ELSEVIER

Discrete Applied Mathematics 57 (1995) 121–132

DISCRETE APPLIED MATHEMATICS

Regularity of matrices in min-algebra and its time- complexity

P. Butkovič

School of Mathematics and Statistics, The University of Birmingham, Edgbaston, Birmingham, B15 2TT, UK

Received 30 August 1992; revised 8 November 1993

Abstract

Let $\mathscr{G} = (G, \otimes, \leqslant)$ be a linearly ordered, commutative group and $\oplus$ be defined by $a \oplus b = \min(a, b)$ for all $a, b \in G$. Extend $\oplus$, $\otimes$ to matrices and vectors as in conventional linear algebra.

An $n \times n$ matrix A with columns $A_1, \ldots, A_n$ is called regular if

$$\sum_{j \in U}^{\oplus} \lambda_j \otimes A_j = \sum_{j \in V}^{\oplus} \lambda_j \otimes A_j$$

does not hold for any $\lambda_1, \ldots, \lambda_n \in G$, $\emptyset \neq U, V \subseteq \{1, 2, \ldots, n\}$, $U \cap V = \emptyset$.

We show that the problem of checking regularity is polynomially equivalent to the even cycle problem.

We also present two other types of regularity which can be checked in $O(n^3)$ operations.

0. Introduction

A wide class of problems in different areas of scientific research, like graph theory, automata theory, scheduling theory, communication networks, etc. can be expressed in an attractive formulation language by setting up an algebra of, say, real numbers in which the operations of multiplication and addition are replaced by arithmetical addition and selection of the greater of the two numbers, respectively. Monograph [3] can be used as a comprehensive guide in this field. Specifically, a significant effort was developed to build up a theory similar to that in linear algebra, i.e. to study systems of linear equations, eigenvalue problems, independence, rank, regularity, dimension, etc. As it turned out there is only a thin barrier separating these concepts and combinatorial properties of matrices. The aim of the present paper is to study the time-complexity of the problem of checking regularity of matrices. Since addition is now not a group operation, there are several non-equivalent ways of defining the regularity. We investigate three different such definitions. Two of these can be checked

SSDI 0166-218X(94)00099-9

efficiently but the third, which plays a central role in minimal-dimensional realisation of the discrete event dynamic systems (see [4]), is shown to be polynomially equivalent to the problem of the existence of an even cycle in digraphs.

1. Notation and definitions

Let $\mathscr{G} = (G, \otimes, \leqslant)$ be a non-trivial linearly ordered, commutative group (LOCG) with neutral element e and $\oplus$ be a binary operation on G given by the formula

$$a \oplus b = \min(a, b) \quad \text{for all } a, b \in G.$$

Note that $\mathscr{G}$ is infinite. By $\mathscr{G}_0$ we denote $(R, +, \leqslant)$, i.e. the additive group of reals with conventional ordering.

Extend $\oplus, \otimes$ to matrices and vectors in the same way as in linear algebra. Concepts and theory similar to those in linear algebra can be developed for $\oplus, \otimes$, see [3]. We shall refer to this as min-algebra.

Throughout the paper we assume that all matrices are $n \times n$ ($n \geqslant 1$ is an integer) and their entries are from G.

We shall denote $\{1, 2, \ldots, n\}$ by N and the set of all permutations of N by P_n. The symbol $|X|$ stands for the number of elements of the set X.

Cyclic permutations will be written in the form $\pi = (i_1 i_2 \ldots i_p)$ where $N' = \{i_1, \ldots, i_p\}$ is some subset of N. The corresponding cycle in the digraph with node set N will be denoted by $(i_1, i_2, \ldots, i_p)$. It is well known that

$$\operatorname{sgn}(\pi) = (-1)^{p-1}.$$

Hence, a cyclic permutation of N' is odd if and only if $|N'|$ is even.

