Strong Linear Independence in Bottleneck Algebra

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ABSTRACT

Let $(B, \leqslant)$ be a dense, linearly ordered set without maximum and minimum and $(\oplus, \otimes) = (\max, \min)$. We say that a matrix A has strongly linearly independent (SLI) columns if for some b the system $A \otimes x = b$ is uniquely solvable. An (n, n) matrix $A = (a_{ij})$ is said (a) to be strongly regular if it has SLI columns; (b) to have a strong permanent if the equality

$$\operatorname{per}(A) = \bigotimes_{i=1}^{n} a_{i,\pi(i)}$$

holds for unique $\pi \in P_n$ [per(A) is $\bigoplus_{\pi \in P_n} \bigotimes_{i=1}^{n} a_{i,\pi(i)}$ and P_n is the set of all permutations of the set $\{1, 2, \dots, n\}$]. We prove: (i) that an (m, n) matrix has SLI columns if and only if it contains an (n, n) submatrix which is strongly regular [we derive an $O(mn \log n)$ algorithm for checking this property], (ii) that every matrix with strong permanent is strongly regular, and (iii) that a solution to the bottleneck assignment problem for strongly regular (n, n) matrices can be found using $O(n^2 \log n)$ operations.

1. INTRODUCTION

The quadruple $\mathscr{B} = (B, \oplus, \otimes, \leqslant)$, or B itself, is called a bottleneck algebra (BA for short) if $(B, \leqslant)$ is a nonempty, linearly ordered set without maximum and minimum and $\oplus, \otimes$ are binary operations on B defined by

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the formulas

$$a \oplus b = \max\{a, b\},$$

$$a \otimes b = \min\{a, b\}.$$

Among the most important interpretations of BA are those based on the following linearly ordered sets ($\leqslant$ is everywhere the obvious order of reals, and $-\infty \leqslant l < u \leqslant +\infty$):

$$((l, u), \leqslant), \tag{1}$$

$$((l, u) \cap Q, \leqslant), \tag{2}$$

$$(Z, \leqslant), \tag{3}$$

$$((l, u) \cap P(\alpha), \leqslant) \tag{4}$$

where Q is the set of rationals, Z is the set of integers, and

$$P(\alpha) = \left\{ \sum_{i=0}^{r} p_i \alpha^i;\ p_0, \dots, p_r \text{ integers},\ r = 0, 1, 2, \dots \right\},$$

α being any (fixed) transcendental number; cf. [10]. We denote by $\mathscr{B}_1, \mathscr{B}_2, \mathscr{B}_3, \mathscr{B}_4$ the BA based on (1)–(4) respectively.

Some practical problems lead to computations in a bottleneck algebra. For example, the permanent of an (n, n) matrix $A = (a_{ij})$ in $\mathscr{B}_1$, i.e.

$$\operatorname{per}(A) = \bigoplus_{\pi} \bigotimes_{i=1}^{n} a_{i, \pi(i)},$$

where the summation is taken over all permutations of the set $\{1, 2, \dots, n\}$, corresponds to a weighted matching in a complete bipartite graph with the maximal possible lowest score. This corresponds to those situations where the overall performance of a team is measured by the worst performance of an individual member—e.g., if each of n workers performs one of n tasks on an assembly line, then the speed of the line equals the speed of the slowest worker; see [5]. The task of finding such an assignment is a special case of the algebraic assignment problem investigated e.g. in [1] and [10].

As another example consider the transportation transmittance problem. If the transportation route consists of two parts UV and VW (say V is a transship point), then the total route transmittance is equal to the minimum of the transmittances of UV and VW. Similarly, in a transportation network with $U_1,\dots,U_l$ as dispatching points, $V_1,\dots,V_m$ as transship points, and $W_1,\dots,W_n$ as destination points, denoting the transmittances of U_iV_j and V_jW_k by a_{ij} and b_{jk}, respectively ($i=1,\dots,l$; $j=1,\dots,m$; $k=1,\dots,n$), we have that the total transportation transmittance between U_i and W_k is equal to

$$c_{ik}=\max_{j=1,\dots,m}\min\{a_{ij},b_{jk}\}$$

for all $i=1,\dots,l$ and $k=1,\dots,n$. This expression can be put in a more convenient form by using the obvious extension of $\oplus$ and $\otimes$ to matrices in $\mathscr{B}_1$:

$$C=A\otimes B,$$

where we denote by A, B, C the matrices $(a_{ij}),(b_{jk}),(c_{ik})$.

2. DEFINITIONS AND BASIC PROPERTIES

Clearly, a bottleneck algebra $(B,\oplus,\otimes,\leqslant)$ is a distributive (infinite) lattice. Among its well-known properties we need to recall that $a\leqslant b$ and $c\leqslant d$ imply

$$a\otimes c\leqslant b\otimes d$$

and

$$a\oplus c\leqslant b\oplus d$$

for all $a,b,c,d\in B$.

The set of all (m,n) matrices over B will be denoted by $B(m,n)$, and $B(m,1)$ by B_m. The elements of B_m will be called vectors. Extend $\oplus$, $\otimes$, and $\leqslant$ to matrices over B as in conventional linear algebra. Many properties of such an extension can be found in [10], and we mention here the following two:

$$\text{if}\quad C\leqslant D,\quad\text{then}\quad A\otimes C\leqslant A\otimes D \text{ and } C\otimes A\leqslant D\otimes A$$

and

$$A\otimes(B\oplus C)=(A\otimes B)\oplus(A\otimes C)$$

whenever the indicated operations exist.

The main results are proved under the assumption of density of the ordering $\leqslant$, that is to say,

$$(\forall x, y \in B) \qquad x < y \ \Rightarrow\ (\exists z \in B)\ x < z < y.$$

Thus $\mathscr{B}_1, \mathscr{B}_2, \mathscr{B}_4$ are dense, whereas $\mathscr{B}_3$ is not.

