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Strong regularity of matrices – a survey of results

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Abstract

Let $\mathscr{G} = (G, \otimes, \leq)$ be a linearly ordered, commutative group and $u \oplus v = \max(u, v)$ for all $u, v \in G$. Extend $\oplus$, $\otimes$ in the usual way on matrices over G. An $m \times n$ matrix A is said to have strongly linearly independent (SLI) columns, if for some b the system of equations $A \otimes x = b$ has a unique solution. If, moreover, $m = n$ then A is said to be strongly regular (SR). This paper is a survey of results concerning SLI and SR with emphasis on computational complexity. We present also a similar theory developed for a structure based on a linearly ordered set where $\oplus$ is maximum and $\otimes$ is minimum.

Keywords: Max-algebra, bottleneck algebra, linear independence, regularity, assignment problem.

Introduction

A wide class of problems in different areas of scientific research, like graph theory, automata theory, scheduling theory, communication networks, etc. can be expressed by an attractive formulation language by setting up an algebra of, say, real numbers in which the operations of multiplication and addition are replaced by arithmetical addition and selection of the greater of the two numbers, respectively. Monographs [9, 16] can be used as a comprehensive guide in this field. Specifically, a significant effort was developed to build up a theory similar to that in linear algebra, i.e., to study systems of linear equations, eigenvalue problems, independence, rank, regularity, dimension, etc. As it turned out there is only a thin barrier separating these concepts and combinatorial properties of matrices.

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The aim of the present paper is to offer a survey of results concerning strong regularity of matrices which, as we show, is closely related to the assignment problem. The emphasis lies in aspects of computational complexity.

We introduce the theory by the following example: Suppose that a (say chemical) factory manufactures products $P_1, \ldots, P_m$ each of which is made out of some of n components prepared on machines $M_1, \ldots, M_n$ (every machine prepares one fixed component to be used in several products). It is known that machine M_j finishes the preparation of the component for product P_i after a_{ij} time units from the beginning of its activity (we set $a_{ij} = -\infty$ if there is no need of the component made on machine M_j to produce P_i).

Denoting by $x_1, \ldots, x_n$ the starting times of the work of machines $M_1, \ldots, M_n$ we have that all the components necessary for making out P_i are prepared in time

$$\max_{j=1,\ldots,n} (a_{ij} + x_j).$$

Any delay of the beginning of processing of the components prepared for products $P_1, \ldots, P_m$ causes losses, on the other hand by technological reasons the processings cannot start at any time but only at specified moments $b_1, \ldots, b_m$. Therefore the question is to find the starting times $x_1, \ldots, x_n$ in order to fulfil the equations

$$\max_{j=1,\ldots,n} (a_{ij} + x_j) = b_i, \quad i = 1, \ldots, m. \tag{0.1}$$

It should be intuitively clear that in some situations there is a certain freedom in moving with $x_1, \ldots, x_n$ (it suffices if some machines just start at any time before certain critical value) however, as it will be apparent later (Section 2), under some circumstances it can happen that all starting times are uniquely determined, i.e., there is no option for them and the system becomes sensitive and instable. This corresponds exactly to situations when the matrix (a_{ij}) has (in terminology introduced in Section 3) strongly linearly independent columns (is strongly regular if $m = n$). Hence the problem of strong linear independence can be interpreted as follows: is the system described by the matrix (a_{ij}) stable for every m-tuple $(b_1, \ldots, b_m)$ of prescribed termination times or can it happen that for some $b_1, \ldots, b_m$ there is no freedom in the choice of when the system should be set in activity?

After setting $\oplus$ for maximum and $\otimes$ for addition, the system (0.1) gets the form

$$\sum_{j=1}^{n}{}^{\oplus}\, a_{ij} \otimes x_j = b_i, \quad i = 1, \ldots, m,$$

which, as was already mentioned, motivated the study of problems which are linear with respect to $\oplus$ and $\otimes$.

Consider now the following small numerical example:

$$\max(1 + x_1, x_2) = 3,$$

$$\max(x_1, x_2) = t,$$

where t is a real parameter. If $t = 3$ then necessarily $x_2 = 3$ but $x_1 \leq 2$ can be arbitrarily small. If $t = 2$ then $x_1 = 2$ but $x_2 \leq 2$ can be arbitrarily small. Finally, if $2 < t < 3$ then necessarily $x_2 = t$ and thus $x_1 = 2$. For $t < 2$ and $t > 3$ the system has evidently no solution. Hence we deduce that the instability arises only for $t \in (2, 3)$ and would not appear if we would consider only integer entries. This explains why the results concerning strong regularity (Section 4) depend on the density of the underlying linearly ordered group.

At last a short introduction to Section 5. Being motivated by several practical interpretations (cf. Section 5) a theory analogous to that mentioned above was developed for problems which are linear with respect to $\oplus = \max$ and $\otimes = \min$, sometimes referred to as a bottleneck algebra. As a consequence of this theory, as we show, some computational complexity results follow, e.g. the bottleneck assignment problem for $n \times n$ matrices is solvable in $O(n^2 \log n)$ operations whenever the optimal permutation is unique.

1. Max-algebra

We assume throughout the paper that $\mathscr{G} = (G, \otimes, \leq)$ is a nontrivial linearly ordered, commutative group (LOCG) with neutral element e.

The symbol $a < b$ means $a \leq b$ and $a \neq b$ for all $a, b \in G$ and (a, b) stands for the open interval $\{c \in G\colon a < c < b\}$. $\mathscr{G}$ is called *dense* if $(a, b) \neq \emptyset$ for all $a, b \in G$, $a < b$ and $\mathscr{G}$ is called *sparse* if it is not dense. An element $a \in G$ is called *positive* if $a > e$.

The iterated product

$$\underbrace{a \otimes a \otimes \cdots \otimes a}_{k \text{ times}}$$

will be denoted by a^k and we set $(a^{-k}) = (a^{-1})^k$ and $a^0 = e$. $\mathscr{G}$ is called *cyclic* if $G = \{g^k\colon k \text{ integer}\}$ for some positive $g \in G$ which is called a *generator of* G.

$\mathscr{G}$ is called *radicable* if for every $a \in G$ and natural number k an element $b \in G$ satisfying $b^k = a$ exists. Such an element b is unique and we denote it by $\sqrt[k]{a}$. (Note that $\sqrt{a}$ stands for $\sqrt[2]{a}$).

