

Tropical tensor product and beyond

Peter Butkovič and Miroslav Fiedler

February 15, 2011

Abstract

Our aim is to introduce the tropical tensor product and investigate its properties. In particular we show its use for solving tropical matrix equations.

1 Tropical prerequisites

The aim of this paper is to introduce the tropical tensor product of matrices and investigate its properties. In particular we show its use for solving tropical matrix equations.

The operations are in max-plus, that is for $a, b \in \overline{\mathbb{R}} := \mathbb{R} \cup \{\varepsilon = -\infty\}$ we define

$$\begin{aligned} a \oplus b &= \max(a, b), \\ a \otimes b &= a + b. \end{aligned}$$

Note that $(\overline{\mathbb{R}}, \oplus, \otimes)$ is a commutative idempotent semiring.

For matrices A, B of compatible sizes we define

$$\begin{aligned} A \oplus B &= (a_{ij} \oplus b_{ij}), \\ A \otimes B &= \left(\sum_k^{\oplus} a_{ik} \otimes b_{kj} \right), \\ \alpha \otimes A &= (\alpha \otimes a_{ij}). \end{aligned}$$

We denote $A \otimes A$ by A^2 , etc... (including scalars).

An $n \times n$ matrix is called *diagonal*, notation $diag(d_1, \dots, d_n)$, or just $diag(d)$, if its diagonal entries are $d_1, \dots, d_n \in \mathbb{R}$ and off-diagonal entries are ε . The matrix $diag(0, \dots, 0)$ of an appropriate order will be called the *unit matrix* and denoted by I . Any matrix which can be obtained from the unit (diagonal) matrix by permuting the rows and/or columns will be called a *permutation matrix* (*generalized permutation matrix*). Note that a matrix in the max-plus setting is invertible if and only if it is a generalized permutation matrix [2].

We denote $N = \{1, 2, \dots, n\}$ and by P_n the set of all permutation of N . *Tropical permanent* is an analogue of the conventional permanent:

$$\text{maper}(A) \stackrel{\text{df}}{=} \sum_{\pi \in P_n} \bigoplus \prod_{i \in N}^{\otimes} a_{i, \pi(i)} = \max_{\pi \in P_n} \sum_{i \in N} a_{i, \pi(i)}.$$

Hence finding $\text{maper}(A)$ amounts to solving the classical assignment problem, that is to finding a permutation $\pi \in P_n$ maximising $w(\pi, A)$ where

$$w(\pi, A) = \sum_{i \in N} a_{i, \pi(i)}.$$

As a direct consequence of the Hungarian method for solving the assignment problem [1] we have the following [2].

Theorem 1 *Let $A \in \overline{\mathbb{R}}^{n \times n}$ and suppose that $w(\pi, A)$ is finite for at least one $\pi \in P_n$. Then diagonal matrices C, D such that*

$$\text{maper}(C \otimes A \otimes D) = 0 \tag{1}$$

and

$$C \otimes A \otimes D \leq 0 \tag{2}$$

exist and can be found in $O(n^3)$ time. Also, the following holds for any diagonal matrices C and D satisfying (1):

$$\begin{aligned} \text{maper}(A) &= \\ &= (\text{maper}(C) \otimes \text{maper}(D))^{-1} \\ &= \left(\prod_{i \in N}^{\otimes} c_{ii} \otimes \prod_{i \in N}^{\otimes} d_{ii} \right)^{-1}. \end{aligned}$$

The maximum cycle mean of $A \in \overline{\mathbb{R}}^{n \times n}$ is

$$\lambda(A) = \max \left\{ \frac{a_{i_1 i_2} + a_{i_2 i_3} + \dots + a_{i_k i_1}}{k}; k, i_1, \dots, i_k \in N \right\}$$

where

Theorem 2 *For every $A \in \overline{\mathbb{R}}^{n \times n}$ there is an $x \in \overline{\mathbb{R}}^n, x \neq \varepsilon$ (eigenvector) such that $A \otimes x = \lambda(A) \otimes x$. If $x \in \mathbb{R}^n$ and $A \otimes x = \lambda \otimes x$ then $\lambda = \lambda(A)$. If A is irreducible then all eigenvectors are finite and hence $\lambda(A)$ is the unique eigenvalue .*

We also denote

$$a \oplus' b = \min(a, b),$$

$$a \otimes' b = a + b \text{ if } \{a, b\} \neq \{-\infty, +\infty\}$$

and

$$(-\infty) \otimes' (+\infty) = +\infty = (+\infty) \otimes' (-\infty).$$

The conjugate of A is $A^\# = -A^T$. It is known that $A \otimes x \leq b \iff x \leq A^\# \otimes' b \stackrel{\text{df}}{=} \bar{x}$ [2] and consequently, a solution to $A \otimes x = b$ exists if and only if \bar{x} is a solution.