Lemma 1.1. *If the permutation π is odd then at least one permutation in the decomposition of π to cyclic permutations is odd, i.e. it is a cyclic permutation of a subset of N of an even size.*

Proof. Trivial. $\square$

Let us denote

$$P_n^+ = \{\pi \in P_n;\ \pi \text{ even}\},$$

$$P_n^- = \{\pi \in P_n;\ \pi \text{ odd}\},$$

$$w(A, \pi) = a_{1,\pi(1)} \otimes a_{2,\pi(2)} \otimes \cdots \otimes a_{n,\pi(n)} \quad \text{for } \pi \in P_n.$$

The task of finding the permanent of A in min-algebra is

$$\operatorname{miper}(A) = \sum_{\pi \in P_n}^{\oplus} w(A, \pi).$$

In $\mathscr{G}_0$ this is obviously equivalent to finding

$$\min_{\pi \in P_n} (a_{1,\pi(1)} + \cdots + a_{n,\pi(n)}),$$

which is well known as the assignment problem for A. Motivated by this, we denote

$$\mathrm{ap}(A) = \{\pi \in P_n;\ w(A,\pi) = \mathrm{miper}(A)\},$$

$$\mathrm{ap}^+(A) = \mathrm{ap}(A) \cap P_n^+,$$

$$\mathrm{ap}^-(A) = \mathrm{ap}(A) \cap P_n^-.$$

Clearly, $\mathrm{ap}^+(A) \cup \mathrm{ap}^-(A) = \mathrm{ap}(A) \neq \emptyset$.

Matrices A and B are said to be *equivalent* ($A \sim B$) if one can be obtained from the other by

(a) permuting the rows and columns,

(b) multiplying of rows and columns by elements of G.

Clearly, $\sim$ constitutes an equivalence relation.

Proof of the following two lemmas is easy.

Lemma 1.2. *If the matrix A is obtained from B by an exchange of two rows (or columns) then there exists a one-to-one mapping between $\mathrm{ap}^+(A)$ and $\mathrm{ap}^-(B)$ as well as between $\mathrm{ap}^-(A)$ and $\mathrm{ap}^+(B)$. Consequently, $|\mathrm{ap}^+(A)| = |\mathrm{ap}^-(B)|$ and $|\mathrm{ap}^-(A)| = |\mathrm{ap}^+(B)|$.*

Lemma 1.3. *If the matrix A is obtained from B by multiplying the rows (or columns) then $\mathrm{ap}^+(A) = \mathrm{ap}^+(B)$ and $\mathrm{ap}^-(A) = \mathrm{ap}^-(B)$.*

As a corollary we have the following lemma.

Lemma 1.4. *If $A \sim B$ then either*

$$|\mathrm{ap}^+(A)| = |\mathrm{ap}^+(B)| \quad \text{and} \quad |\mathrm{ap}^-(A)| = |\mathrm{ap}^-(B)|,$$

or

$$|\mathrm{ap}^+(A)| = |\mathrm{ap}^-(B)| \quad \text{and} \quad |\mathrm{ap}^-(A)| = |\mathrm{ap}^+(B)|.$$

In any case $|\mathrm{ap}(A)| = |\mathrm{ap}(B)|$.

Matrix $A = (a_{ij})$ is called *normal* in $\mathscr{G}$ if

$$a_{ij} \geqslant a_{ii} = e \quad \text{for all } i,j \in N.$$

Clearly, $\mathrm{id} \in \mathrm{ap}(A)$ if A is normal (id stands for identical permutation).

The Hungarian method [7] for solving the assignment problem for the matrix A enables us to find in $O(n^3)$ operations a normal matrix $B \sim A$.

Let us denote the columns of A by $A_1, \ldots, A_n$. They will be called *linearly dependent* in $\mathscr{G}$ if

$$\sum_{j \in U}^{\oplus} \lambda_j \otimes A_j = \sum_{j \in V}^{\oplus} \lambda_j \otimes A_j \tag{1.1}$$

holds for some $\lambda_1, \ldots, \lambda_n \in G$, $U, V \neq \emptyset$, $U \cap V = \emptyset$, $U \cup V = N$. (Note that $U \cup V = N$ can be replaced equivalently by $U \cup V \subseteq N$.) Columns of A are called *linearly independent* in $\mathscr{G}$ if they are not linearly dependent in $\mathscr{G}$. Matrix A is called *regular* in $\mathscr{G}$ if its columns are linearly independent.

In what follows we omit "in $\mathscr{G}$" when no confusion can arise.