One can easily verify that in a radicable group $a < b$ implies

$$a < \sqrt{a \otimes b} < b$$

and hence we have

Proposition 1.1. *Every radicable LOCG is dense.*

On the other hand for $g > e$ and arbitrary integer k we have $g^k < g^{k+1}$ yielding

Proposition 1.2. *Every cyclic LOCG is sparse.*

As examples we recall some well-known linearly ordered, commutative groups:

$$\mathcal{G}_1 = (\mathbb{Q}, +, \leq),$$

$$\mathcal{G}_2 = (\mathbb{Z}, +, \leq),$$

$$\mathcal{G}_3 = (\mathbb{Z}_2, +, \leq),$$

$$\mathcal{G}_4 = (\mathbb{R}^+, \cdot, \leq),$$

$$\mathcal{G}_5 = (\mathbb{Q}^+, \cdot, \leq),$$

$$\mathcal{G}_6 = (\mathbb{Z} \times \mathbb{Z}, +, \leq'),$$

where $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{Z}_2$, $\mathbb{R}^+$ and $\mathbb{Q}^+$ is the set of rationals, integers, even integers, positive reals and positive rationals, respectively. The signs $+, \cdot$ and $\leq$ stand here for conventional arithmetic operations and ordering, respectively (in the case of $\mathcal{G}_6$ the addition is to be applied componentwise). The ordering of $\mathcal{G}_6$ is defined by the formula $(a, b) \leq (c, d)$ iff $a < c$ or $a = c$ and $b \leq d$.

We see by inspection that

(i) $\mathcal{G}_1$, $\mathcal{G}_4$ are radicable,

(ii) $\mathcal{G}_5$ is dense but not radicable,

(iii) $\mathcal{G}_2$, $\mathcal{G}_3$ are cyclic,

(iv) $\mathcal{G}_6$ is sparse but not cyclic.

Bounded subsets of linearly ordered, commutative groups need not necessarily have an infimum, however the following holds:

Proposition 1.3. *The set of all positive elements of a LOCG has an infimum.*

Proof. If $\leq$ is dense then e is evidently the infimum.

It suffices now to show that $(e, a) = \emptyset$ for some positive $a \in G$ whenever $\leq$ is sparse. Suppose on the contrary that $(e, a) \neq \emptyset$ for all $a > e$ and let $c, d \in G$, $c < d$ be arbitrary. Since $(e, d \otimes c^{-1}) \neq \emptyset$ we have that an element $b \in G$ satisfying

$$e < b < d \otimes c^{-1}$$

exists and hence

$$c < b \otimes c < d,$$

a contradiction to the sparseness of $\mathcal{G}$. $\square$

For any $\mathcal{G}$ the infimum mentioned in Proposition 1.3 will be denoted by $\alpha(\mathcal{G})$ or only α if no confusion can arise. Clearly, $\alpha(\mathcal{G}) = e$ if $\mathcal{G}$ is dense, $\alpha(\mathcal{G}) > e$ if $\mathcal{G}$ is sparse and $\alpha(\mathcal{G}) = g$ if $\mathcal{G}$ is cyclic with generator g. Note that $\alpha(\mathcal{G}_6) = [0, 1]$.

Let us introduce the operation $\oplus$ on G by the formula

$$a \oplus b = \max\{a, b\} \quad \text{for all } a, b \in G.$$

Clearly, associativity, commutativity of $\oplus$ and $\otimes$ as well as distributivity in the conventional sense hold. From the early 60's an effort [8, 9, 14, 15, 16] was devoted for developing a systematic theory on algebraic problems linear with respect to such a couple of operations based on LOCG or similar structures under various names (like path algebra, extremal algebra, max-algebra). We will make use of the last mentioned expression.

We deal with matrices in max-algebra as well as with permutations related to them. For convenience let us introduce the following notation. If $m, n \geq 1$ are integers and S a set, we denote by $S(m, n)$ the set of all $m \times n$ matrices with entries from S. The symbol S_m stands for the set $S(m, 1)$ and its elements will be called *vectors*. We put $M = \{1, 2, \ldots, m\}$, $N = \{1, 2, \ldots, n\}$; P_n will denote the set of all permutations of N and C_n the set of all cyclic permutations (briefly *cycles*) of nonempty subsets of N. If $A = (a_{ij}) \in G(n, n)$ and $\sigma = (i_1, \ldots, i_l) \in C_n$ then l, the length of σ, will be denoted by $l(\sigma)$ and $w_A(\sigma)$, the weight of σ with respect to A, is defined as

$$a_{i_1 i_2} \otimes a_{i_2 i_3} \otimes \cdots \otimes a_{i_{l-1} i_l} \otimes a_{i_l i_1}.$$

Since every $\pi \in P_n$ can be decomposed into pairwise disjoint cycles, say $\sigma_1, \cdots, \sigma_s \in C_n$, $w_A(\pi)$, the weight of π with respect to A, can be defined by

$$w_A(\pi) = w_A(\sigma_1) \otimes w_A(\sigma_2) \otimes \cdots \otimes w_A(\sigma_s) \tag{1.1}$$

and one can easily see that then

$$w_A(\pi) = a_{1, \pi(1)} \otimes a_{2, \pi(2)} \otimes \ldots \otimes a_{n, \pi(n)}. \tag{1.2}$$

The identity permutation will be denoted by id.

2. Linear equations and the eigenproblem in max-algebra

We extend $\oplus$ and $\otimes$ to operations between matrices and between matrices and scalars as in conventional linear algebra, i.e., supposing that $A = (a_{ij})$, $B = (b_{ij})$ are matrices over G of an appropriate size and $a \in G$ we define

$$A \oplus B = (a_{ij} \oplus b_{ij}),$$

$$A \otimes B = \left(\sum_k^{\oplus} a_{ik} \otimes b_{kj} \right)$$

and

$$a \otimes A = (a \otimes a_{ij}).$$

The ordering is extended componentwise.

We can now describe the system of equations linear with respect to $\oplus$, $\otimes$ in the following equivalent standard ways:

$$\sum_{j\in N}^{\oplus} a_{ij}\otimes x_j = b_i, \quad i\in M, \tag{2.1}$$

$$\sum_{j\in N}^{\oplus} a^{(j)}\otimes x_j = b, \tag{2.2}$$

$$A\otimes x = b, \tag{2.3}$$

where $A = (a_{ij})\in G(m, n)$, $b = (b_1, \ldots, b_m)^{\mathrm{T}} \in G_m$, $x = (x_1, \ldots, x_n)^{\mathrm{T}}$ and $a^{(j)}$ denotes the jth column of A. (The symbol T denotes transposition.)

Solution methods for such systems of equations are well known [9, 15, 16] and we briefly recall some of the results. For convenience we use the following notation:

$$S(A, b) = \{x\in G_n\colon A\otimes x = b\} \quad \text{for } A\in G(m, n),\ b\in G_m,$$

$$\bar{x}_j = (\max(a_{ij}\otimes b_i^{-1}))^{-1} \quad \text{for } j\in N,$$

$$M_j(A, b) = \{i\in M\colon a_{ij}\otimes b_i^{-1} = \bar{x}_j^{-1}\}.$$

The symbol $|X|$ stands for the cardinality of the set X.

Theorem 2.1. *Let* $x = (x_1, \ldots, x_n)^{\mathrm{T}} \in G_n$. *Then* $x\in S(A, b)$ *if and only if*

(a) $x \le \bar{x}$ *and*

(b) $\bigcup_{j\in N_x} M_j(A, b) = M$, *where* $N_x = \{j\in N\colon x_j = \bar{x}_j\}$.

Proof. Can be found e.g. in [9, 15]. $\square$

Corollary 2.2. *The following three statements are equivalent*:

(i) $S(A, b) \neq \emptyset$,

(ii) $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)^{\mathrm{T}} \in S(A, b)$,

(iii) $\bigcup_{j\in N} M_j(A, b) = M$.