2 Tensor product

Let $A = (a_{ij}) \dots m \times n, B = (b_{ij}) \dots r \times s$. The *tensor product* of A and B is the following $mr \times ns$ matrix:

$$A \boxtimes B = \begin{pmatrix} A \otimes b_{11} & A \otimes b_{12} & \cdots & A \otimes b_{1s} \\ A \otimes b_{21} & A \otimes b_{22} & \cdots & A \otimes b_{2s} \\ \cdots & \cdots & \cdots & \cdots \\ A \otimes b_{r1} & A \otimes b_{r2} & \cdots & A \otimes b_{rs} \end{pmatrix}.$$

Note that $(A \boxtimes B)^T = A^T \boxtimes B^T$ and the tensor product of two diagonal matrices of order n is a diagonal matrix of order n^2 . In particular the tensor product of two unit matrices is a unit matrix.

Matrices A and B are called *product compatible* if the product $A \otimes B$ is well defined, that is the number of columns of A is equal to the number of rows of B . The proof of the following theorem follows the lines of the proof of the analogous statement in [3].

Theorem 3 *If the matrices A and C are product compatible and also B and D are product compatible then $A \boxtimes B$ and $C \boxtimes D$ are product compatible and*

$$(A \boxtimes B) \otimes (C \boxtimes D) = (A \otimes C) \boxtimes (B \otimes D). \quad (3)$$

Proof. If B is $p \times q$ and D is $q \times r$ then the LHS of (3) is

$$\begin{pmatrix} A \otimes b_{11} & A \otimes b_{12} & \cdots & A \otimes b_{1q} \\ A \otimes b_{21} & A \otimes b_{22} & \cdots & A \otimes b_{2q} \\ \cdots & \cdots & \cdots & \cdots \\ A \otimes b_{p1} & A \otimes b_{p2} & \cdots & A \otimes b_{pq} \end{pmatrix} \otimes \begin{pmatrix} C \otimes d_{11} & C \otimes d_{12} & \cdots & C \otimes d_{1r} \\ C \otimes d_{21} & C \otimes d_{22} & \cdots & C \otimes d_{2r} \\ \cdots & \cdots & \cdots & \cdots \\ C \otimes d_{q1} & C \otimes d_{q2} & \cdots & C \otimes d_{qr} \end{pmatrix}.$$

Using blockwise tropical matrix multiplication we get that this is the same as

$$\begin{pmatrix} A \otimes C \otimes (b_{11} \otimes d_{11} \oplus \dots \oplus b_{1q} \otimes d_{q1}) & \cdots & A \otimes C \otimes (b_{11} \otimes d_{1r} \oplus \dots \oplus b_{1q} \otimes d_{qr}) \\ \cdots & \cdots & \cdots \\ A \otimes C \otimes (b_{p1} \otimes d_{11} \oplus \dots \oplus b_{pq} \otimes d_{q1}) & \cdots & A \otimes C \otimes (b_{p1} \otimes d_{1r} \oplus \dots \oplus b_{pq} \otimes d_{qr}) \end{pmatrix},$$

which is $(A \otimes C) \boxtimes (B \otimes D)$. ■

Theorem 4 *If A and B are invertible then $A \boxtimes B$ is invertible too and*

$$(A \boxtimes B)^{-1} = A^{-1} \boxtimes B^{-1}.$$

Proof. By Theorem 3 we have

$$(A \boxtimes B)^{-1} \otimes (A^{-1} \boxtimes B^{-1}) = (A \otimes A^{-1}) \boxtimes (B \otimes B^{-1}) = I \boxtimes I = I.$$

■

Theorem 5 If $A \otimes x = \lambda \otimes x$ and $B \otimes y = \mu \otimes y$ then

(a) $(A \boxtimes B) \otimes (x \boxtimes y) = \lambda \otimes \mu \otimes (x \boxtimes y)$ and

(b) $\alpha \otimes \lambda \oplus \beta \otimes \mu$ is an eigenvalue of $\alpha \otimes (A \boxtimes I) \oplus \beta \otimes (I \boxtimes B)$ for any $\alpha, \beta \in \mathbb{R}$.