Suppose m and n are positive integers. The set of all permutations of the set $\{1, 2, \dots, n\}$ is denoted by P_n; id means the identity permutation. If $A = (a_{ij}) \in B(m,n)$, $\sigma \in P_m$, $\pi \in P_n$, then $A(\sigma, \pi)$ denotes the matrix $C = (c_{ij}) \in B(m,n)$ such that $c_{ij} = a_{\sigma(i), \pi(j)}$ for all i and j. If $\sigma \in P_n$, $A = (a_{ij}) \in B(n,n)$, then the weight of σ with respect to A, i.e.

$$a_{1,\sigma(1)} \otimes \cdots \otimes a_{n,\sigma(n)},$$

is denoted by $w(A,\sigma)$. Since $\mathrm{per}(A) = \bigoplus_{\sigma \in P_n} w(A,\sigma) = \max_{\sigma \in P_n} w(A,\sigma)$, we have $\mathrm{per}(A) = w(A,\sigma)$ for at least one $\sigma \in P_n$. We say that A has a strong permanent if $\mathrm{per}(A) = w(A,\sigma)$ holds for only one $\sigma \in P_n$ (or, equivalently: if the corresponding bottleneck assignment problem has unique solution). Note that several efficient algorithms can be used in order to check the uniqueness of the algebraic assignment problem solution—some of them can be found in [2] and [4].

By $\max(A)$ we denote the set

$$\{\sigma \in P_n;\ w(A,\sigma) = \mathrm{per}(A)\},$$

and for any set S the symbol $|S|$ means the number of its elements. Hence the property of possessing a strong permanent can be expressed by the equality $|\max(A)| = 1$.

In the following we deal with (m,n) matrices, and we assume everywhere that m and n are given positive integers. For short we denote $\{1, 2, \dots, m\}$ by M and $\{1, 2, \dots, n\}$ by N; A_i, where $i \in M$, stands for row i of the matrix A.

3. CLASSIFICATION OF MATRICES

Systems of simultaneous linear equations (or briefly, linear systems) of the form

$$A \otimes x = b, \tag{5}$$

where $A \in B(m,n)$, $b \in B_m$, have been treated e.g. in [9] and [10]. But these works do not provide any information about the size of the solution set. We give now the answer in the case when $\leqslant$ is dense. For this purpose denote the solution set of (5) by $S(A,b)$ and

$$T(A) = \{|S(A,b)|;\ b \in B_m\}.$$

LEMMA 1. *$0 \in T(A)$ for every matrix $A = (a_{ij}) \in B(m,n)$.*

Proof. It suffices to take $b = (q,\dots,q)^T \in B_m$, where q is an arbitrary element of B greater than $\max\{a_{ij};\ i \in M,\ j \in N\}$. ■

LEMMA 2. *$\infty \in T(A)$ for every matrix $A = (a_{ij}) \in B(m,n)$.*

Proof. Let b be the first column of A. Then every vector $x = (x_1,\dots,x_n)^T \in B_n$ with $x_1 \geqslant \max_{i\in M} a_{i1}$ and $x_j \leqslant \min_{i \in M} a_{i1}$ for $j \neq 1$ is an element of $S(A,b)$. ■

LEMMA 3. *Let $\leqslant$ be dense. If $|S(A,b)| > 1$, then $|S(A,b)| = \infty$ for every $A \in B(m,n)$ and $b \in B_m$.*

Proof. Suppose that $x, y \in S(A,b)$, $x \neq y$. Then also $z = x \oplus y \in S(A,b)$ since $A \otimes (x \oplus y) = (A \otimes x) \oplus (A \otimes y) = b \oplus b = b$. Clearly, $x \leqslant z$, $y \leqslant z$ and either $x \neq z$ or $y \neq z$. Let us suppose without loss of generality that $x \neq z$. Then it follows from the assumption of density that an infinite number of vectors v satisfying $x \leqslant v \leqslant z$ exists. But each of these vectors is in $S(A,b)$, because

$$b = A \otimes x \leqslant A \otimes v \leqslant A \otimes z = b.$$ ■

As a consequence of Lemmas 1–3 we have

THEOREM 1. *Let $\leqslant$ be dense. Then*

$$\{0,\infty\} \subseteq T(A) \subseteq \{0,1,\infty\} \tag{6}$$

for every matrix A.

REMARK 1. It is easy to see that both classes of matrices indicated in (6) are actually nonempty. Nevertheless, the following example shows that Theorem 1 does not hold true in general without the assumption of density. To see this put, in $\mathscr{B}_3$,

$$A = \begin{pmatrix} 3 & 0 \\ 2 & 3 \\ 0 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix},$$

and suppose that $x = (x_1, x_2)^T \in S(A, b)$. Then it follows from the first equation that $x_1 = 2$, and the inequalities $x_2 \leqslant 2$ and $x_2 \geqslant 1$ can be derived from the second and third equation, respectively. Hence, $S(A, b) = \{(2,1)^T, (2,2)^T\}$.

Note that after introducing the notation $S(A, b)$ and $T(A)$ in linear algebra, we get a classification of matrices which can be conveniently described using the concept of rank: supposing that A is a real (m, n) matrix, clearly $T(A) \subseteq \{0, 1, \infty\}$, and denoting the rank of A by $r(A)$, we have that $0 \in T(A)$ means $r(A) < m$, $\infty \in T(A)$ means $r(A) < n$, and $1 \in T(A)$ means $r(A) = n$, so that e.g. $\{1, \infty\} \not\subseteq T(A)$ for any matrix A. All possibilities for $T(A)$ in the linear as well as in the bottleneck case are described in Table 1.

In linear algebra several methods for computing the rank and hence also for finding $T(A)$ exist (Gaussian elimination etc.). A similar question arises in the bottleneck case. The results of this paper enable one to decide effectively whether $T(A) = \{0, \infty\}$ or $T(A) = \{0, 1, \infty\}$ for a given matrix A in a dense bottleneck algebra—or, equivalently, whether A has full column rank.