Corollary 2.3. $|S(A, b)| = 1$ *if and only if*

(a) $\bigcup_{j\in N} M_j(A, b) = M$ *and*

(b) $\bigcup_{j\in N} M_j(A, b) \neq M$ *for every* $N' \subseteq N$, $N' \neq N$.

The vector $\bar{x}$ can be computed from the definition in $\mathrm{O}(mn)$ operations and hence Theorem 2.1 enables to decide in $\mathrm{O}(mn)$ operations whether $S(A, b)$ is empty or nonempty and in the latter case to establish one solution to the system (2.3) (which is $\bar{x}$).

Example 2.4. Consider the system (2.1) in $\mathscr{G}_1$ with the matrix

$$A = \begin{pmatrix} 1 & 5 & 3 \\ 1 & 4 & 7 \\ -1 & 4 & 3 \\ 5 & 7 & 6 \\ 0 & 6 & 0 \end{pmatrix}$$

and three different right-hand side vectors:

$$b^{(1)} = (6, 9, 5, 10, 7)^{\mathrm{T}},$$
$$b^{(2)} = (7, 9, 5, 10, 7)^{\mathrm{T}},$$
$$b^{(3)} = (6, 9, 5, 8, 7)^{\mathrm{T}}.$$

We find easily from the definitions:

$$M_1(A, b^{(1)}) = \{1, 4\}, \quad M_2(A, b^{(1)}) = \{1, 3, 5\}, \quad M_3(A, b^{(1)}) = \{2, 3\},$$
$$\bar{x} = (5, 1, 2)^{\mathrm{T}}, \qquad S(A, b^{(1)}) = \{\bar{x}\};$$
$$M_1(A, b^{(2)}) = \{4\}, \quad M_2(A, b^{(2)}) = \{3, 5\}, \quad M_3(A, b^{(2)}) = \{2, 3\},$$
$$\bar{x} = (5, 1, 2)^{\mathrm{T}}, \qquad S(A, b^{(2)}) = \emptyset;$$
$$M_1(A, b^{(3)}) = \{4\}, \quad M_2(A, b^{(3)}) = \{1, 3, 4, 5\}, \quad M_3(A, b^{(3)}) = \{2, 3, 4\},$$
$$\bar{x} = (3, 1, 2)^{\mathrm{T}}, \qquad S(A, b^{(3)}) = \{(t, 1, 2)\colon t \le 3\}.$$

It is easy to see that a nontrivial LOCG has neither maximum nor minimum (because $a^{k+1} > a^k$ for $a \in G$, $a > e$ and integer k). Based on Theorem 2.1 we then immediately have

Theorem 2.5. *If the system* (2.3) *has more than one solution then it has an infinite number of solutions, i.e.,* $|S(A, b)| \in \{0, 1, \infty\}$ *for all* $A \in G(m, n)$ *and* $b \in G_m$.

An intensive effort was devoted also to the eigenproblem in max-algebra [9, 14, 16] which has various economical interpretations (cf. e.g. [8]) and can be formulated as follows: given $A \in G(n, n)$, find $x \in G_n$, called extremal eigenvector of A, and $\lambda \in G$, called extremal eigenvalue of A, satisfying

$$A \otimes x = \lambda \otimes x. \tag{2.4}$$

We mention here only one basic result which will be useful later.

Theorem 2.6. *Let* $\mathscr{G}$ *be radicable. Then for every* $A \in G(n, n)$ *there exists a unique* $\lambda \in G$ *satisfying* (2.4) *and* λ *equals the maximum cycle mean of the matrix* A, *i.e.:*

$$\lambda = \sum_{\sigma \in C_n}^{\oplus} \sqrt[l(\sigma)]{w_A(\sigma)}.$$

The extremal eigenvalue of A will be denoted $\lambda(A)$.

Example 2.7. Consider the eigenproblem for the matrix

$$A=\begin{pmatrix}1 & 3 & -2\\ 2 & 0 & 1\\ 4 & 1 & -3\end{pmatrix}$$

in $\mathscr{G}_1$. From Theorem 2.6 we compute

$$\lambda(A)=\max\left\{1,0,-3,\frac{3+2}{2},\frac{1+1}{2},\frac{4-2}{2},\frac{3+1+4}{3},\frac{2+1-2}{3}\right\}=\frac{8}{3}.$$

Hence $\lambda=8/3$ is the extremal eigenvalue of A and one can easily verify that for instance $x=(1/3,0,5/3)^{\mathrm{T}}$ is a corresponding extremal eigenvector.

Several papers are devoted to developing efficient algorithms for computing $\lambda(A)$, e.g. [3, 10, 14]. The least computational complexity ($\mathrm{O}(n^3)$) for general matrices has the algorithm presented in [12]. Methods for finding extremal eigenvectors can be found e.g. in [9, 16].

3. Linear independence, rank and regularity in max-algebra

There are several nonequivalent ways of introducing linear independence in max-algebra. One natural definition is as follows. The vectors $a^{(1)},a^{(2)},\dots,a^{(n)}\in G_m$ are said to be *linearly dependent* if some of them can be expressed as a linear combination of the others, i.e., for some $k\in N$ the coefficients $x_1,\dots,x_{k-1},x_{k+1},\dots,x_n$ satisfying

$$a^{(k)}=\sum_{i\in N-\{k\}}^{\oplus} x_i\otimes a^{(i)}$$

exist, and they are said to be *linearly independent* if they are not linearly dependent. It was shown in [9] that such definitions lead to dimensional anomalies, e.g. for $m>2$ an arbitrarily large set of linearly independent vectors can be constructed.

We say that $a^{(1)},\dots,a^{(n)}\in G_m$ are *strongly linearly independent* (SLI), if some $b\in G_m$ can be uniquely expressed as a linear combination of $a^{(1)},\dots,a^{(n)}$, i.e., if the system

$$\sum_{j\in N}^{\oplus} a^{(j)}\otimes x_j=b$$

has a unique solution. If, moreover $m=n$, then the matrix $A=(a^{(1)},\dots,a^{(n)})$ is called *strongly regular* (SR). It was proved in [9] that SLI vectors are linearly independent.

The concept of strong linear independence is in a correspondence with the concept of full column rank in conventional linear algebra. For this purpose we denote for any $A\in G(m,n)$ by $T(A)$ the set

$$\{|S(A,b)|:\ b\in G_m\}.$$

Theorem 2.5 says that $T(A) \subseteq \{0, 1, \infty\}$ for all $A \in G(m, n)$. We show that actually there are only two possibilities for $T(A)$.

Theorem 3.1. *For every $A \in G(m, n)$ with $m \geq 2, n \geq 2$ there is either $T(A) = \{0, \infty\}$ or $T(A) = \{0, 1, \infty\}$.*

Proof. Let $A \in G(m, n)$, $m \geq 2$, $n \geq 2$. Due to Theorem 2.5 it suffices to show that $S(A, b) = \emptyset$ for some $b \in G_m$ and $|S(A, b)| = \infty$ for some $b' \in G_m$.

