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# Sign-nonsingular matrices and matrices with unbalanced determinant in symmetrised semirings

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## Abstract

The operations $\oplus$ and $\otimes$ are defined by $a \oplus b = \max\{a, b\}$, $a \otimes b = a + b$ over the set of reals extended by $-\infty$. The columns $a^{(1)}, \ldots, a^{(n)}$ of an $n \times n$ matrix $A$ are said to be linearly dependent in max-algebra if

$$\sum_{j \in U}^{\oplus} \lambda_j \otimes a^{(j)} = \sum_{j \in V}^{\oplus} \lambda_j \otimes a^{(j)}$$

holds for some $\lambda_1, \ldots, \lambda_n \in R$; $U, V \neq \emptyset$; $U \cap V = \emptyset$; $U \cup V = \{1, \ldots, n\}$. We prove that there is a close relationship between sign-nonsingular (SNS) $(0, 1, -1)$ matrices and matrices with unbalanced determinant in symmetrised semirings. Given a matrix $A$ we then show how to construct a $(0, 1, -1)$ matrix $\tilde{A}$ such that $A$ has columns linearly dependent in max-algebra if and only if $\tilde{A}$ is not SNS. Also, it follows that if the system $A \otimes x = B \otimes x$ has a nontrivial solution then $\tilde{C}$ is not SNS, where $C = A \Theta B$ (here $\Theta$ stands for the subtraction in the symmetrised semiring). As another corollary we have a new, independent proof that the problems of checking whether a matrix is SNS and that of deciding whether a digraph contains a cycle of even length, are polynomially equivalent. 

*Keywords:* Sign-nonsingular matrix; Linear dependence; Max-algebra; Symmetrised semining; Even cycle

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PII: S 0 0 2 4 - 3 7 9 5 ( 9 9 ) 0 0 2 0 1 - 3

## 1. Preliminary results

In this paper we deal with the relationship between two different types of square matrices. One of them are sign-nonsingular matrices [3,8], the other one are matrices over a symmetrised semiring with unbalanced determinant [7]. Given a matrix $A$ we show how to construct a $(0, 1, -1)$ matrix $\tilde{A}$ such that $A$ has columns linearly dependent in max-algebra if and only if $\tilde{A}$ is not sign-nonsingular. We also present a necessary solvability condition for two-sided linear systems in max-algebra [5]. It also follows that SNS (that is the problem: given a matrix, is it sign-nonsingular?) and EVEN CYCLE (that is the problem: given a digraph, does it contain a cycle of even length?) are polynomially equivalent, the result proved in [3] in a completely different way.

For any set $X$ and positive integers $m, n$ the symbol $X(m, n)$ stands for the set of $m \times n$ matrices over $X$ and $X_n$ denotes the set of $n$-tuples (vectors) over $X$. The extended set of reals $R \cup \{-\infty\}$ will be denoted by $\bar{R}$.

The operations $\oplus$ and $\otimes$ are defined by

$$a \oplus b = \max\{a, b\}, \quad a \otimes b = a + b$$

for all $a, b \in \bar{R}$. Note that $(\bar{R}, \oplus, \otimes)$ is a commutative semiring [1,7]. $\oplus$ and $\otimes$ are extended to matrices and vectors over $\bar{R}$ as in the conventional linear algebra: If $A = (a_{ij}), B = (b_{ij}) \in \bar{R}(n, n)$, then

$$A \oplus B = (a_{ij} \oplus b_{ij}),$$

$$A \otimes B = \left( \sum_{k}^{\otimes} a_{ik} \otimes b_{kj} \right)$$

$$a \otimes A = (a \otimes a_{ij}).$$

The theory dealing with properties of matrices that are “linear” with respect to $\oplus$ and $\otimes$ is called max-algebra [4,5,7].