We shall say that the columns of the matrix $A \in B(m, n)$ are strongly linearly independent (SLI for short) if $1 \in T(A)$; moreover, if $m = n$, then A will be called strongly regular. It is clear that one column is always SLI (it

TABLE 1

$T(A)$	Linear case	Dense bottleneck case
$\{0\}$	—	—
$\{1\}$	$n = r(A) = m$	—
$\{\infty\}$	$m = r(A) < n$	—
$\{0, 1\}$	$n = r(A) < m$	—
$\{0, \infty\}$	$r(A) < \min(m, n)$	Columns not SLI
$\{1, \infty\}$	—	—
$\{0, 1, \infty\}$	—	Columns SLI

suffices to take $b=(q,\dots,q)^T$ where $q\in B$ is less than the least element of the column).

A characterization of strong regularity was presented in [3]; moreover, an efficient algorithm for checking this property was derived. We recall briefly the main result of [3].

A matrix $A=(a_{ij})\in B(n,n)$ for $n>1$ is said to be trapezoidal if

$$a_{kk} > \bigoplus_{i=1}^{k} \bigoplus_{j=i+1}^{n} a_{ij}$$

for all $k\in N$. Every (1,1) matrix is trapezoidal by definition. Matrices $A,C\in B(m,n)$ are said to be equivalent ($A\sim C$) if one of them can be obtained from the other by permuting the rows and columns. It is evident that equivalence constitutes an equivalence relation.

THEOREM 2. *Let $A\in B(n,n)$. Then a necessary condition for A to be strongly regular is the existence of a trapezoidal matrix equivalent to A. Moreover, if $\leqslant$ is dense, then this condition is also sufficient.*

The concepts of SLI and strong regularity were introduced originally in [6] in the same way as here, but in a structure where $\otimes$ is defined by the assumption that $(B,\otimes,\leqslant)$ is a linearly ordered, commutative group (and $\oplus$ plays the same role as in BA). We shall refer to this structure as the "group case." Strong regularity in the group case was treated in [2], and the main results for square matrices can be formulated as follows:

(i) a necessary and sufficient condition for the columns of an (m,n) matrix A to be SLI is the existence of a strongly regular submatrix of A of order n;

(ii) a necessary condition for a square matrix A to be strongly regular is that A has a strong permanent, and moreover, if $\leqslant$ is dense, then this condition is also sufficient.

The aim of the present paper is to prove in a dense bottleneck algebra:

(i) the same necessary and sufficient condition for the columns to be SLI as in the group case, as well as an efficient method for checking this property;

(ii) that the concepts of strong regularity and strong permanent are connected here too, though they are not equivalent in general—more precisely, strong permanent is a sufficient condition for strong regularity, but it is necessary only for matrices of order 2.

4. AN (m, n) MATRIX WITH SLI COLUMNS CONTAINS A STRONGLY REGULAR SUBMATRIX OF ORDER n

Throughout this section we suppose that $A = (a_{ij}) \in B(m, n)$, $b = (b_1, \dots, b_m)^T \in B_m$ are given, and we denote for all $j \in N$

$$M_j = \{ i \in M;\ a_{ij} > b_i \},$$

$$\tilde{M}_j = \{ i \in M;\ a_{ij} = b_i \},$$

$$\bar{x}_j = \min\{ b_i;\ i \in M_j \},$$

$$I_j = \{ i \in M_j;\ b_i = \bar{x}_j \},$$

$$K_j = \{ i \in \tilde{M}_j;\ b_i \leqslant \bar{x}_j \},$$

$$L_j = I_j \cup K_j.$$

We recall some results proved in [3] which will be helpful in the later theory.

LEMMA 4. *If* $|S(A, b)| = 1$, *then*

(a) $M_j \neq \varnothing$ *for all* $j \in N$ *and*
(b) $(\bar{x}_1, \dots, \bar{x}_n)^T \in S(A, b)$.

LEMMA 5. *Let* $m = n$. *Then* $|S(A, b)| = 1$ *if and only if the relations*

$$a_{i,\pi(i)} > b_i > \bigoplus_{j \in N - \{i\}} a_{i,\pi(j)} \otimes b_j \qquad \text{for all} \quad i \in M$$

are satisfied by at least one $\pi \in P_n$.

LEMMA 6. *If* $|S(A, b)| = 1$, *then* $\{L_1, \dots, L_n\}$ *is a minimal covering of* M.

Proof. At first suppose $\{L_1, \dots, L_n\}$ is not a covering, say $i \in M - \bigcup_{j \in N} L_j$. Then $A_i \otimes \bar{x} < b_i$, since for every $j \in N$ either $a_{ij} < b_i$, or $a_{ij} = b_i$ but $\bar{x}_j < b_i$ (because $i \notin K_j$), or $a_{ij} > b_i$ but $\bar{x}_j < b_i$ (because $i \notin I_j$).

Now suppose $\{L_1, \ldots, L_n\}$ is a covering of M but not minimal, say $\bigcup_{j \in N - \{k\}} L_j = M$ for some $k \in N$. Then every vector $x = (\bar{x}_1, \ldots, \bar{x}_{k-1}, \alpha, \bar{x}_{k+1}, \ldots, \bar{x}_n)^T$ with $\alpha < \bar{x}_k$ is in $S(A, b)$, because $A_i \otimes x \leqslant A_i \otimes \bar{x} = b_i$ for $i \in M$, and equality follows from the existence of $l \in N - \{k\}$ for which $i \in L_l$, because in this case either $a_{il} > b_i = \bar{x}_l$ or $a_{il} = b_i \leqslant \bar{x}_l$. ■

Lemma 6 shows that if $|S(A, b)| = 1$, then for every $k \in N$ an index $i(k) \in M$ satisfying

$$i(k) \in L_k - \bigcup_{j \in N - \{k\}} L_j$$

exists. Naturally, we can permute the rows of A so that $i(k) \in N$ for all $k \in N$ and so that the right hand side constants of the first n equations are ranked nondecreasingly. Finally, since $i(1), \ldots, i(n)$ are pairwise different, we can permute the columns of A so that $i(k) = k$ for all $k \in N$. Now we say that the system (5) is in a normal form. Thus for any system (5) in normal form we have

$$b_1 \leqslant b_2 \leqslant \cdots \leqslant b_n$$

and

$$i \in L_i - \bigcup_{j \in N - \{i\}} L_j$$

for all $i \in N$.