Proof. (a) By Theorem 3 we have

$$\begin{aligned} (A \boxtimes B) \otimes (x \boxtimes y) &= \\ &= (A \otimes x) \boxtimes (B \otimes y) \\ &= (\lambda \otimes x) \boxtimes (\mu \otimes y) \\ &= \lambda \otimes \mu \otimes (x \boxtimes y). \end{aligned}$$

(b) By Theorem 3 we have

$$\begin{aligned} (\alpha \otimes (A \boxtimes I) \oplus \beta \otimes (I \boxtimes B)) \otimes (x \boxtimes y) &= \\ &= \alpha \otimes (A \boxtimes I) \otimes (x \boxtimes y) \oplus \beta \otimes (I \boxtimes B) \otimes (x \boxtimes y) \\ &= \alpha \otimes (A \otimes x) \boxtimes (I \otimes y) \oplus \beta \otimes (I \otimes x) \boxtimes (B \otimes y) \\ &= \alpha \otimes (\lambda \otimes x) \boxtimes y \oplus \beta \otimes x \boxtimes (\mu \otimes y) \\ &= \alpha \otimes \lambda \otimes (x \boxtimes y) \oplus \beta \otimes \mu \otimes (x \boxtimes y), \end{aligned}$$

from which the statement follows. ■

Corollary 6 If A and B have finite eigenvectors, in particular if they are irreducible, then $\lambda(A \boxtimes B) = \lambda(A) \otimes \lambda(B)$.

Proof. By Theorem 2 $\lambda(A)$ and $\lambda(B)$ is the eigenvalue of A and B respectively with associated finite eigenvectors, say x and y . By Theorem 5 $x \boxtimes y$ is a finite eigenvector of $A \boxtimes B$ with the associated eigenvalue $\lambda(A) \otimes \lambda(B)$. But the only eigenvalue of $A \boxtimes B$ associated with a finite eigenvector is $\lambda(A \boxtimes B)$ (Theorem 2). ■

If A is a matrix then $vec(A)$ will stand for the vector whose components are formed by the first column of A followed by the second column and so on.

Theorem 7 Matrix equation

$$A_1 \otimes X \otimes B_1 \oplus A_2 \otimes X \otimes B_2 \oplus \dots \oplus A_r \otimes X \otimes B_r = C, \quad (4)$$

where A_i, B_i and C are of compatible sizes, is equivalent to the vector-matrix system

$$(A_1 \boxtimes B_1^T \oplus A_2 \boxtimes B_2^T \oplus \dots \oplus A_r \boxtimes B_r^T) \otimes vec(X) = vec(C).$$

Proof. It is sufficient to prove the statement for $r = 1$. Let X_1, X_2, \dots be the columns of X . Let

$$B = \begin{pmatrix} b_{11} & \dots & b_{1p} \\ \dots & \dots & \dots \\ b_{n1} & \dots & b_{np} \end{pmatrix}.$$

Then

$$\begin{aligned}
\text{vec}(A \otimes X \otimes B) &= \\
&= \text{vec}((A \otimes X_1, \dots, A \otimes X_n) \otimes B) \\
&= \begin{pmatrix} A \otimes X_1 \otimes b_{11} \oplus A \otimes X_2 \otimes b_{21} \oplus \dots \oplus A \otimes X_n \otimes b_{n1} \\ A \otimes X_1 \otimes b_{12} \oplus A \otimes X_2 \otimes b_{22} \oplus \dots \oplus A \otimes X_n \otimes b_{n2} \\ \dots \\ A \otimes X_1 \otimes b_{1p} \oplus A \otimes X_2 \otimes b_{2p} \oplus \dots \oplus A \otimes X_n \otimes b_{np} \end{pmatrix} \\
&= \begin{pmatrix} A \otimes b_{11} & A \otimes b_{21} & \dots & A \otimes b_{n1} \\ A \otimes b_{12} & A \otimes b_{22} & \dots & A \otimes b_{n2} \\ \dots & \dots & \dots & \dots \\ A \otimes b_{1p} & A \otimes b_{2p} & \dots & A \otimes b_{np} \end{pmatrix} \otimes \begin{pmatrix} X_1 \\ X_2 \\ \dots \\ X_n \end{pmatrix} \\
&= (A \boxtimes B^T) \otimes \text{vec}(X)
\end{aligned}$$

■

Corollary 8 Matrix equation (4) has a solution if and only if

$$D \otimes (D^\# \otimes' \text{vec}(C)) = \text{vec}(C),$$

where

$$D = A_1 \boxtimes B_1^T \oplus A_2 \boxtimes B_2^T \oplus \dots \oplus A_r \boxtimes B_r^T.$$

Lemma 9 If P and Q are diagonal matrices of order n and $A \in \overline{\mathbb{R}}^{n \times n}$ then

$$\text{maper}(A \boxtimes Q) = (\text{maper}(A))^n \otimes (\text{maper}(Q))^n$$

and

$$\text{maper}(P \boxtimes A) = (\text{maper}(P))^n \otimes (\text{maper}(A))^n.$$