Lemma 1.5. *If $A \sim B$ then A is regular if and only if B is regular.*

Proof. Trivial. $\square$

2. Criterion of regularity

Theorem 2.1. (a) *A is regular if and only if*

$$\text{either } \operatorname{ap}^+(A) = \emptyset \text{ or } \operatorname{ap}^-(A) = \emptyset. \tag{2.1}$$

(b) *Moreover, if $\pi \in \operatorname{ap}^+(A)$, $\sigma \in \operatorname{ap}^-(A)$ are known then the linear dependence of the form* (1.1) *can be found in* $O(n^2)$ *operations.*

Proof. A proof of (a) was partly given in [5]. We modify those ideas to give a complete proof and to prove at the same time the computational complexity bound in (b).

First we show that if A is not regular then $\operatorname{ap}^+(A) \neq \emptyset$ and $\operatorname{ap}^-(A) \neq \emptyset$. Due to Lemma 1.4 it suffices to prove this property for any matrix equivalent to A.

Permute the columns of the matrix

$$(\lambda_1 \otimes A_1, \ldots, \lambda_n \otimes A_n)$$

in such a way that the left-hand side of (1.1) contains only its first (say k) columns and denote this matrix by $\bar{A} = (\bar{A}_1, \ldots, \bar{A}_n) = (\bar{a}_{ij})$.

$$\text{Then } \sum_{j \leqslant k}^{\oplus} \bar{A}_j = \sum_{j > k}^{\oplus} \bar{A}_j = (c_1, c_2, \ldots, c_n)^{\mathrm{T}} \tag{2.2}$$

for some $c_1, \ldots, c_n \in G$. Let $\hat{A} = (\hat{a}_{ij})$ be defined by

$$\hat{a}_{ij} = c_i^{-1} \otimes \bar{a}_{ij} \quad \text{for all } i, j \in N$$

and $B = (b_{ij})$ be obtained from $\hat{A}$ by an arbitrary permutation of the rows such that $\mathrm{id} \in \operatorname{ap}(B)$. Then B has the following properties:

$$b_{ij} \geqslant e \quad \text{for all } i,j \in N, \tag{2.3}$$

$$(\forall i)(\exists j_1 \leqslant k)(\exists j_2 > k) b_{ij_1} = e = b_{ij_2}. \tag{2.4}$$

(Note that B may not be normal.)

Now construct a sequence of indices $i_1, i_2, \ldots$ as follows: $i_1 = 1$; if i_r is already defined and $i_r \leqslant k$ then i_{r+1} is arbitrary $j > k$ such that $b_{i_r j} = e$ and if $i_r > k$ then $i_{r+1} = j \leqslant k$ such that $b_{i_r j} = e$.

By finiteness, $i_r = i_s$ for some r, s and $s < r$. Let r, s be the first such indices and denote

$$L = \{i_s, i_{s+1}, \ldots, i_{r-1}\}.$$

Clearly, if $i_s \leqslant k$ then $i_{s+1} > k$, $i_{s+2} \leqslant k$, $i_{s+3} > k, \ldots$ and hence (using a similar reason if $i_s > k$) $|L|$ is even.

Set

$$\pi(i_t) = i_{t+1} \quad \text{for } t = s,\ s+1, \ldots, r-1,$$

$$\pi(i) = i \quad \text{for } i \in N \backslash L.$$

Hence

$$\begin{aligned} w_B(\pi) &= \prod_{i \notin L}{}^{\oplus} b_{ii} \otimes \prod_{i \in L}{}^{\oplus} b_{i,\pi(i)} \\ &= \prod_{i \notin L}{}^{\oplus} b_{ii} \otimes \prod_{i \in L}{}^{\oplus} e \quad \text{(by (2.4))} \\ &\leqslant \prod_{i \in N}{}^{\oplus} b_{ii} \\ &= w_B(\mathrm{id}) \\ &\leqslant w_B(\pi) \quad \text{(by optimality of id).} \end{aligned}$$

Therefore, $\pi \in \mathrm{ap}(B)$ and denoting $\pi' = \pi | L$ we have $\mathrm{sgn}(\pi) = \mathrm{sgn}(\pi') = -1$ since $|L|$ is even and π' is a cyclic permutation of L. Hence $\mathrm{id} \in \mathrm{ap}^+(B)$ and $\pi \in \mathrm{ap}^-(B)$.