Lemma 7. *If $|S(A, b)| = 1$ and* (5) *is in normal form, then*

$$a_{ii} \geqslant b_i \geqslant \bigoplus_{j \in N - \{i\}} a_{ij} \otimes b_j \tag{7}$$

holds for all $i \in N$.

Proof. The first inequality in (7) for $i \in N$ follows from $i \in L_i$.

To prove the second inequality for any $i \in N$, take an index $j \in N - \{i\}$ with $a_{ij} > b_i$ (otherwise the assertion is trivial). Thus $b_i \geqslant \min\{b_l;\ a_{lj} > b_l\} = \bar{x}_j$. But $j \in L_j$, and thus $\bar{x}_j \geqslant b_j$, both when $j \in K_j$ and when $j \in I_j$. Hence $b_i \geqslant b_j$, which yields that

$$a_{ij} \otimes b_j \leqslant b_i.$$

■

THEOREM 3. *A sufficient condition for A to have SLI columns is that A contains a strongly regular submatrix of order n. Moreover, if $\leqslant$ is dense, then this condition is also necessary.*

Proof. Suppose A contains a strongly regular submatrix C of order n, and without loss of generality let C consist of the first n rows of A. Denote by y and $c = (c_1, \dots, c_n)^T$ the vectors satisfying $S(C, c) = \{y\}$. Put $b_i = c_i$ for $i = 1, \dots, n$, and $b_i = A_i \otimes y$ for $i = n+1, \dots, m$. Then evidently $S(A, b) = \{y\}$.

Now suppose $\leqslant$ is dense and $|S(A, b)| = 1$ for some $b = (b_1, \dots, b_m)^T \in B_m$. Without loss of generality let $A \otimes x = b$ be in normal form. We show that the submatrix of A consisting of its first n rows is strongly regular. According to Lemma 5 it is sufficient to find $d_1, \dots, d_n \in B_n$ satisfying

$$a_{ii} > d_i > \bigoplus_{j \in N - \{i\}} a_{ij} \otimes d_j \tag{8}$$

for all $i \in N$. We take arbitrary $d_1, \dots, d_n$ fulfilling the following conditions:

$$b_1 > d_1 > \bigoplus_{j=2}^{n} a_{1j},$$

$$b_i > d_i > d_{i-1} \oplus \bigoplus_{j=i+1}^{n} a_{ij}$$

for $i = 2, \dots, n$. Clearly, in order to guarantee that $d_1, \dots, d_n$ are well defined it suffices to prove the inequality

$$b_i > \bigoplus_{j=i+1}^{n} a_{ij}$$

for all $i \in N$. But $b_i \leqslant a_{il}$ for some $l > i$ would imply $b_i > \bar{x}_l$, because neither $i \in K_l$ nor $i \in I_l$. Thus, using $\bar{x}_l \geqslant b_l$ (since $l \in L_l$), we get $b_i > b_l$, which is impossible for $l > i$.

Now it remains to verify (8) for every $i \in N$. The first inequality follows from $a_{ii} \geqslant b_i$ (since $i \in L_i$), and one can easily see that

$$\begin{aligned} d_i > \bigoplus_{j=1}^{i-1} d_j \oplus \bigoplus_{j=i+1}^{n} a_{ij} &\geqslant \bigoplus_{j=1}^{i-1} a_{ij} \otimes d_j \oplus \bigoplus_{j=i+1}^{n} a_{ij} \otimes d_j \\ &= \bigoplus_{j \in N - \{i\}} a_{ij} \otimes d_j. \end{aligned}$$

■

The natural number $r(A)$ defined by the relation

$$r(A) = \max\{r;\ \text{there exists } C \in B(r,r),\ C \text{ a strongly regular submatrix of } A\}$$

will be called the rank of the matrix A. Hence we have immediately

COROLLARY of Theorem 3. *Let $\leqslant$ be dense. Then the columns of A are SLI if and only if $r(A) = n$.*

5. AN ALGORITHM FOR CHECKING STRONG LINEAR INDEPENDENCE

Theorems 2 and 3 imply that assuming the density of $\leqslant$, $A \in B(m,n)$ has SLI columns if and only if A is equivalent to a matrix containing a trapezoidal submatrix of order n. Using this result, we derive an algorithm for checking strong linear independence which uses successive reductions of the problem for an (m,n) matrix to the same problem for an $(m-1,n-1)$ matrix. As a result, our algorithm provides a trapezoidal (n,n) submatrix or indicates that such a submatrix does not exist.

We note at first that according to our definitions every matrix with just one column has SLI columns and any of its $(1,1)$ submatrices is strongly regular. Furthermore, it follows from Lemma 6 that $m \geqslant n$ is a necessary condition for (5) to be uniquely solvable (and hence also for the columns to be SLI).

For $A = (a_{ij}) \in B(m,n)$ with $n \geqslant 2$, $v \in B$, and $i \in M$, we denote in this section

$$P_v(A) = \left\{i \in M;\ (\exists k \in N)\ (\forall j \in N - \{k\})\ a_{ik} > v \geqslant a_{ij}\right\}$$

and $v(i) = \min\{v;\ i \in P_v(A)\}$ (we put $\min \varnothing = +\infty$). [Clearly, $v(i) = +\infty$ if the maximal element of A_i is not unique, and $v(i)$ is equal to the second greatest element of A_i otherwise.] If $\{v \in B;\ i \in P_v(A)\} \neq \varnothing$, we denote by $k(i)$ the unique index $k \in N$ for which $a_{ik} > v(i)$.

LEMMA 8. *Let $A, C \in B(m,n)$ and $A \sim C$. Then $|P_v(A)| = |P_v(C)|$ for all $v \in B$.*

The proof is trivial.

LEMMA 9. *Let $A \in B(m, n)$ and $n \geqslant 2$. If A is equivalent to a matrix $\begin{pmatrix} C \\ D \end{pmatrix}$ where $C \in B(n, n)$ is trapezoidal, then $P_v(A) \neq \varnothing$ for some $v \in B$.*

Proof. Since $C = (c_{ij})$ is trapezoidal, we have

$$c_{11} > \bigoplus_{j=2}^{n} c_{1j}.$$

Hence, denoting $\bigoplus_{j=2}^{n} c_{1j}$ by v, we get

$$1 \in P_v(C) \subseteq P_v\left(\begin{pmatrix} C \\ D \end{pmatrix}\right).$$

The rest follows from Lemma 8. ∎

For $A = (a_{ij}) \in B(n, n)$ we denote $\bigotimes_{i \in N} a_{ii}$ by $d(A)$.