Proof. $A \boxtimes Q$ is the blockdiagonal matrix

$$\begin{pmatrix} A \otimes q_{11} & \dots & \dots \\ \dots & A \otimes q_{22} & \dots \\ \dots & \dots & \ddots \end{pmatrix}.$$

Hence

$$\begin{aligned}
\text{maper}(A \boxtimes Q) &= \text{maper}(A \otimes q_{11}) \otimes \text{maper}(A \otimes q_{22}) \otimes \dots \\
&= \text{maper}(A) \otimes q_{11}^n \otimes \text{maper}(A) \otimes q_{22}^n \otimes \dots \\
&= (\text{maper}(A))^n \otimes (q_{11} \otimes q_{22} \otimes \dots)^n \\
&= (\text{maper}(A))^n \otimes (\text{maper}(Q))^n.
\end{aligned}$$

The other statement is proved similarly, since $P \boxtimes A$ is also a block-diagonal matrix. ■

Theorem 10 If $A \in \overline{\mathbb{R}}^{n \times n}$, $B \in \overline{\mathbb{R}}^{m \times m}$ then

$$\text{maper}(A \boxtimes B) = (\text{maper}(A))^n \otimes (\text{maper}(B))^m.$$

Proof. Let P, Q, R, S be diagonal matrices (see Theorem 1) such that

$$\begin{aligned} C &\leq 0 \\ \text{maper}(C) &= 0 \end{aligned}$$

where $C = P \otimes A \otimes Q$ and

$$\begin{aligned} D &\leq 0 \\ \text{maper}(D) &= 0 \end{aligned}$$

where $D = R \otimes B \otimes S$. Hence

$$\text{maper}(A) = (\text{maper}(P) \otimes \text{maper}(Q))^{-1}$$

and

$$\text{maper}(B) = (\text{maper}(R) \otimes \text{maper}(S))^{-1}$$

By two applications of Theorem 3 we have

$$\begin{aligned} C \boxtimes D &= \\ &= (P \otimes A \otimes Q) \boxtimes (R \otimes B \otimes S) = \\ &= (P \boxtimes R) \otimes ((A \otimes Q) \boxtimes (B \otimes S)) \\ &= (P \boxtimes R) \otimes (A \boxtimes B) \otimes (Q \boxtimes S). \end{aligned}$$

It is sufficient now to show that $\text{maper}(C \boxtimes D) = 0$ because by Lemma 9 and Theorem 1 then

$$\begin{aligned} \text{maper}(A \boxtimes B) &= (\text{maper}(P \boxtimes R) \otimes \text{maper}(Q \boxtimes S))^{-1} \\ &= ((\text{maper}(P))^n \otimes (\text{maper}(R))^n \otimes (\text{maper}(Q))^n \otimes (\text{maper}(S))^n)^{-1} \\ &= (\text{maper}(A))^n \otimes (\text{maper}(B))^m. \end{aligned}$$

Clearly, $E = C \boxtimes D \leq 0$ and so we only need to identify a permutation, say $\tau \in P_{n^2}$ such that $e_{i, \tau(i)} = 0$ for all $i = 1, \dots, n^2$. Recall that

$$C \boxtimes D = \begin{pmatrix} C \otimes d_{11} & C \otimes d_{12} & \dots \\ C \otimes d_{21} & \dots & \\ \dots & & \end{pmatrix}.$$

Since $C \leq 0$ and $\text{maper}(C) = 0$ there is a $\pi \in P_n$ such that $c_{i, \pi(i)} = 0$ for all $i \in N$ and similarly there is a $\sigma \in P_n$ such that $d_{i, \sigma(i)} = 0$ for all $i \in N$. Let $i \in \{1, \dots, n^2\}$, $i = kn + j$, $0 \leq k \leq n - 1$, $1 \leq j \leq n$. Set $\tau(i) = (\sigma(k + 1) - 1)n + \pi(j)$. Then

$$e_{i, \tau(i)} = c_{j, \pi(j)} \otimes d_{k+1, \sigma(k+1)} = 0 \otimes 0 = 0.$$

■

References

- [1] R.E. Burkard, Dell'Amico M and Martello S: Assignment problems. SIAM, Philadelphia 2009
- [2] P. Butkovic: Max-linear Systems: Theory and Algorithms, Springer Monographs in Mathematics, Springer-Verlag 2010
- [3] M. Fiedler: Special Matrices and Their Applications in Numerical Mathematics, Dover 2008