Suppose now that $\pi \in \mathrm{ap}^+(A)$, $\sigma \in \mathrm{ap}^-(A)$ are known. For an optimal primal solution (say π) the corresponding optimal dual solution can be found in $\mathrm{O}(n)$ time. Hence, we can find in $\mathrm{O}(n)$ operations $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in G$ such that in

$$A' = (a'_{ij}) = (\alpha_i \otimes a_{ij} \otimes \beta_j)$$

all elements are greater than or equal to e and $\mathrm{miper}(A') = e$. Exchange successively column i and column $\pi(i)$ of A' for $i = 1, 2, \ldots, n$ (this needs $\mathrm{O}(n^2)$ time). Then for the arising matrix A'' we have $\mathrm{id} \in \mathrm{ap}(A'')$ and a permutation $\sigma' \in \mathrm{ap}^-(A'')$ can be derived from σ in $\mathrm{O}(n)$ time (in fact $\sigma' = \sigma \circ \pi^{-1}$). The odd cyclic permutation in the decomposition of σ' (see Lemma 1.1) can be found in $\mathrm{O}(n)$ time. By a simultaneous permutation (say ρ) of the rows and columns of A'' (in $\mathrm{O}(n^2)$ time) it can be achieved that this cycle is $(12 \ldots k)$ for some even number $k \geqslant 2$. The arising normal matrix B is of the form as shown in Fig. 1.

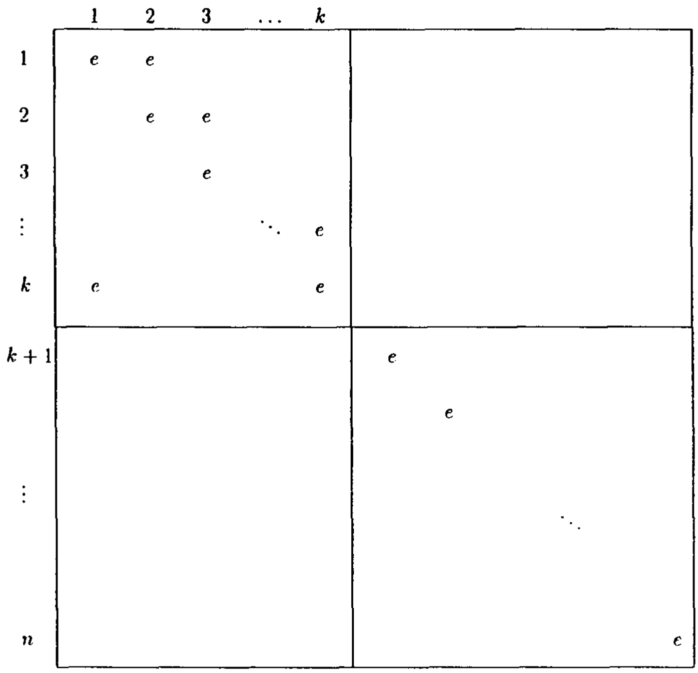

Fig. 1.

Assign the indices $1, 3, \ldots, k-1$ to U, $2, 4, \ldots, k$ to V and set $\lambda_1 = \lambda_2 = \cdots = \lambda_k = \mathrm{e}$.

If $k = n$ then (1.1) is satisfied for B. Let $k < n$. As we shall see all $\lambda_{k+1}, \ldots, \lambda_n$ will be set to non-negative values, therefore (1.1) will hold for the first k rows of B independently of the assignment of the columns $k+1, \ldots, n$ to U and V. To ensure equality in the rows $k+1, \ldots, n$ we compute first

$$L_i = \min_{j \in U} b_{ij} \otimes \lambda_j, \tag{2.5}$$

$$R_i = \min_{j \in V} b_{ij} \otimes \lambda_j \tag{2.6}$$

(in $\mathrm{O}(n^2)$ operations).