LEMMA 10. *Let $d \in B$, $A = (a_{ij}) \in B(m, n)$, and $n \geqslant 2$. Suppose that A can be written blockwise in the form $\begin{pmatrix} C \\ D \end{pmatrix}$ where $C \in B(n, n)$ is trapezoidal and $d(C) > d$. Then the set*

$$V(d) = \left\{ v \in B;\ a_{l, k(l)} > d \text{ for some } l \in P_v(A) \right\}$$

is nonempty. Furthermore, if $v_0 = \min V(d)$ and $i \in P_{v_0}(A)$ is an arbitrary index satisfying $a_{i, k(i)} > d$, then permutations $\sigma \in P_m$, $\pi \in P_n$ with the following properties exist:

$$\sigma(1) = i,$$

$$\pi(1) = k(i),$$

$$A(\sigma, \pi) = \begin{pmatrix} C' \\ D' \end{pmatrix},$$

where $C' \in B(n, n)$ is trapezoidal and $d(C') > d$.

Proof. $V(d) \neq \varnothing$, since $a_{1, k(1)} = a_{11} > d$ and thus at least $\bigoplus_{j=2}^{n} a_{1j} \in V(d)$.

Suppose that v_0, i, and $k(i)$ fulfill the assumptions of Lemma 10. Denote $k(i)$ by j. Clearly, $j \leqslant i$ for $i \leqslant n$, since C is trapezoidal (actually equality holds), and for $i > n$ trivially because $j \leqslant n \leqslant m$.

If $j=i$, we put $\sigma=(i\cdots 21)(i+1)\cdots(m)$, $\pi=(i\cdots 21)(i+1)\cdots(n)$, and

$$A'=(a'_{ij})=A(\sigma,\pi)=\begin{pmatrix} C' \\ D' \end{pmatrix}, \qquad C'\in B(n,n).$$

Then the relations

$$a'_{ss}=a_{s-1,s-1}>\bigoplus_{q=2}^{n} a_{1q}\geqslant v_0=\bigoplus_{q\in N-\{i\}} a_{iq}=\bigoplus_{q=2}^{n} a'_{1q}$$

for all $s=2,3,\dots,i$ follow from the assumption that C is trapezoidal and from the choice of v_0. Clearly,

$$a'_{11}=a_{ii}>\bigoplus_{q\in N-\{i\}} a_{iq}=\bigoplus_{q=2}^{n} a'_{1q},$$

and since

$$\begin{aligned}
a'_{rs}&=a_{r-1,s-1} &&\text{for all}\quad r,s\in\{2,\dots,i\},\\
a'_{rs}&=a_{r-1,s} &&\text{for all}\quad r\in\{2,\dots,i\},\quad s\in\{i+1,\dots,n\},\\
a'_{rs}&=a_{rs} &&\text{for all}\quad r\in\{i+1,\dots,m\},\quad s\in\{i+1,\dots,n\},
\end{aligned}$$

we can summarize that C' is trapezoidal and $d(C')>d$.

Similarly, if $j<i$, we put

$$\begin{aligned}
\sigma&=(ij\cdots 21)(j+1)\cdots(i-1)(i+1)\cdots(m),\\
\pi&=(j\cdots 21)(j+1)\cdots(n),\\
A'&=(a'_{ij})=A(\sigma,\pi)=\begin{pmatrix} C' \\ D' \end{pmatrix}, \qquad C'\in B(n,n).
\end{aligned}$$

One can derive in the same way as before that C' is trapezoidal and $d(C')>d$. ■

LEMMA 11. *Let $A\in B(m,n)$ and $n\geqslant 2$. Suppose that A can be written blockwise in the form*

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

where $A_{11} \in B(r, r)$, $1 \leqslant r < n$ *and*

$$a_{ii} > \bigoplus_{k=1}^{i} \bigoplus_{l=k+1}^{n} a_{kl}$$

for $i = 1, \dots, r$. *Then the following two statements are equivalent*:

(a) *a trapezoidal matrix*

$$C = \begin{pmatrix} A_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \in B(n, n)$$

and a matrix D *such that*

$$A \sim \begin{pmatrix} C \\ D \end{pmatrix}$$

exist;

(b) *a trapezoidal matrix* $C' \in B(n - r, n - r)$ *and a matrix* D' *such that*

$$A_{22} \sim \begin{pmatrix} C' \\ D' \end{pmatrix} \quad \text{and} \quad d(C') > \bigoplus_{k=1}^{r} \bigoplus_{l=k+1}^{n} a_{kl}$$

exist.

Proof. To prove (b) $\Rightarrow$ (a) it suffices to put $C_{22} = C'$ and to take for C_{12} and C_{21} the matrices obtained by permuting correspondingly the columns and rows of A_{12} and A_{21}, respectively.

In order to prove the converse implication, put $C' = C_{22}$ and for D' take the matrix obtained from the matrix consisting of the last $m - n$ rows of A_{22} by permuting correspondingly its columns and rows. ∎

Now it is not difficult to compile an $O(mn^2)$ algorithm for checking strong linear independence, based on Lemmas 9–11. Lemma 9 shows that a necessary condition for the columns of A to be SLI is the existence of a unique maximal element in at least one row of A. Lemma 10 implies that on choosing the maximal element in row r, say a_{rs}, for which the second greatest, say a_{rq} ($a_{rq} < a_{rs}$), is as small as possible, A is equivalent to $\begin{pmatrix} C \\ D \end{pmatrix}$ where C is trapezoidal and a_{rs} is in its first row and column. Lemma 11 enables us now to transform our problem to the submatrix A' arising from A

by deleting the rth row and the sth column. It follows again from Lemma 10 that in some row of A' the unique maximal element greater than a_{rq} exists whenever columns of A are SLI. The procedure continues in this way until the whole trapezoidal submatrix of order n is found or at some step it is not possible to continue because no row exists with unique maximal element greater than all known superdiagonal elements.