The columns $a^{(1)}, \ldots, a^{(n)}$ of a matrix $A \in \bar{R}(n, n)$ are said to be linearly dependent in max-algebra [6] if

$$\sum_{j \in U}^{\oplus} \lambda_j \otimes a^{(j)} = \sum_{j \in V}^{\oplus} \lambda_j \otimes a^{(j)}$$

holds for some $\lambda_1, \ldots, \lambda_n \in R$; $U, V \neq \emptyset$; $U \cap V = \emptyset$; $U \cup V = N = \{1, \ldots, n\}$. (The expression “in max-algebra” will be omitted in this paper.) They are called linearly independent if they are not linearly dependent.

LD will mean the following problem: Given a square matrix, are its columns linearly dependent?

$P_n$ stands for the set of all permutations of the set $N$ and for $\pi \in P_n$ and we define $w(A, \pi) = \prod_{i \in N}^{\otimes} a_{i,\pi(i)}$. Hence the max-algebraic permanent of $A$, per $(A) =$

$\sum^{\oplus}_{\pi\in P_n} w(A,\pi) = \max_{\pi\in P_n} \sum_{i\in N} a_{i,\pi(i)}$, is the optimal value of the assignment problem for $A$.

We denote $ap(A) = \{\pi \in P_n;\, w(A,\pi) = \operatorname{per}(A)\}$ (the set of optimal permutations to the assignment problem) and

$$P_n^+ = \{\pi \in P_n;\, \pi\,\text{even}\};\ P_n^- = \{\pi \in P_n;\, \pi\,\text{odd}\};$$

$$ap^+(A) = ap(A) \cap P_n^+;\ ap^-(A) = ap(A) \cap P_n^-.$$

Clearly, for every matrix $A$ either $ap^+(A) \neq \emptyset$ or $ap^-(A) \neq \emptyset$.

The following two known results will be useful.

**Theorem 1.1** (Gondran–Minoux [6]). *The columns of $A \in \bar{R}(n,n)$ are linearly dependent if and only if both $ap^+(A) \neq \emptyset$ and $ap^-(A) \neq \emptyset$.*

**Theorem 1.2** [2]. *LD is polynomially equivalent to EVEN CYCLE.*

Systems of simultaneous linear equations $A \otimes x = b$ and $A \otimes x = B \otimes x$ have been studied by several authors [1,4,5,7,9]. Motivations from the area of machine scheduling can be found in [4,5]. No polynomial method is available for the second type of systems. This was one of the reasons for introducing and studying symmetrised semirings [7]. We now give a brief account of some results of this theory.

Denote $S = \bar{R} \times \bar{R}$ and extend $\oplus$ and $\otimes$ to $S$ as follows:

$$(a,a') \oplus (b,b') = (a \oplus b, a' \oplus b'),$$
$$(a,a') \otimes (b,b') = (a \otimes b \oplus a' \otimes b', a \otimes b' \oplus a' \otimes b).$$

It is easy to check that $\varepsilon = (-\infty,-\infty)$ is the neutral element w.r.t. $\oplus$ and $(0,-\infty)$ is the neutral element w.r.t. $\otimes$.

If $x = (a,b)$ then $\Theta x$ stands for $(b,a)$, $x\Theta y$ means $x \oplus (\Theta y)$, the modulus of $x$ is $|x| = a \oplus b$, the balance operator is $x^{\bullet} = x\Theta x = (|x|,|x|)$. The following identities hold:

$$\Theta(\Theta x) = x,$$
$$\Theta(x \oplus y) = (\Theta x) \oplus (\Theta y),$$
$$\Theta(x \otimes y) = (\Theta x) \otimes y.$$

Let $x = (x',x'')$, $y = (y',y'')$. We say that $x$ balances $y$ (notation $x\nabla y$) if $x' \oplus y'' = x'' \oplus y'$. Though $\nabla$ is reflexive and symmetric, it is not transitive.

If $x = (a,b)$ then $x$ is called sign-positive [sign-negative], if $a > b$ [$a < b$] or $x = \varepsilon$; $x$ is called signed if it is either sign-positive or sign-negative; $x$ is called balanced if $a = b$, otherwise it is called unbalanced. Thus, $\varepsilon$ is the only element of $S$ that is both signed and balanced.