We set $b = (b_1, e, e, \ldots, e)^{\mathrm{T}} \in G_m$ where b_1 is an arbitrary element of G less than

$$\min\{a_{1j} \otimes a_{ij}^{-1} \colon i \in M,\ i \neq 1, j \in N\}$$

since then

$$a_{1j} \otimes b_1^{-1} > a_{ij} \quad \text{for all } i \in M - \{1\},\ j \in N$$

yielding that

$$M_j(A, b) = \{1\} \quad \text{for all } j \in N$$

and thus

$$\bigcup_{j \in N} M_j(A, b) = \{1\} \neq M,$$

which by Corollary 2.2 implies $S(A, b) = \emptyset$.

We set $b' = a^{(1)}$ because then $M_1(A, b') = M$ and by Corollary 2.3 (putting $N' = \{1\} \neq N$) and by Corollary 2.2 we have that $|S(A, b')| > 1$ which using Theorem 2.5 completes the proof. □

Now we can compare the situation with that in conventional linear algebra as it is done in Table 1. Here $r(A)$ denotes the usual rank of the matrix A in linear algebra which enables one to describe matrices with specified $T(A)$. It is an easy exercise to show that classes of matrices $T(A) = \{0\}, \{1, \infty\}$ or $\{0, 1, \infty\}$ are empty in the case of conventional linear algebra.

The correspondence to the situation in classical linear algebra becomes more apparent after introducing the concept of rank of a matrix in max-algebra as follows:

$$r(A) = \max\{k \colon \exists (k \times k) \text{ SR submatrix of } A\}.$$

Theorem 3.2. *Let $A \in G(m, n)$. The columns of A are SLI if and only if $r(A) = n$, i.e., A has full column rank.*

Proof. Can be found in [4]. □

Theorem 3.2 reduces the question of SLI into the problem of strong regularity which can be solved efficiently as it is shown in the next section. However, this

Table 1

$T(A)$	Linear algebra	Max-algebra
$\{0\}$	–	–
$\{1\}$	$m = r(A) = n$	–
$\{\infty\}$	$m = r(A) < n$	–
$\{0, 1\}$	$n = r(A) < m$	–
$\{0, \infty\}$	$r(A) < \min(m, n)$	Columns not SLI
$\{1, \infty\}$	–	–
$\{0, 1, \infty\}$	–	Columns SLI

transformation is apparently not polynomial and it should be noted at this place that no efficient method for checking SLI as well as for computing the rank is known to the author.

We say that $A = (a_{ij}) \in G(m, n)$ is equivalent to $B = (b_{ij}) \in G(m, n)$, denoted by $A \sim B$, if B can be obtained from A by a sequence of operations of the following types:

(i) permuting the rows and/or columns,

(ii) multiplying (in the sense of $\otimes$) of the rows and/or columns by constants from G.

Clearly, $\sim$ constitutes an equivalence relation on $G(m, n)$.

Theorem 3.3. *If $A, B \in G(m, n)$, $A \sim B$ and A has SLI columns then also B has SLI columns.*

Proof. Trivial. $\square$

4. Strong regularity of matrices in max-algebra

In this section we summarize and unify the results of the preceding research concerned with finding efficient algorithms for checking the strong regularity of matrices.

We begin by a combinatorial aspect of this problem. Let $A = (a_{ij}) \in G(n, n)$. If A is SR then by Corollary 2.3 for some $b \in G_n$ the sets

$$M_1(A, b),\ M_2(A, b), \ldots, M_n(A, b) \tag{4.1}$$

(being subsets of N) form a minimal covering of N. As it is known from combinatorics this is possible if and only if the sets (4.1) are one-element and pairwise disjoint, i.e., for some permutation π of the set N we have

$$M_{\pi(j)}(A, b) = \{j\} \quad \text{for all } j \in N,$$

which is equivalent to

$$a_{j,\pi(j)} \otimes b_j^{-1} > a_{i,\pi(j)} \otimes b_i^{-1} \quad \text{for all } i, j \in N \text{ and } i \neq j. \tag{4.2}$$

By other words the problem of the strong regularity of A is equivalent to the question: can we multiply (in the sense of $\otimes$) the rows of A by constants in such a way that every column maximum will then be achieved in only one row and the maxima of any two different columns will lie in different rows?

It can be seen easily that (4.2) implies

$$w_A(\pi) > w_A(\tau) \tag{4.3}$$

for every $\tau \in P_n - \{\pi\}$. Hence the max-algebraic permanent of A

$$\operatorname{per}(A) = \sum_{\tau \in P_n}{}^{\oplus} w_A(\tau) \tag{4.4}$$

is achieved by unique permutation from P_n. In [4] matrices with this property are said to have a strong permanent.

On the other hand, the problem of finding $\operatorname{per}(A)$ defined by (4.4) is actually a generalisation of the (linear) assignment problem

$$\max\left\{ \sum_{j \in N} a_{j,\tau(j)} \colon \tau \in P_n \right\}$$

to which (4.4) turns in every subgroup of the additive group of reals (e.g. in $\mathscr{G}_1, \mathscr{G}_2, \mathscr{G}_3$). Being motivated by this we denote

$$\operatorname{ap}(A) = \{\pi \in P_n \colon w_A(\pi) = \operatorname{per}(A)\}$$

and hence $|\operatorname{ap}(A)| = 1$ means the same as "A has a strong permanent".

It is an easy exercise to prove that the following holds:

Proposition 4.1. *Let $A, B \in G(n, n)$. If $A \sim B$ then $|\operatorname{ap}(A)| = |\operatorname{ap}(B)|$.*

As it follows from (4.3) we have

Proposition 4.2. *If A is SR then A has a strong permanent.*

The converse implication does not hold in general; the matrix

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

in $\mathscr{G}_2$ is a possible counterexample. However, using a series of intermediate results the following was proved in [4].

Theorem 4.3. *Let $\mathscr{G}$ be dense and $A \in G(n, n)$. Then A is SR if and only if A has a strong permanent.*

The matrix $A = (a_{ij}) \in G(n, n)$ will be called *normal* if

$$a_{ij} \le a_{ii} = e \quad \text{for all } i, j \in N.$$

Clearly, $\mathrm{id} \in \mathrm{ap}(A)$ for every normal matrix A.

For solving the problem of checking the strong permanent we recall that the Hungarian method (e.g. [13]) for the solution of the assignment problem (AP) transforms an arbitrary matrix to an equivalent normal matrix in $O(n^3)$ operations. Note that the Hungarian method does not use the density of the group. If $A = (a_{ij}) \in G(n, n)$ then D_A will denote a digraph with node set N in which an arc (i, j) exists if and only if $a_{ij} = e$ and $i \ne j$. It was shown in [2] that the optimal solution to AP for a normal matrix A is unique if and only if D_A is acyclic. Hence the uniqueness of the optimal solution to AP can be checked by standard algorithms for testing whether a digraph is acyclic in only $O(n^2)$ operations. Using Proposition 4.1 this enables one to answer the question of SR in the dense case in $O(n^3) + O(n^2) = O(n^3)$ operations. In addition, a vector b for which $|S(A, b)| = 1$ can be found in a dense LOCG by a procedure derived in [4] in $O(n^3)$ operations.