We present a more sophisticated version of this algorithm with smaller computational complexity, achieved by rearranging of each row of A in nonincreasing order. This enables us to avoid the repeated computation of the greatest and the second greatest element of each row and saves in this way a significant number of evaluations. We must, of course, distinguish infeasible and feasible rows and columns, depending on whether they have or have not been already chosen for the trapezoidal submatrix being formed.

In the algorithm written below in pidgin ALGOL (for a description of this informal language we recommend e.g. [8]) we denote by ro(A) the matrix arising from A by reordering of each row nonincreasingly. The variables fc(j) and fr(i) indicate the feasibility or infeasibility of the jth column and the ith row, respectively; g1(i) and g2(i) express the column indices of the greatest and the second greatest element in the ith row; respectively. In the variables $r(l)$, $c(l)$ the row and column indices of the desired trapezoidal submatrix are collected.

TRAPEZOIDAL ALGORITHM.

Input: An (m, n) matrix $A = (a_{ij})$ of elements of a bottleneck algebra with $m \geqslant n \geqslant 2$.

Output: "yes" for the variable named answer and a trapezoidal (n, n) submatrix $T = (t_{ij})$ of A, if A has SLI columns; "no" for the variable answer otherwise.

begin
  for all $j \in N$ **do** fc(j) := "yes";
  for all $i \in M$ **do** fr(i) := "yes", g1(i) := 1, g2(i) := 2;
  answer := "no", $d := \min\{a_{ij};\ i \in M,\ j \in N\}$, $B \equiv (b_{ij}) :=$ ro(A);
  for all $(i, j) \in M \times N$ **do** $p_i(j)$:= column index of b_{ij} in A;
  for $l = 1, \ldots, n-1$ **do**
  begin
    for all $i \in M$ with fr(i) = "yes" **do**
    begin
      while fc(p_i(g1(i))) = "no" **do** g1(i) := g1(i) + 1;
      g2(i) := max(g1(i) + 1, g2(i));
      (**comment**: ensure the monotonicity of g2(i))

while $\mathrm{fc}(p_i(\mathrm{g2}(i))) =$ "no" **do** $\mathrm{g2}(i) := \mathrm{g2}(i) + 1$
end
$\mathscr{P} := \{ b_{i,\mathrm{g2}(i)};\ i \in M,\ b_{i,\mathrm{g2}(i)} \neq b_{i,\mathrm{g1}(i)} > d,\ \mathrm{fr}(i) = \text{"yes"} \}$;
if $\mathscr{P} = \varnothing$ **then** stop;
find min $\mathscr{P}$, say $b_{r,\mathrm{g2}(r)}$;
$r(l) := r,\ c(l) := p_r(\mathrm{g1}(r)),\ d := d \oplus b_{r,\mathrm{g2}(r)}$;
$\mathrm{fr}(r) :=$ "no", $\mathrm{fc}(p_r(\mathrm{g1}(r))) :=$ "no"
end
find $\max\{ b_{i,\mathrm{g2}(i)};\ i \in M,\ \mathrm{fr}(i) = \text{"yes"} \}$, say $b_{r,\mathrm{g2}(r)}$;
if $b_{r,\mathrm{g2}(r)} \leqslant d$ **then** stop;
$r(n) := r,\ c(n) := p_r(\mathrm{g2}(r))$;
for all $(i, j) \in N \times N$ **do** $t_{ij} = a_{r(i), c(j)}$;
answer := "yes"
end

THEOREM 4. *The trapezoidal algorithm is correct and terminates after using at most $O(mn \log n)$ arithmetic operations and comparisons. It enables, in particular, using not more than $O(n^2 \log n)$ operations to find a trapezoidal matrix equivalent to a given square matrix of order n or to indicate that such a matrix does not exist.*

Proof. The correctness is shown by Lemmas 9–11.

In order to estimate the computational complexity, realize first that to arrange each row nonincreasingly we need no more than $O(n \log n)$ operations (cf. [7]), and hence $O(mn \log n)$ is an upper bound for the reordering of all rows. We show that all other steps do not require more than $O(mn)$ operations and comparisons. The variables $\mathrm{g1}(i), \mathrm{g2}(i)$ for all $i \in M$ increase monotonically from 1, 2 respectively to at most n, and thus their evaluation does not need more than $O(mn)$ operations. The number of all other operations in one loop (when l is fixed) is not greater than $4m$ (for compiling and minimizing $\mathscr{P}$) plus 5 [for $r(l), c(l)$ and for redefining $d, \mathrm{fr}(r), \mathrm{fc}(p_r(\mathrm{g1}(r)))$]. It remains to recall that the number of loops is at most n and that T can be compiled by $O(n^2) \leqslant O(mn)$ operations. ■

EXAMPLE 1. We illustrate the algorithm by its application to the following matrix A in $\mathscr{B}_1$ with arbitrary $l < 0$, $u > 5$:

$$A = \begin{pmatrix} 5 & 5 & 3 & 1 \\ 2 & 0 & 3 & 4 \\ 1 & 0 & 5 & 0 \\ 4 & 3 & 2 & 1 \\ 2 & 1 & 1 & 5 \\ 3 & 4 & 0 & 4 \end{pmatrix}.$$

In this case our algorithm gives successively:

$$B=\begin{pmatrix}5&5&3&1\\4&3&2&0\\5&1&0&0\\4&3&2&1\\5&2&1&1\\4&4&3&0\end{pmatrix},\qquad (p_i(j))=\begin{pmatrix}1&2&3&4\\4&3&1&2\\3&1&2&4\\1&2&3&4\\4&1&3&2\\2&4&1&3\end{pmatrix};$$

$$d=0,\qquad \min\mathscr{P}=1,\qquad r(1)=3,\qquad c(1)=3;$$

$$d=1,\qquad \min\mathscr{P}=2,\qquad r(2)=2,\qquad c(2)=4;$$

$$d=2,\qquad \min\mathscr{P}=3,\qquad r(3)=4,\qquad c(3)=1;$$

$$r(4)=1,\qquad c(4)=2;$$

$$T=\begin{pmatrix}5&0&1&0\\3&4&2&0\\2&1&4&3\\3&1&5&5\end{pmatrix},\qquad \text{answer}=\text{“yes”}.$$

