Supereigenvectors

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April 13, 2012

This text is in max-plus, $\otimes$ is sometimes omitted.

Supereigenvectors are the elements of

$$V^{*}(A,\lambda)=\left\{x\in\overline{\mathbb{R}}^{n};Ax\geq\lambda x,x\neq\varepsilon\right\},$$

where $A\in\overline{\mathbb{R}}^{n\times n}$ and $\lambda\in\overline{\mathbb{R}}$.

We also denote

$$\begin{aligned} V(A,\lambda) &= \left\{x\in\overline{\mathbb{R}}^{n};Ax=\lambda x,x\neq\varepsilon\right\}, \\ V_{*}(A,\lambda) &= \left\{x\in\overline{\mathbb{R}}^{n};Ax\leq\lambda x,x\neq\varepsilon\right\}, \\ N &= \{1,...,n\}. \end{aligned}$$

$A(J)$ is an abbreviation for $A(J,J)$.

Lemma 1 *If $Ax\geq\lambda x, x$ finite then $\lambda\leq\lambda(A)$.*

Proof. Take any $i=i_{1.}$ Then

$$\lambda+x_{i_1}\leq a_{i_1i_2}+x_{i_2}$$

for some i_2. Similarly

$$\lambda+x_{i_2}\leq a_{i_2i_3}+x_{i_3}$$

for some i_3 and so on. By finiteness and by omitting, if necessary, a few first indices we get for some k :

$$\lambda+x_{i_k}\leq a_{i_ki_1}+x_{i_1}.$$

After adding up and simplifying we have

$$\lambda\leq\frac{a_{i_1i_2}+...+a_{i_ki_1}}{k}\leq\lambda(A).$$

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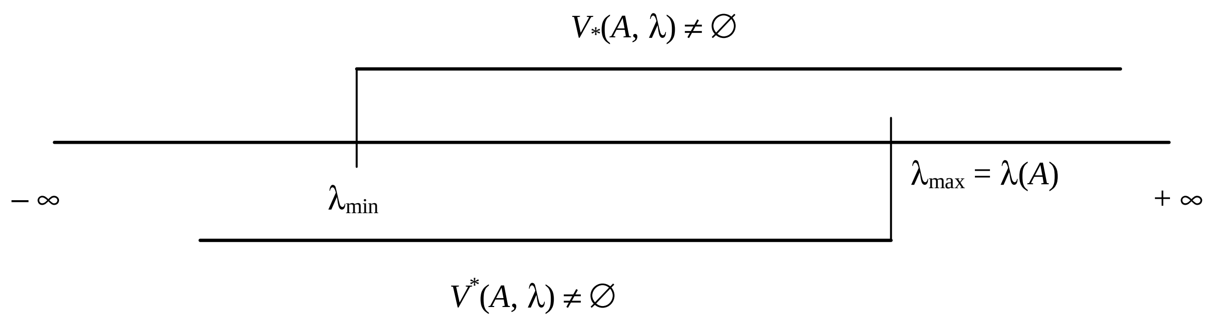
Proposition 2 *$V^{*}(A,\lambda)\neq\emptyset$ if and only if $\lambda\leq\lambda(A)$.*



Proof. Suppose first $Ax \geq \lambda x, x \neq \varepsilon$. Let $J = \mathrm{supp}\,(x)$, then

$$A(J)\,x(J) \geq \lambda x(J).$$

By Lemma 1 we have $\lambda \leq \lambda(A(J)) \leq \lambda(A)$.

Suppose now $\lambda \leq \lambda(A)$. Let $x \in V(A, \lambda(A))$. Then $x \neq \varepsilon$ and

$$A \otimes x = \lambda(A) \otimes x \geq \lambda \otimes x.$$

■

Corollary 3 *If $\lambda(A) = \varepsilon$ and $V^*(A, \lambda) \neq \emptyset$ then $\lambda = \varepsilon$ and $V^*(A, \varepsilon) = \overline{\mathbb{R}}^n - \{\varepsilon\}$.*

We will now assume that $\lambda(A) > \varepsilon$ and so without loss of generality $\lambda(A) = 0 \geq \lambda$.

Proposition 4 *For every $J \subseteq N$ there exists an $x \in V^*(A, \lambda)$, where $\lambda = \lambda(A(J))$ and $x(N - J) = \varepsilon$.*

Proof. Let $J \subseteq N$. Then there exists a $z \neq \varepsilon$ such that $A(J)\,z = \lambda(A(J))\,z$. Set $x(J) = z$ and $x(N - J) = \varepsilon$. Then

$$\begin{aligned} A \otimes x &= \begin{pmatrix} A(J,J) & A(J,N-J) \\ A(N-J,J) & A(N-J,N-J) \end{pmatrix} \begin{pmatrix} x(J) \\ \varepsilon \end{pmatrix} \\ &= \begin{pmatrix} \lambda(A(J))\,x(J) \\ A(N-J,J)\,x(J) \end{pmatrix} \\ &\geq \lambda(A(J)) \begin{pmatrix} x(J) \\ \varepsilon \end{pmatrix} \\ &= \lambda(A(J))\,x. \end{aligned}$$

■

Proposition 5 *If* $A \otimes x \geq \lambda(A(J))\, x,\ x \neq \varepsilon$, *where* $J = \operatorname{supp}(x)$ *then there exists a critical cycle* $(i_1, i_2, ..., i_k)$ *in* $A(J)$ *such that*

$$A(C)\, x(C) = \lambda(A(J))\, x(C), \tag{1}$$

where $C = \{i_1, i_2, ..., i_k\}$.

Proof. If $\lambda(A(J)) = \varepsilon$ then every cycle is critical and at least one component, say i, of $A(J) \otimes x$ is ε because has an ε column. Then we can take $C = \{i\}$.

Let us now suppose that $\lambda(A(J)) > \varepsilon$ and denote $\lambda = \lambda(A(J))$. Let $i_1 \in J$. Then

$$\lambda + x_{i_1} \leq \max_j (a_{i_1 j} + x_j) = a_{i_1 i_2} + x_{i_2}$$

for some $i_2 \in J$. Similarly we have

$$\lambda + x_{i_2} \leq \max_j (a_{i_2 j} + x_j) = a_{i_2 i_3} + x_{i_3}$$

for some $i_3 \in J$, and so on. By finiteness and by omitting, if necessary, a few first indices we get for some k :

$$\lambda + x_{i_k} \leq \max_j (a_{i_k j} + x_j) = a_{i_k i_1} + x_{i_1}.$$

After adding up and simplifying we have

$$\lambda \leq \frac{a_{i_1 i_2} + ... + a_{i_k i_1}}{k} \leq \lambda(A(J)).$$

Hence none of the inequalities can be strict and (1) follows. ∎

Questions:

1. How to find all solutions?
2. Generators of $V^*(A, \lambda)$?
3. Basis of $V^*(A, \lambda)$?