Let

$$I = \{i > k;\ L_i \neq R_i\} \tag{2.7}$$

and $s \in I$ be an (arbitrary) index satisfying

$$L_s \oplus R_s = \min_{i \in I} (L_i \oplus R_i). \tag{2.8}$$

Set $V' = V \cup \{s\}$, $U' = U$ if $L_s < R_s$ and set $V' = V$, $U' = U \cup \{s\}$ if $L_s > R_s$. In both cases take $\lambda_s = L_s \oplus R_s$. Denote

$$L'_i = \min_{j \in U'} b_{ij} \otimes \lambda_j,$$

$$R'_i = \min_{j \in V'} b_{ij} \otimes \lambda_j.$$

Since $b_{ss} \otimes \lambda_s = e \otimes \lambda_s = L_s \oplus R_s$ we then get $L'_s = R'_s$. At the same time

$$b_{is} \otimes \lambda_s \geqslant e \otimes \lambda_s = L_s \oplus R_s \tag{2.9}$$

holds for all $i > k$ and therefore $L_i = R_i \leqslant L_s \oplus R_s$ implies $L'_i = R'_i$. Let q be defined by

$$L'_q \oplus R'_q = \min_{i \in I'} (L'_i \oplus R'_i), \quad q \in I', \ I' = \{i > k;\ L'_i \neq R'_i\}.$$

Then,

$$L'_q \oplus R'_q \geqslant L_s \oplus R_s, \tag{2.10}$$

because either $L'_q \oplus R'_q = b_{qs} \otimes \lambda_s$ and then (2.10) follows from (2.9), or $L'_q \oplus R'_q < b_{qs} \otimes \lambda_s$, implying $q \in I$ and thus (2.10) follows from (2.8). This also shows that if we continue in this way after resetting $U' \to U$, $V' \to V$, $L'_i \to L_i$, $R'_i \to R_i$, $I' \to I$, $q \to s$ then the process will be monotone ($L_s \oplus R_s$ will be non-decreasing) and in the row in which the equality was already achieved this will never be spoiled. Hence, after at most $n - k$ repetitions $I = \emptyset$. If $U \cup V = N$ then (1.1) is completely satisfied, otherwise for all $j \in N \backslash V \cup U$ we set

$$\lambda_j = \max_{i > k} L_i$$

and assign j to V or U arbitrarily.

Obviously, all computations for assigning j and setting λ_j are $O(n)$, hence, the overall performance for finding the linear dependence for B is $O(n^2)$. It remains to apply ρ^{-1} and π to the set of column indices and to $\lambda_1, \ldots, \lambda_n$ (in $O(n)$ time) in order to find the decomposition (1.1) for A'.

To obtain the same for A we finally multiply $\lambda_1, \ldots, \lambda_n$ by $\beta_1^{-1}, \ldots, \beta_n^{-1}$. This completes the proof of both parts of Theorem 2.1. □

We illustrate the algorithm presented in the proof of Theorem 2.1 on the following example in $\mathscr{G}_0$ (points indicate arbitrary non-negative reals and the development of L_i, R_i ($i = 5, 6, 7, 8, 9$) is expressed for convenience to the left of the matrix (see Fig. 2). Note that here we have $k = 4$, $n = 9$. Applying the method, we obtain successively:

$$I = \{5, 7, 9\}, \quad s = 7, \quad V := V \cup \{7\}, \quad \lambda_7 = 1,$$

$$I = \{5, 9\}, \quad s = 5, \quad U := U \cup \{5\}, \quad \lambda_5 = 2,$$

$$I = \{6, 9\}, \quad s = 6, \quad V := V \cup \{6\}, \quad \lambda_6 = 3,$$

$$I = \emptyset, \quad \lambda_8 = \lambda_9 = 4, \quad U := U \cup \{8, 9\} \quad \text{(say)}.$$

Hence, we have found $U = \{1, 3, 5, 8, 9\}$, $V = \{2, 4, 6, 7\}$.

