & 1 \end{pmatrix}.$$

Since $w'(1,2,1) = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3} < 1$ we conclude that the considered system has only trivial solutions.

**Remark.** The condition in Theorem 2 is not sufficient because the system (II) with

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \\ 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$$

has only trivial solutions but the matrix $(A \otimes A^*) \oplus' (B \otimes B^*)$ is dually definite.

## 4. A condition which is in one special case necessary and sufficient

Obviously, an idempotent matrix need not to be definite (e.g. zero matrix).

**Proposition 11.** *An idempotent matrix $A \in G^0(n,n)$ having at least one column from $G_n$ is definite.*

**Proof.** Let $A(j) \in G_n$ be the $j$th column of the idempotent matrix $A$. Then $A \otimes A(j) = A(j)$ and thus $A(j) \in V(A)$. The assertion follows now from Proposition 4. □

**Theorem 3.** *Let $A, B \in G(n,n)$ be increasing and idempotent. Then the following statements are equivalent:*

(i) *there exists a nontrivial solution of the system* (II),
(ii) $A \oplus B$ *is definite,*
(iii) *there exists a nontrivial solution of the system* (I).

**Proof.** (i)$\Rightarrow$(ii) Let $\binom{X}{Y} \in G_{2n}^0$ be the nontrivial solution of (II). Thus the vector $Z$ defined by

$$A \otimes X = Z = B \otimes Y$$

is an element of $G_n$ (for details see the beginning of the proof of Theorem 1). The matrices $A, B$ are definite according to Proposition 11 and $\Gamma(A) = A$, $\Gamma(B) = B$ because $A, B$ are idempotent. The definiteness of $A, B$ yields that their diagonal elements are less or equal to 1 (these elements are weights of the circuits of the length 1 in $\Delta(A)$ and $\Delta(B)$). Thus, Proposition 6 gives that all diagonal elements of $A$ and $B$ are 1. According to Proposition 5 then $Z \in V(A) \cap V(B)$ and due to Proposition 7 also $Z \in V(A \oplus B)$. Finally, from Proposition 4 we get that $A \oplus B$ is definite.

(ii)$\Rightarrow$(iii) Let $A \oplus B$ be definite. Since $A, B \in G(n, n)$, Proposition 4 yields that $V(A \oplus B) \neq \emptyset$. But $A, B$ are increasing and thus $V(A) \cap V(B) \neq \emptyset$ (Corollary 9). Hence, there exists $X \in V(A) \cap V(B)$, i.e. $A \otimes X = X$ and $B \otimes X = X$. We see that $X$ is the nontrivial solution of (I).

(iii)$\Rightarrow$(i) This implication is trivial. □

**Corollary.** *If* $A, B \in G(n, n)$ *are projection matrices, then the statements in Theorem* 3 *are equivalent.*

**Proof.** Every projection matrix is increasing according to Proposition 2 and idempotent due to Proposition 3. □

## References

[1] G. Birkhoff, Lattice Theory (AMS, Providence, RI, 1973).
[2] P. Butkovič, On properties of solution sets of extremal linear programs, in: R.E. Burkard, R.A. Cunninghame-Green, U. Zimmermann, eds., Combinatorial and Algebraic Methods in Operations Research, Proc. Workshop Alg. Struct. Op. Res. (Bad Honnef, 1982), Ann. Discrete Math., to appear.
[3] R.A. Cunninghame-Green, Minimax Algebra, Lecture Notes in Econom. Math. Syst. 166 (Springer, Berlin, 1979).
[4] M. Gondran, Algèbre des chemins et algorithmes, Bull. Direction Études Rech. Sér. C Math. Inform. (1975) 57–64.
[5] N.N. Vorobyev, Extremal algebra of positive matrices, Elektron. Informationsverarb. Kybernet. 3 (1967) 39–71 (in Russian).
[6] K. Zimmermann, Extremální algebra (Výzkumná publikace Ekonomicko-matematické laboratoře při Ekonomickém ústavě ČSAV, 46, Praha, 1976) (in Czech).
[7] U. Zimmermann, Linear and Combinatorial Optimization in Ordered Algebraic Structures, Ann. Discrete Math. 10 (North-Holland, Amsterdam, 1981).