Example 4.4. Consider the normal matrix

$$A = \begin{bmatrix} 0 & -4 & 0 & -6 \\ -1 & 0 & -3 & 0 \\ -2 & -1 & 0 & 0 \\ 0 & -2 & -5 & 0 \end{bmatrix}$$

in any subgroup of $\mathscr{G}_1$ containing all entries of A. Since D_A contains the cycle (1, 3, 4), we deduce that A is not strongly regular by Proposition 4.2.

Example 4.5. Consider the matrix

$$A = \begin{bmatrix} -12 & -8 & -14 & -4 \\ -2 & -2 & -4 & -6 \\ -8 & 0 & -4 & -2 \\ -10 & -4 & -6 & -4 \end{bmatrix}$$

in $\mathscr{G}_1$. By adding constants 2, 0, 0, 2 to the rows and 2, 0, 4, 2 to the columns the Hungarian method transforms A to a normal matrix (suitable permutation of columns is already included):

$$C = \begin{pmatrix} 0 & -8 & -6 & -8 \\ -4 & 0 & -2 & 0 \\ 0 & -6 & 0 & 0 \\ 0 & -6 & -2 & 0 \end{pmatrix}.$$

Since D_C is acyclic, we conclude that C (and A) has a strong permanent and hence A is SR in $\mathscr{G}_1$. However it is not clear immediately whether A is also SR in sparse subgroups of $\mathscr{G}_1$.

In order to solve the problem of checking SR in the case of sparse LOCG we now adopt the method developed in [2] for cyclic LOCG. Thereafter we show how it can be unified with the dense case.

Let $-\infty$ be an element adjoined to G and let us introduce the following rules for $-\infty$:

$$a \geq -\infty$$

and

$$a \otimes -\infty = -\infty \otimes a = -\infty$$

for all $a \in G'$, where $G' = G \cup \{-\infty\}$. By $G'(n, n)$ we denote the set of $n \times n$ matrices with entries from G'. Given $A = (a_{ij}) \in G'(n, n)$ the symbol $\tilde{A}$ will stand for the matrix $(\tilde{a}_{ij})$ such that

$$\begin{aligned} \tilde{a}_{ii} &= -\infty \quad && \text{for all } i \in N, \\ \tilde{a}_{ij} &= a_{ij} && \text{for all } i, j \in N,\ i \neq j. \end{aligned}$$

Evidently, we can easily extend already introduced operations $\oplus$ and $\otimes$ between matrices over G to matrices over G'. For $A \in G'(n, n)$ we denote by A^k (k natural) the (necessarily associative) iterated product

$$\underbrace{A \otimes A \otimes \cdots \otimes A}_{k \text{ times}}$$

and by $\Gamma(A)$ the matrix $A \oplus A^2 \oplus \cdots \oplus A^n$. The element of $\Gamma(A)$ in its ith row and jth column will be denoted by $\Gamma_{ij}(A)$ (for $i, j \in N$).

It is known that the elements of A^k express the weights of heaviest paths consisting of k arcs between any two nodes of the complete n-node digraph the arcs of which are weighted by the elements of A, and that the Floyd–Warshall algorithm (see e.g. [9, 13]) applied to A gives as a result $\Gamma(A)$ in $O(n^3)$ operations.

Due to the previous discussion we can suppose without loss of generality that the matrix, the strong regularity of which is to be checked, is normal.

Theorem 4.6. *Let $\mathscr{G}$ be sparse and $A \in G(n, n)$ be normal. Then A is SR if and only if*

$$\Gamma_{ii}(\alpha \otimes \tilde{A}) \leq e \quad \text{for all } i \in N. \tag{4.5}$$

Proof. It follows all the lines of the proof of the same assertion for cyclic groups in [2] since that proof does not fully use the cyclicity of $\mathscr{G}$ but only the existence of the minimal positive element. □

Theorem 4.6 enables to compile an $O(n^3)$ method for checking SR in the sparse case ($O(n^3)$ operations for transforming A to a normal matrix $+ O(n^3)$ for computing $\Gamma(\alpha \otimes \tilde{A}) + O(n)$ for the final test of (4.5)). Moreover, as shown in [2], every column of $\Gamma(\alpha \otimes \tilde{A})$ is an instance of the vector b for which $|S(A, b)| = 1$.

Example 4.7. Consider the same matrix as in Example 4.5 but in $\mathscr{G}_2$. Here $\alpha = 1$ and hence

$$\Gamma(\alpha \otimes \tilde{C}) = \begin{pmatrix} -3 & -7 & -5 & -4 \\ 2 & -4 & 0 & 1 \\ 2 & -4 & 0 & 1 \\ 1 & -5 & -1 & 0 \end{pmatrix}$$

Thus C (and A) is SR and taking (say) $d = (-3, 2, 2, 1)^{\mathrm{T}}$ we have that the system $C \otimes x = d$ has unique solution. One can then easily find a vector b for which $A \otimes x = b$ has unique solution:

$$b = (-3 - 2, 2, 2, 1 - 2)^{\mathrm{T}} = (-5, 2, 2, -1)^{\mathrm{T}}.$$

Example 4.8. Consider the same matrix as in Examples 4.5 and 4.7 but in $\mathscr{G}_3$. Here $\alpha = 2$ and hence

$$\alpha \otimes \tilde{C} = \begin{pmatrix} -\infty & -6 & -4 & -6 \\ -2 & -\infty & 0 & 2 \\ 2 & -4 & -\infty & 2 \\ 2 & -4 & 0 & -\infty \end{pmatrix} = (d_{ij}).$$

Clearly $\Gamma_{33}(\alpha^{-1} \otimes \tilde{C}) \geq d_{34} + d_{43} = 2 > 0$ and thus A is not SR in $\mathscr{G}_3$.

In the following theorem we unify the results both for the dense and sparse LOCG.

Theorem 4.9. *Let $A \in G(n, n)$ be normal. Then A is SR if and only if*

$$\Gamma_{ii}(\alpha \otimes \tilde{A}) < \alpha \quad \textit{for all } i \in N. \tag{4.6}$$

Proof. It is easy to verify that the theorem statement holds for $n = 1$; therefore we suppose $n > 1$.

If $\mathscr{G}$ is sparse then (4.6) follows immediately from Theorem 4.6.

If $\mathscr{G}$ is dense then $\alpha = e$ and hence (4.6) sounds:

$$\Gamma_{ii}(\tilde{A}) < e \quad \text{for all } i \in N,$$

which means that $w_A(\sigma) < e$ for all $\sigma \in C_n$, $l(\sigma) \geq 2$. Since every $\pi \in P_n - \{\mathrm{id}\}$ can be decomposed to pairwise disjoint cycles at least one of which has length 2 or more, we derive from (1.1) that

$$w_A(\pi) < e$$

for every $\pi \in P_n - \{\mathrm{id}\}$ and hence $\mathrm{ap}(A) = \{\mathrm{id}\}$.