L_i	R_i		U	V	U	V					
0	0		0	0	.	.	.	.	.	.	.
0	0		.	0	0	.	.	.	.	.	.
0	0		.	.	0	0	.	.	.	.	.
0	0		0	.	.	0	0	.	.	.	.
$\cancel{4}\ 2$	$\cancel{3}\ 2$		8	3	4	4	0	1	1	.	.
$\cancel{5}\ 3$	$\cancel{5}\ 3$		5	8	6	5	1	0	6	.	.
1	$\cancel{3}\ 1$		1	3	7	4	0	2	0	.	.
0	0		0	0	3	5	2	4	3	0	.
$\cancel{6}\ 4$	$\cancel{7}\ \cancel{5}\ 4$		6	7	8	7	2	1	4	.	0
		$\lambda_j =$	0	0	0	0	2	3	1	4	4

Fig. 2.

3. REGULARITY is polynomially equivalent to EVEN CYCLE

Consider the following two problems:

REGULARITY: Given a linearly ordered, commutative group $\mathscr{G}$ and the matrix A, is A regular in $\mathscr{G}$?

EVEN CYCLE: Given a digraph, does it contain a cycle of even length?

It was pointed out by several authors [6, 8–10] that neither a polynomial-time algorithm for solving EVEN CYCLE is known, nor NP-completeness of it was proved.

The following simple lemma will be useful.

Lemma 3.1. *Let $D = (N, E)$ be a digraph, $N = \{1, 2, \dots, n\}$ and $A = (a_{ij})$ be an $n \times n$ zero-one matrix defined as follows:*

$$a_{ii} = 0 \quad \textit{for } i \in N;$$

$$\textit{if } i \neq j \textit{ then } a_{ij} = 0 \iff (i,j) \in E.$$

Then D contains an even cycle if and only if $w_A(\pi) = 0$ in $\mathscr{G}_0$ for some $\pi \in P_n^-$.

Proof. Let $(i_1, \dots, i_k)$ be an even cycle in D and $\pi \in P_n$ be defined as follows:

$$\pi(i_r) = i_{r+1} \quad \text{for } r = 1, 2, \dots, k-1,$$

$$\pi(i_k) = i_1,$$

$$\pi(i) = i \quad \text{for } i \notin \{i_1, \dots, i_k\}.$$

Then $w_A(\pi) = 0$ in $\mathscr{G}_0$ and $\pi \in P_n^-$ since π is a product of $n - k$ trivial cycles and cyclic permutation $(i_1 i_2 \ldots i_k)$ which is odd.

Let $w_A(\pi) = 0$ in $\mathscr{G}_0$ for some $\pi \in P_n^-$ and let $\pi = \pi_1 \circ \cdots \circ \pi_s$ be its decomposition to cyclic permutations. Then at least one of $\pi_1, \ldots, \pi_s$, say $\pi_t = (i_1 i_2, \ldots, i_k)$ is an odd cyclic permutation, hence $(i_1, \ldots, i_k)$ is an even cycle in D (Lemma 1.1). □

Theorem 3.1. *REGULARITY and EVEN CYCLE are polynomially equivalent.*

Proof. Suppose A is given. By the Hungarian method we find a normal matrix $B \sim A$. Since $\mathrm{id} \in \mathrm{ap}^+(B)$, by Theorem 2.1 the matrix B (and hence by Lemma 1.5 also A) is not regular if and only if

$$w(B, \pi) = e \quad \text{for some } \pi \in P_n^- . \tag{3.1}$$

Let $C = (c_{ij})$ be an $n \times n$ zero-one matrix defined by

$$c_{ij} = 0 \quad \text{if } b_{ij} = e,$$

$$c_{ij} = 1 \quad \text{if } b_{ij} > e.$$

Clearly, C is a normal matrix in $\mathscr{G}_0$ and (3.1) holds if and only if

$$w(C, \pi) = 0 \tag{3.2}$$

or, equivalently (Lemma 3.1), the digraph $D = (N, \{(i,j);\ c_{ij} = 0\})$ contains an even cycle. Hence, A is not regular if and only if D contains an even cycle and D can be constructed from A in

$$\begin{array}{rl} \mathrm{O}(n^3) & \text{(for the Hungarian method)} \\ +\,\mathrm{O}(n^2) & \text{(construction of } C \text{ and } D) \\ \hline =\mathrm{O}(n^3) & \text{operations} \end{array}$$