EXAMPLE 2. In the same bottleneck algebra we check the strong linear independence of columns of the matrix

$$A=\begin{pmatrix}1&2&3\\2&1&2\\0&1&1\\0&2&4\end{pmatrix}.$$

Here we get

$$B=\begin{pmatrix}3&2&1\\2&2&1\\1&1&0\\4&2&0\end{pmatrix},\qquad (p_i(j))=\begin{pmatrix}3&2&1\\1&3&2\\3&2&1\\3&2&1\end{pmatrix};$$

$$d=0,\qquad \min\mathscr{P}=2,\qquad r(1)=1,\qquad c(1)=3;$$

$$d=2,\qquad \mathscr{P}=\varnothing,\qquad \text{answer}=\text{“no”}.$$

6. MATRICES WITH STRONG PERMANENT ARE STRONGLY REGULAR

In the following, for $A = (a_{ij}) \in B(n, n)$ we denote by $p(A)$ the set

$$\{ i \in N;\ (\exists k \in N)\ (\forall j \in N - \{k\})\ a_{ik} \geqslant \operatorname{per}(A) > a_{ij} \},$$

and the ith row of A will be called a permanent row whenever $i \in p(A)$.

LEMMA 12. *If A and C are equivalent matrices, then*

(a) $\operatorname{per}(A) = \operatorname{per}(C)$,
(b) $|\max(A)| = |\max(C)|$,
(c) *A is strongly regular if and only if C is strongly regular,*
(d) $|p(A)| = |p(C)|$.

The proof is elementary.

Note that assertion (b) yields in particular that the property of possessing a strong permanent is also an invariant in the class of equivalent matrices.

LEMMA 13. *Let $A = (a_{ij}) \in B(n, n)$. If A is trapezoidal, then* $\operatorname{per}(A) = a_{11} \otimes a_{22} \otimes \cdots \otimes a_{nn}$ *(and hence* $\mathrm{id} \in \max(A)$*).*

Proof. Let $a_{rr} = \bigotimes_{i \in N} a_{ii}$, and take an arbitrary $\pi \in P_n$. We have to show that

$$a_{i, \pi(i)} \leqslant a_{rr} \tag{9}$$

for at least one $i \in N$. If $\pi(i) > i$ for some $i \in R = \{1, \dots, r\}$, then $a_{i, \pi(i)} < a_{rr}$, since A is trapezoidal. If $\pi(i) \leqslant i$ for all $i \in R$, then, of course, $\pi(i) = i$ for all $i \in R$, yielding equality in (9) for $i = r$. ■

LEMMA 14. *Every matrix with strong permanent contains at least one permanent row.*

Proof. Suppose $A = (a_{ij}) \in B(n, n)$ has no permanent row. Without loss of generality (by Lemma 12) we assume that $\mathrm{id} \in \max(A)$. We have to show that $|\max(A)| \geqslant 2$. Obviously, $l > 1$ satisfying $a_{1l} \geqslant \operatorname{per}(A)$ exists, because otherwise the first row would be permanent. Let $C = (c_{ij})$ be the matrix obtained from A by interchanging the second and lth rows as well as the columns with the same indices. Then, naturally, $\mathrm{id} \in \max(C)$ and $c_{12} \geqslant \operatorname{per}(C)$. If $c_{21} \geqslant \operatorname{per}(C)$, then evidently $(1\,2)(3)\cdots(n)$ is in $\max(C)$, and it is different from id, implying $|\max(A)| = |\max(C)| \geqslant 2$. Now suppose that $c_{21} < \operatorname{per}(C)$. Since C also has no permanent row (by Lemma 12), there exists l such that $2 < l \leqslant n$ and $c_{2l} \geqslant \operatorname{per}(C)$. Let $D = (d_{ij})$ be the matrix obtained from C by interchanging the third and lth rows as well as the columns with the same indices. Then again $\mathrm{id} \in \max(D)$, $d_{23} \geqslant \operatorname{per}(D)$, and D has no permanent row. Therefore we can continue by distinguishing whether $d_{3j} \geqslant \operatorname{per}(D)$ for some $j \in \{1,2\}$ or for $j \in \{4,\dots,n\}$. After a finite number (at most $n-2$) of such steps we obtain a matrix, say $Z = (z_{ij}) \sim A$, for which $\mathrm{id} \in \max(Z)$, $z_{i,i+1} \geqslant \operatorname{per}(Z)$ for $i = 1,\dots,k$ ($k \leqslant n-1$), and $z_{k+1,j} \geqslant \operatorname{per}(Z)$ for some $j \in \{1,2,\dots,k-1\}$. But then evidently $(1)(2)\cdots(j-1)(j\,j+1\cdots k\,k+1)(k+2)(k+3)\cdots(n)$ belongs to $\max(Z)$, and it is different from id. Thus $|\max(A)| = |\max(Z)| \geqslant 2$. ■

LEMMA 15. *Let $A = (a_{ij}) \in B(n,n)$. Suppose that A can be written blockwise in the form*

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where $A_{11} \in B(r,r)$, $1 \leqslant r < n$, and

$$\operatorname{per}(A) > \bigoplus_{i=1}^{r} \bigoplus_{j=i+1}^{n} a_{ij}. \tag{10}$$

Then A is equivalent to a trapezoidal matrix if and only if A_{22} is equivalent to a trapezoidal matrix.