Conversely, if $\mathrm{ap}(A) = \{\mathrm{id}\}$ then $w_A(\sigma) < e$ for every $\sigma \in C_n$ with $l(\sigma) \geq 2$ because otherwise σ can be completed by cycles of length 1 to a permutation $\pi \neq \mathrm{id}$, $w_A(\pi) = e$, which would be then an other element of $\mathrm{ap}(A)$. The theorem statement now follows from Theorem 4.3. □

Corollary 4.10. *Let $\mathcal{G}$ be radicable and $A \in G(n, n)$ be normal. Then A is SR if and only if*

$$\lambda(\tilde{A}) < e. \tag{4.7}$$

Proof. By Proposition 1.1, $\mathcal{G}$ is dense and thus (4.6) sounds:

$$\Gamma_{ii}(\tilde{A}) < e \quad \text{for all } i \in N$$

which is equivalent to (4.7) because $a < e$ if and only if $\sqrt[k]{a} < e$ for all $a \in G$ and $k \in Z$, $k \geq 1$. □

5. Strong regularity in bottleneck algebra

In this section we summarize the results concerning the strong regularity of matrices in a structure where $\otimes$ stands for minimum and $\oplus$ for maximum. More precisely, we suppose that $(B, \leq)$ is a nonempty, linearly ordered set without maximum and minimum and we define binary operations $\oplus$, $\otimes$ on B as follows:

$$a \oplus b = \max(a, b), \tag{5.1}$$

$$a \otimes b = \min(a, b) \tag{5.2}$$

for all $a, b \in B$. The theory dealing with problems which are linear with respect to $\oplus$ and $\otimes$ as defined by (5.1) and (5.2) is called a *bottleneck algebra based on* $(B, \leq)$ or shortly, a bottleneck algebra (BA). Clearly, basic properties of the operations $\oplus$ and $\otimes$ follow immediately from the fact that the quadruple $(B, \leq, \oplus, \otimes)$ is an infinite distributive lattice and we will use them without an explicit formulation.

Investigation of the strong regularity of matrices in BA is not only theoretically motivated but as it turns out it enables to formulate some computational complexity results.

We extend $\oplus$, $\otimes$ and $\leq$ on matrices from $B(m, n)$ in the same way as in max-algebra and hence the equation

$$A \otimes x = b$$

is a more formidable expression for the system of equations

$$\max_{j \in N} \min(a_{ij}, x_j) = b_i, \quad i \in M. \tag{5.3}$$

Some practical problems can be conveniently expressed using the concepts of BA. Consider, for example, the following transmittance problem. If the transportation route consists of two parts UV and VW (say V is a transshipment point), then the total route transmittance is equal to the minimum of the transmittances of UV and VW. Similarly, in a transportation network with $U_1, \ldots, U_m$ as dispatching points, $V_1, \ldots, V_l$ as transship points, and $W_1, \ldots, W_n$ as destination points, denoting the transmittances of U_iV_k and V_kW_j by a_{ik} and b_{kj}, respectively ($i = 1, \ldots, m$; $k = 1, \ldots, l$; $j = 1, \ldots, n$) we have that the total transportation transmittance between U_i and W_j is equal to

$$c_{ij} = \max_{k=1,\ldots,l} \min(a_{ik}, b_{kj})$$

for all $i \in M$, $j \in N$. This relation can be written as $C = A \otimes B$ in BA based on the set of reals with conventional ordering, where A, B, C denote the matrices $(a_{ik}), (b_{kj}), (c_{ij})$.

As another example consider the permanent of $A = (a_{ij}) \in B(n, n)$ in the same BA:

$$\mathrm{per}(A) = \sum_{\pi \in P_n}^{\oplus} \prod_{i \in N}^{\otimes} a_{i,\pi(i)} = \max_{\pi \in P_n} \min_{i \in N} a_{i,\pi(i)}.$$

Hence to compute per(A) means now to find a weighted matching in a complete bipartite graph with the maximal possible lowest score. This corresponds to those situations where the overall performance of a team is measured by the worst performance of an individual member, e.g., if each of n workers performs one of n tasks on an assembly line, then the speed of the line equals the speed of the slowest worker. The task of finding such an assignment is called *bottleneck assignment problem* (BAP). An $O(n^{2.5} \log n)$ algorithm for solving this problem follows immediately from the $O(n^{2.5})$ algorithm for finding maximum matching in a bipartite graph [13] and using the binary search. An $O(n^{2.5}\sqrt{\log n})$ algorithm for solving BAP is also known [11].

As a consequence of the results which we now present, the BAP can be solved in only $O(n^2 \log n)$ operations whenever the optimal permutation is unique.

Formal similarity of the systems of equations linear with respect to $\oplus$, $\otimes$ in both max-algebra and bottleneck algebra leads to a question whether the same or similar results as in Sections 3 and 4 can now be proved. For this purpose the notation $S(A, b)$ and $T(A)$ as well as the relation $<$ and density are introduced in the same way as in max-algebra. Note that the system of linear equations in bottleneck algebra (5.3) can be solved by an $O(mn)$ algorithm developed in [15].

Consider the system $A \otimes x = b$ for

$$A = \begin{pmatrix} 3 & 0 \\ 2 & 3 \\ 0 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

in BA based on $(Z, \leq)$ and suppose that $x = (x_1, x_2)^{\mathrm{T}} \in S(A, b)$. Then it follows from the first equation that $x_1 = 2$, and the inequalities $x_2 \leq 2$ and $x_2 \geq 1$ can be derived from the second and third equation, respectively. Hence,

$$S(A, b) = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\},$$

and thus an analogue of Theorem 2.5 does not hold in BA. However, the following was proved in [1].

Theorem 5.1. *Let* $\leq$ *be dense on* B. *Then*

$$\{0, \infty\} \subseteq T(A) \subseteq \{0, 1, \infty\} \quad \textit{for all } A \in B(m, n).$$

This result motivates us to define *strong linear independence* (SLI), *strong regularity* (SR) and *rank* formally in the same way as in max-algebra.

Theorem 5.2. *A sufficient condition for* A *to have SLI columns is that* $r(A) = n$. *Moreover, if* $\leq$ *is dense on* B, *then this condition is also necessary.*

Proof. Can be found in [1]. □

As it was presented in Section 4, the problem of SR in max-algebra can be solved by an $O(n^3)$ algorithm whereas the problem of SLI of columns of a rectangular matrix remains still open. On the other hand, in BA on a dense set both problems are solvable simultaneously. To show this we define the following concepts.

Matrix $A = (a_{ij}) \in B(m, n)$, $n \geq 2$ is said to be *trapezoidal*, if

$$a_{kk} > a_{ij} \tag{5.4}$$

for all $k = 1, 2, \ldots, \min(m, n)$; $i = 1, \ldots, k$; $j = i + 1, \ldots, n$.

Matrices $A, C \in B(m, n)$ are called similar (notation $A \approx C$), if one of them can be obtained from the other by permuting its rows and columns. Clearly, $\approx$ constitutes an equivalence relation on $B(m, n)$.