To transform polynomially EVEN CYCLE to REGULARITY, suppose that a digraph $D = (N, E)$, $N = \{1, 2, \ldots, n\}$, $E \subseteq N \times N$, is given. Let $A = (a_{ij})$ be an $n \times n$ zero-one matrix defined by

$$a_{ii} = 0 \quad \text{for all } i \in N;$$

$$\text{for } i \neq j\text{:} \quad a_{ij} = 0 \iff (i,j) \in E.$$

Clearly, in $\mathscr{G}_0$ we have $\mathrm{id} \in \mathrm{ap}^+(A)$ and by Lemma 3.1 $\mathrm{ap}^-(A) \neq \emptyset \iff D$ contains an even cycle. It follows now from Theorem 2.1 that A is not regular in $\mathscr{G}_0 \iff D$ contains an even cycle. It remains to mention that A was constructed from D in $\mathrm{O}(n^2) \leqslant \mathrm{O}((|N| + |E|)^2)$ operations. □

4. Other types of regularity

At least we mention briefly two other types of regularity.

Matrix A with columns $A_1, \ldots, A_n$ is called *weakly regular* (WR) if

$$A_k = \sum_{\substack{j \in N \\ j \neq k}}^{\oplus} \lambda_j \otimes A_j$$

does not hold for any $k \in N$ and $\lambda_1, \ldots, \lambda_{k-1}, \lambda_{k+1}, \ldots, \lambda_n \in G$.

Matrix A is called *strongly regular* (SR) if for some vector b the system of equations

$$A \otimes x = b$$

has a unique solution.

Lemma 4.1. *If $A \sim B$ then A is SR (WR) if and only if B is SR (WR).*

Proof. Can be done straightforwardly from the definitions. □

Clearly, regularity implies weak regularity and it will follow from a later result that strong regularity implies regularity.

Both weak and strong regularities (the first under a different name) were introduced in [3]. At the same place an $O(n^3)$ method, the so-called $\mathscr{A}$-test, for checking weak regularity was presented.

Investigations concerning strong regularity were summarised in [1]. We present now some of the results showing that strong regularity can be essentially also checked in $O(n^3)$ operations thus making our inability of checking regularity efficiently more striking.

Matrix $A = (a_{ij})$ is said to be *strictly normal* if

$$a_{ij} > a_{ii} = e \quad \text{for all } i, j \in N,\ i \neq j.$$

Clearly, $\mathrm{ap}(A) = \{\mathrm{id}\}$ for every strictly normal matrix A.

It was shown in [3] and elsewhere that a necessary and sufficient condition that A be strongly regular is that $A \sim B$, where B is strictly normal. Using Lemma 1.4 we then have the following theorem.

Theorem 4.1. *If A is SR then $|\mathrm{ap}(A)| = 1$.*

Corollary. *If A is SR then A is regular.*

Proof of Corollary. If $|\mathrm{ap}(A)| = 1$ then either $\mathrm{ap}^+(A) = \emptyset$ or $\mathrm{ap}^-(A) = \emptyset$ and the result follows now from Theorem 2.1. □

The condition of strong regularity in Theorem 4.1 is not sufficient in general e.g. the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

in the additive group of integers is not equivalent to a strictly normal matrix though $\mathrm{ap}(A) = \{\mathrm{id}\}$. However, considering the same matrix in the additive group of rationals after subtracting $\frac{1}{2}$ from column 2 and adding $\frac{1}{2}$ to row 2 we get

$$B = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$$

which is strictly normal.

This observation was generalised as follows.

Theorem 4.2. *If $\mathscr{G}$ is dense (i.e. if $a < b$ then $a < c < b$ for some $c \in G$) and $|\mathrm{ap}(A)| = 1$ then A is SR.*

Proof. Can be found in [2]. □

Clearly $\mathscr{G}_0$ is dense as well as $\mathscr{G}_1 = (Q, +, \leqslant)$.

A typical class of non-dense LOCG are cyclic groups, like $\mathscr{G}_2 = (Z, +, \leqslant)$. A simple example of a LOCG which is neither dense nor cyclic is

$$\mathscr{G}_3 = (Z \times Z, +, \leqslant^*),$$

where $(a,b) \leqslant^* (c,d)$ if and only if $a < c$ or $a = c$ and $b \leqslant d$.