Proof. Let $C = (c_{ij})$ be a trapezoidal matrix, $C \sim A$. Then $A = C(\sigma, \pi)$ for some $\sigma, \pi \in P_n$. Since a_{11} is the only element in the first row of A greater than or equal to $\operatorname{per}(A)$ and

$$\operatorname{per}(A) = \operatorname{per}(C) = c_{11} \otimes \cdots \otimes c_{nn} \tag{11}$$

(by Lemmas 12 and 13), we have $\sigma(1) = \pi(1)$, yielding $\sigma(i) \neq \pi(1)$ for all $i \in N - \{1\}$. In particular, $\sigma(2) \neq \pi(1)$, and thus, since (11) holds and A is trapezoidal, the only value of j satisfying $\sigma(2) = \pi(j)$ is $j = 2$. Hence $\sigma(i) \neq \pi(2)$ for all $i \in N - \{2\}$, and proceeding in this way, we derive that $\sigma(i) = \pi(i)$ for all $i = 1,\dots,r$. Denote by D the matrix arising from C on deleting the rows and columns with indices $\sigma(1),\dots,\sigma(r)$. Clearly, D as a principal submatrix of C is also trapezoidal, and $D \sim A_{22}$.

Now let us suppose that A_{22} is equivalent to a trapezoidal matrix. Since a permutation of the last $n-r$ rows and columns of A does not change the validity of the assumptions, we may assume without loss of generality that A_{22} is trapezoidal, i.e.,

$$a_{kk} > \bigoplus_{i=r+1}^{k} \bigoplus_{j=i+1}^{n} a_{ij} \tag{12}$$

for all $k \in N - R$, where $R = \{1,2,\dots,r\}$. Now it suffices to show that A is trapezoidal too. If $\pi \in \max(A)$, then it follows from (10) that π is the identity on R. Thus if $\mathrm{id} \notin \max(A)$, i.e. $w(A,\mathrm{id}) < w(A,\sigma)$ for some $\sigma \in \max(A)$, then $\sigma \restriction R$ is the identity and $\sigma \restriction N - R$ is a permutation, say σ', satisfying

$$w(A_{22},\sigma') > w(A_{22},\mathrm{id}),$$

which contradicts Lemma 13. Hence

$$a_{kk} \geqslant \mathrm{per}(A) > \bigoplus_{i=1}^{r} \bigoplus_{j=i+1}^{n} a_{ij}$$

for all $k \in N$. This and (12) complete the proof. ∎

THEOREM 5. Let $A \in B(n,n)$. A sufficient condition for A to be equivalent to a trapezoidal matrix is that A has a strong permanent. If n is (less than or) equal to 2, then this condition is also necessary.

Proof. By induction on n. If $A \in B(2,2)$ has a strong permanent, then take any $C = (c_{ij})$ equivalent to A in which c_{12} is the minimal element of C. Hence $c_{11} \otimes c_{22} \geqslant c_{12} \otimes c_{21} = c_{12}$. But this inequality is in fact strict, since C has a strong permanent (Lemma 12), implying that $c_{11} > c_{12}$ and $c_{22} > c_{12}$.

Now suppose $A = (a_{ij}) \in B(n,n)$, $n > 2$, and A has a strong permanent. Hence (by Lemma 14) A has at least one permanent row. Let $C = (c_{ij})$ be any matrix equivalent to A for which

$$c_{11} \geqslant \mathrm{per}(C) > c_{1j} \tag{13}$$

holds for all $j \in N - \{1\}$. Let us write C in the form

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}, \qquad \text{where} \quad C_{11} = (c_{11}).$$

It follows from (13) that $\mathrm{per}(C) = c_{11} \otimes \mathrm{per}(C_{22})$. Clearly, $|\max(C_{22})| \geqslant 2$ would imply $|\max(C)| \geqslant 2$, which contradicts the assumption of a strong permanent of A (by Lemma 12). Hence $C_{22} \in B(n-1, n-1)$ has a strong permanent, and thus it is equivalent to a trapezoidal matrix by the induction hypothesis. But (13) shows also that all assumptions of Lemma 15 are fulfilled. Hence C (and A) is equivalent to a trapezoidal matrix, too.

It remains to prove the necessary condition for matrices of order 2. Without loss of generality (Lemma 12) we suppose that $A = (a_{ij}) \in B(2,2)$ is trapezoidal. Hence $a_{11} > a_{12}$, $a_{22} > a_{12}$, and thus

$$\mathrm{per}(A) = a_{11} \otimes a_{22} > a_{12} \otimes a_{21},$$

implying that $\max(A) = \{\mathrm{id}\}$. ∎

By Theorem 2 we have immediately the desired corollary.

Corollary of Theorems 2 and 5. *Let $A \in B(n, n)$.*

(a) *If $\leqslant$ is dense and A has a strong permanent, then A is strongly regular.*

(b) *If n is (less than or) equal to 2 and A is strongly regular, then A has a strong permanent.*

Remark 2. Trapezoidal matrices of order $n > 2$ need not have a strong permanent; e.g., the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

in $\mathscr{B}_1$ with $l < 0$, $u > 2$ is trapezoidal, but it does not have a strong permanent.

Remark 3. Assertion (a) of the Corollary of Theorems 2 and 5 does not hold in general without the assumption of density. This can be demonstrated by the matrix

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

in $\mathscr{B}_3$, which has a strong permanent (and is trapezoidal) but is not strongly regular (this can be verified elementarily from the definition).

THEOREM 6. *The bottleneck assignment problem can be solved using not more than $O(n^2 \log n)$ operations for every matrix of order n equivalent to a trapezoidal matrix. In particular, this is true for all matrices with strong permanent.*

Proof. If A is a square matrix of order n and a trapezoidal matrix equivalent to A exists, then at least one such matrix, say T, can be found in $O(n^2 \log n)$ operations by Theorem 4. But $\mathrm{per}(A) = \mathrm{per}(T)$ (by Lemma 12), and the latter value can be computed in $O(n)$ operations (Lemma 13). Moreover, $r^{-1}c$ is obviously a solution to the bottleneck assignment problem for A, where r and c are permutations found by the trapezoidal algorithm.

The second assertion follows immediately from this result and from Theorem 5. ∎

We notice finally that one can investigate properties of "weakly trapezoidal matrices" defined in the same way as trapezoidal ones but replacing $>$ by $\geqslant$. It is not difficult to show that the trapezoidal algorithm can be appropriately modified to this class of matrices, and hence also Theorem 6 can be extended to weakly trapezoidal matrices.

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