Proposition 5.3. *If* A *has SLI columns and* $A \approx C$ *then also* C *has SLI columns.*

Proof. Follows immediately from the definitions. □

Theorem 5.4. *Let $A \in B(n, n)$. Then a necessary condition for A to be SR is the existence of a trapezoidal matrix similar to A. Moreover, if $\leq$ is dense on B, then this condition is also sufficient.*

Proof. Can be found in [5]. $\square$

Corollary 5.5. *Let $\leq$ be dense on B and $A \in B(m, n)$, $m \geq n$. Then A has SLI columns if and only if $A \approx T$, T trapezoidal.*

Proof. The statement follows immediately from Theorems 5.4 and 5.2. $\square$

If $\leq$ is dense on B, then the problem of checking SLI of the columns of a matrix over B is turned by Corollary 5.5 to the question whether this matrix is similar to a trapezoidal one. In order to derive an algorithm for checking this property realize that for every trapezoidal matrix $A = (a_{ij}) \in B(m, n)$ we have

$$a_{11} > a_{1j} \quad \text{for all } j \in N.$$

Hence a row of an arbitrary matrix can be considered as a candidate for being the first row (up to the order of its elements) of a similar trapezoidal matrix only if it has unique maximal element. For convenience, we say that a row of a matrix is *regular*, if it has unique maximal element. In general not every regular row can become the first row of a similar trapezoidal matrix. The precise specification is given in the theorem below. At first we denote by $d(A)$ the least diagonal element of A and by m_i and m_i' we denote the greatest and second greatest element of the regular row i.

Theorem 5.6. *Let $d \in B$ and $A = (a_{ij}) \in B(m, n)$ be similar to a trapezoidal matrix T with $d(T) > d$. Let the kth row of A be regular and satisfy*

(1) $m_k > d$,

(2) $m_k' = \min\{m_i'$: *ith row is regular and* $m_i > d\}$.

Then A is similar to a trapezoidal matrix T', $d(T') > d$ in which its first row is the kth row of A (up to the order of its elements).

Proof. Can be found in [1]. $\square$

Theorem 5.6 enables to compile an algorithm for checking SLI. It is based on the fact that a necessary condition for the columns of A to be SLI is the existence of at least one regular row in A. Due to Theorem 5.6 we choose an element, say a_{kl}, which is unique maximal in its row and for which the second greatest, say $a_{kl'}$ (where $a_{kl'} < a_{kl}$), is as small as possible and we proceed by considering the same for the submatrix $A(k, l)$ arising from A by deleting its kth row and lth column. It follows again from Theorem 5.6 that in some row of $A(k, l)$ the unique maximal element greater than $a_{kl'}$ exists whenever the columns of A are SLI. The procedure continues in this way until the whole trapezoidal matrix is found or at some step it is not possible to continue because no row exists with unique maximal element greater than all known superdiagonal elements. Clearly, the algorithm stops whenever it finds all rows of the trapezoidal $n \times n$ submatrix.

Trapezoidal algorithm.

Input: $A = (a_{ij}) \in B(m, n)$ with $m \geq n \geq 2$.

Output: "yes" for the variable named answer and a trapezoidal submatrix $T = (t_{ij}) \in B(n, n)$, if A has SLI columns; "no" for the variable answer otherwise.

1 $d := \min\{a_{ij}: i \in M, j \in N\}$, $s := 1$, answer := "no";

2 $R := \{i \in M$: row i of A is regular and $m_i > d\}$;
if $R = \emptyset$ **then** stop;

3 Let k be an arbitrary index satisfying $m'_k = \min\{m'_i: i \in R\}$ and let l, l' be defined by the formulas $a_{kl} = \max\{a_{kj}: j \in N\}$, $a_{kl'} = \max\{a_{kj}: j \in N - \{l\}\}$;
(comment: m'_k and l' are undefined for $s = n$)

4 $\pi(s) := k$, $\tau(s) := l$;
if $s = n$ **then go to** 5;
$M := M - \{k\}$, $N := N - \{l\}$;
$d := d \oplus a_{kl'}$, $A := A(k, l)$;
$s := s + 1$, **go to** 2;

5 answer := "yes";
$t_{ij} := a_{\pi(i), \tau(j)}$ for all $i, j = 1, \ldots, n$;
stop.

Theorem 5.7. *The trapezoidal algorithm is correct and terminates after using at most* $O(mn^2)$ *operations.*

Proof. Correctness follows from Theorem 5.6.

The number of loops 2–4 is at most n and in every loop $O(mn)$ operations suffice to find the necessary regular rows. Steps 1 and 5 are $O(n^2)$. $\square$

Example 5.8. Consider the matrix

$$A = \begin{pmatrix} 5 & 5 & 3 & 1 \\ 2 & 0 & 3 & 4 \\ 1 & 0 & 5 & 0 \\ 4 & 3 & 2 & 1 \\ 2 & 1 & 1 & 5 \\ 3 & 4 & 0 & 4 \end{pmatrix}$$

in an arbitrary dense subset of the set of reals containing all entries of A.

The trapezoidal algorithm gives successively:

$$s = 1, d = 0, R = \{2, 3, 4, 5\}, k = 3 = \pi(1), m'_3 = 1, l = 3;$$

$$s = 2, d = 1, R = \{2, 4, 5\}, k = 2 = \pi(2), m'_2 = 2, l = 4;$$

$$s = 3, d = 2, R = \{4, 6\}, k = 4 = \pi(3), m'_4 = 3, l = 1;$$

$$s = 4, d = 3, R = \{1, 6\}, k = 1 = \pi(4), l = 2.$$

Hence we have

$$T = \begin{pmatrix} 5 & 0 & 1 & 0 \\ 3 & 4 & 2 & 0 \\ 2 & 1 & 4 & 3 \\ 3 & 1 & 5 & 5 \end{pmatrix}.$$

Example 5.9. Consider the matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 2 & 4 \end{pmatrix}$$

in the same bottleneck algebra as in Example 5.8.

Hence we get

$$s = 1, d = 0, R = \{1, 4\}, k = 1 = \pi(1), m'_1 = 2, l = 3;$$

$$s = 2, d = 2, R = \emptyset, \text{answer} = \text{"no"}, \text{stop}.$$

In [1] a more sophisticated version of this algorithm is presented and it was shown in the same paper that the pre-ordering of the rows of A leads to a reduction of the computational complexity to $O(mn \log n)$.