Clearly, in a LOCG a bounded set may not have an infimum in general. However, it is not difficult to prove the following statement [1].

Lemma 4.2. *Let $\mathscr{G}$ be a non-trivial LOCG. Then the set $\{a \in G;\ a > e\}$ has an infimum.*

The infimum mentioned in Lemma 4.2 will be denoted by $\alpha(\mathscr{G})$ or only α. Evidently $\alpha(\mathscr{G}) = e$ if and only if $\mathscr{G}$ is dense, $\alpha(\mathscr{G}) = g$ if $\mathscr{G}$ is cyclic with generator $g > e$, $\alpha(\mathscr{G}_3) = [0,1]$.

The *metric matrix* corresponding to A is

$$\Gamma(A) = A \oplus A^2 \oplus \cdots \oplus A^n$$

and its entry in row i and column j will be denoted by $\Gamma_{ij}(A)$. $\Gamma(A)$ can be computed by the Floyd–Warshall algorithm in $O(n^3)$ operations provided that the digraph associated with A has no negative cycles.

We adjoin $+\infty$ to G by the rules

$$a \leqslant +\infty \quad \text{for all } a \in G, \qquad a \otimes \infty = \infty \otimes a = \infty,$$

$\boxed{\text{STRONG REGULARITY}} \Rightarrow \boxed{\text{REGULARITY}} \Rightarrow \boxed{\text{WEAK REGULARITY}}$

($cc = O(n^3)$) (Polyn. equivalent to EVEN CYCLE) ($cc = O(n^3)$)

Fig. 3.

and we denote by $\tilde{A}$ the matrix arising from A after replacing all diagonal elements by ∞.

Theorem 4.3. *Let A be normal. Then A is SR $\Leftrightarrow$ $\Gamma_{ii}(\alpha \otimes \tilde{A}) > \alpha$ for all $i \in N$.*

Proof. Can be found in [1]. □

Theorem 4.3 shows that SR of a normal matrix can be checked in $O(n^3)$ operations, whenever $\alpha(\mathscr{G})$ is known. Using Lemma 4.1 and by the Hungarian method which enables us to find an equivalent normal matrix in $O(n^3)$ time we have then the same result for an arbitrary matrix.

Finally, we summarise our observations (*cc* stands for computational complexity) as shown in Fig. 3.

References

[1] P. Butkovič, Strong regularity of matrices – a survey of results, Discrete Appl. Math. 48 (1994) 45–68.

[2] P. Butkovič and H. Hevery, A condition for the strong regularity of matrices in the minimax algebra, Discrete Appl. Math. 11 (1985) 209–222.

[3] R.A. Cuninghame-Green, Minimax algebra, Lecture Notes in Economics and Mathematical Systems 166 (Springer, Berlin, 1979).

[4] R.A. Cuninghame-Green and P. Butkovič, Discrete-event dynamic systems: the strictly convex case, Math. Industrial Systems, to appear.

[5] M. Gondran and M. Minoux, L'indépendance linéaire dans les dioïdes, Bull. Direction Etudes Rech. Ser. C, Math. Inform. 1 (1978) 67–90.

[6] J. van Leeuwen, Algorithms and complexity, in: Handbook of Theoretical Computer Science Vol. A (Elsevier, Amsterdam 1990).

[7] C.H. Papadimitrion and K. Steiglitz, Combinatorial Optimization – Algorithms and Complexity (Prentice-Hall, Englewood Cliffs, NJ, 1982).

[8] C. Thomassen, Even cycles in directed graphs, European J. Combin. 6 (1985) 85–89.

[9] C. Thomassen, The even cycle problem for directed graphs, J. Amer. Math. Soc. 5 (1992) 217–229.

[10] V.V. Vazirani and M. Yannakakis, Pfaffian orientations, 0/1 permanents, and even cycles in directed graphs, in: T. Lepistö and A. Salomaa, eds., Proceedings of the 15th International Colloqium Automata, Languages and Programming, Lecture Notes in Computer Science 317 (Springer, Berlin, 1988) 667–681.