In connection with the corresponding results in max-algebra a natural question arises, namely whether there is any relation between SR and strong permanent in BA too. To answer this question consider at first the permanent of a trapezoidal matrix $A = (a_{ij}) \in B(n, n)$. Let $a_{qq} = d(A)$ and $\pi \in P_n - \{\mathrm{id}\}$. If $\pi(i) > i$ for some $i \in Q = \{1, 2, \dots, q\}$ then $a_{i,\pi(i)} < a_{qq}$ and hence $w_A(\pi) < w_A(\mathrm{id})$. If $\pi(i) \leq i$ for all $i \in Q$ then $\pi(i) = i$ for all $i \in Q$ and hence $w_A(\pi) \leq a_{qq} = w_A(\mathrm{id})$. We proved

Proposition 5.10. *Let $A = (a_{ij}) \in B(n, n)$ be trapezoidal. Then $\mathrm{id} \in \mathrm{ap}(A)$ (and hence $\mathrm{per}(A) = \min_{i \in N} a_{ii}$).*

The connection between SR and strong permanent is in BA not as strong as in max-algebra even if the ordering is dense, e.g. the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

in BA based on the set of reals is trapezoidal (and hence SR) but it does not have a strong permanent since $\mathrm{ap}(A) = \{\mathrm{id}, (1)(23)\}$. However, we have

Theorem 5.11. *Let $A \in B(n, n)$. A sufficient condition for A to be equivalent to a trapezoidal matrix is that A have a strong permanent. This condition is also necessary for matrices of order $n = 2$.*

Proof. Can be found in [1]. □

Corollary 5.12. *Let $\leq$ be dense on B and $A \in B(n, n)$. If A has a strong permanent then A is SR.*

Proof. The statement follows from Theorems 5.4 and 5.11. □

Note that the statement of Corollary 5.12 does not remain true after omitting the assumption of density as it is shown by the matrix

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

in BA on the set of integers. Here A has a strong permanent (and is trapezoidal) but it is not SR.

To see that A is not SR suppose that $(\bar{x}_1, \bar{x}_2)^{\mathrm{T}} \in S(A, b)$ for some $b = (b_1, b_2)^{\mathrm{T}}$; if $\bar{x}_1 > 0$ then $\bar{x}_1 \geq 1$ and $(x_1, \bar{x}_2)^{\mathrm{T}} \in S(A, b)$ for all $x_1 \geq \bar{x}_1$. By the same argument for $\bar{x}_2$ we can assume that $\bar{x}_1 \leq 0$, $\bar{x}_2 \leq 0$ and hence the left-hand side value in each equation is $\max(\bar{x}_1, \bar{x}_2)$, yielding $b_1 = b_2$. Therefore either $(x_1, \bar{x}_2)^{\mathrm{T}} \in S(A, b)$ for all $x_1 \leq \bar{x}_1$ or $(\bar{x}_1, x_2)^{\mathrm{T}} \in S(A, b)$ for all $x_2 \leq \bar{x}_2$.

Corollary 5.13. *The bottleneck assignment problem can be solved using no more than $O(n^2 \log n)$ operations for every matrix of order n similar to a trapezoidal matrix. In particular, this is true for all matrices with strong permanent.*

Proof. It follows from Proposition 5.10, Theorem 5.11 and from the $O(n^2 \log n)$ version of the trapezoidal algorithm in [1]. □

As we have just seen, the negative answer in checking the uniqueness of the optimal solution to BAP for the matrix A is not helpful in deciding whether A is SR. However the question whether BAP has one or more optimal solutions can be interesting itself and as it now turns out, it can be answered by less operations than it is necessary to use for finding the optimal solution in general.

Theorem 5.14. *Let $A = (a_{ij}) \in B(n, n)$ be a matrix found by the trapezoidal algorithm. A necessary and sufficient condition for A to have a strong permanent is that*

$$d(A) > a_{ij} \quad \text{for all } i, j, \in N,\ i < j. \tag{5.5}$$

Proof. If $d(A) > a_{ij}$ then $w_A(\pi) < d(A) = w_A(\mathrm{id})$ for every $\pi \in P_n - \{\mathrm{id}\}$ because $\pi(i) > i$ for at least one $i \in N$.

Suppose now that $d(A) \leq a_{rs}$ for some $r, s \in N$, $r < s$. Without loss of generality we may assume

$$a_{rs} = \max\{a_{rj}: r < j\}$$

and clearly $a_{rs} < a_{rr}$. Let $C = (c_{ij})$ be a matrix arising from A by deleting its first $r - 1$ rows and first $r - 1$ columns. Then a_{rs} lies in its first row.

Consider now an arbitrary row of C, say the kth. If it is not regular then $c_{kk} \leq c_{kt}$ for some $t \neq k$ and clearly $c_{kk} > a_{rs}$. If it is regular then the existence of an index $t \neq k$ satisfying $a_{rs} \leq c_{kt}$ is a consequence of the work of the trapezoidal algorithm which in Step 3 chooses a regular row of the remaining matrix with the least second greatest element. Hence in every row of C at least one nondiagonal element greater than or equal $d(A)$ exists. Thus in the set $\{r, r + 1, \ldots, n\}$ there exists a cycle σ, $l(\sigma) \geq 2$, with $w_A(\sigma) \geq d(A)$ because this situation corresponds to a digraph without loops in which a leaving arc from each node exists. The cycle σ completed by loops to a permutation π yields that $w_A(\pi) \geq d(A) = \mathrm{per}(A)$ and thus $\pi \in \mathrm{ap}(A)$, $\pi \neq \mathrm{id}$. □

Corollary. 5.15. *For every $A \in B(n, n)$ it is possible to check in $O(n^2 \log n)$ operations whether the bottleneck assignment problem for A has a unique optimal solution.*

Proof. Apply the trapezoidal algorithm on A. If it terminates by "no" then by Theorem 5.11 the optimal solution to BAP for A is not unique. If it terminates by finding a trapezoidal matrix similar to A then it suffices to check the condition (5.5) which can be done in $O(n^2)$ steps. □

We summarize the main results in Tables 2 and 3. Here the letters N and S stand for the words "necessary" and "sufficient", respectively; T means a trapezoidal matrix.

In constrast to max-algebra, in BA the ordering is not necessarily discrete (i.e., every element is a predecessor and a successor of some other element), if it is not dense. This has motivated a special research of discrete BA in [6, 7]. In the latter work an $O(n^2 \log n)$ algorithm for checking SR of matrices is proved and also the eigenproblem in BA is studied.

Table 2

	Max-algebra		Bottleneck algebra	
	General	Dense	General	Dense
Full column rank is for SLI	N, S	N, S	S	N, S
SP is for SR	N	N, S	neither	S
Similarity to T is for SR	–	–	N	N, S
Efficient algorithm for checking SLI	?	?	?	$O(mn \log n)$
Efficient algorithm for checking SR	$O(n^3)$	$O(n^3)$	?	$O(n^2 \log n)$

Table 3

	Max-algebra	Bottleneck algebra
Algorithm for per(A)	$O(n^3)$	$O(n^{2.5}\sqrt{\log n})$
Algorithm for per(A) if strong	$O(n^3)$	$O(n^2 \log n)$
Algorithm for checking SP	$O(n^3) + O(n^2)$	$O(n^2 \log n) + O(n^2)$

As indicated in Tables 2 and 3 some questions remain still open as a challenge for further research. We would like to draw the attention to three particular questions:

(1) Is it possible in max-algebra to avoid checking SR of all $\binom{m}{n}$ square submatrices of the matrix A of order n to check SLI of its columns?

(2) Is it possible to develop a faster algorithm for solving the linear assignment problem for matrices with strong permanent than for general matrices (as it is in the case of the bottleneck assignment problem)?

(3) Is it possible to check the strong permanent in max-algebra by a faster algorithm than the algorithm for solving AP (as it is in BA)?

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