Due to the bijective semiring morphism: $t \mapsto (t,-\infty)$ we will identify (when appropriate) the elements of $\bar{R}$ and the sign-positive elements of $S$ of the form $(t,-\infty)$. Conversely, a sign-positive element $(a,b)$ will be identified with $a \in \bar{R}$. So, for instance 3 may denote the real number as well as the element $(3,-\infty)$ of $S$. Thus by this convention we write $3\Theta 2 = 3$, $3\Theta 7 = \Theta 7$, $3\Theta 3 = 3^{\bullet}$.

The following are easily proved:

$$a \nabla b, c \nabla d \quad \Rightarrow \quad a \oplus c \nabla b \oplus d$$
$$a \nabla b \quad \Rightarrow \quad a \otimes c \nabla b \otimes c$$
$$a \nabla b \text{ and } a = (a', a''), \quad b = (b', b'') \text{ are sign-positive} \quad \Rightarrow \quad a' = b'$$

The operations $\oplus$ and $\otimes$ are extended to matrices and vectors over $S$ in the same way as in conventional linear algebra. A vector is called sign-positive [sign-negative, signed] if all its components are sign-positive [sign-negative, signed]. The properties mentioned above hold if they are appropriately modified for vectors. For more details see [7].

**Proposition 1.1** [7]. *To every solution of the system $A \otimes x = B \otimes x$ there exists a sign-positive solution to the system of linear balances $(A \Theta B) \otimes x \nabla \varepsilon$, and conversely.*

We now define the determinant of matrices in symmetrised semirings. The sign of a permutation $\sigma$ is $\operatorname{sgn}(\sigma) = 0$ if is even and it is $\Theta 0$ if $\sigma$ is odd. The determinant of $A = (a_{ij}) \in S(n, n)$ is $\det(A) = \sum_{\sigma}^{\oplus} \operatorname{sgn}(\sigma) \prod_{i \in N}^{\otimes} a_{i,\sigma(i)}$. It will be convenient to say that a permutation $\sigma$ contains the elements $a_{1,\sigma(1)}, \ldots, a_{n,\sigma(n)}$ and that its weight is $\operatorname{sgn}(\sigma) \prod_{i \in N}^{\otimes} a_{i,\sigma(i)}$.

**Theorem 1.3** [7]. *Let $A \in S(n, n)$. Then the system of balances $A \otimes x \nabla \varepsilon$ has a signed nontrivial (i.e. $\neq \varepsilon$) solution if and only if* $\det(A) \nabla \varepsilon$.

$A \in S(n, n)$ is said to have balanced determinant if $\det(A) \nabla \varepsilon$, otherwise it is said to have unbalanced determinant.

**Corollary of Proposition 1.1 and Theorem 1.3.** *Let $A, B \in \bar{R}(n, n)$ and $C = A \Theta B$. Then a necessary condition that the system $A \otimes x = B \otimes x$ have a nontrivial solution is that $C$ has balanced determinant.*

We can reformulate the statement of Theorem 1.1 as follows:

**Theorem 1.1$'$.** *The columns of $A \in \bar{R}(n, n)$ are linearly dependent if and only if $A$ (as a matrix from $S(n, n)$) has balanced determinant.*

## 2. New results

An $n \times n$ matrix with diagonal elements $d_1, d_2, \ldots, d_n$ and off-diagonal elements equal to $-\infty$ is called diagonal and denoted by $\operatorname{diag}(d_1, d_2, \ldots, d_n)$.

**Lemma 2.1.** *If $D, A \in S(n, n)$ and $D$ is diagonal then* $\det(A \otimes D) = \det(A) \otimes \det(D)$.

**Proof.** Trivial. □

**Lemma 2.2.** *If $D, A, B \in S(n,n)$, $B = A \otimes D$ and $D$ is diagonal with unbalanced diagonal entries then $A$ has balanced determinant if and only if $B$ has balanced determinant.*

**Proof.** Immediate consequence of Lemma 2.1. □

A matrix from $S(n,n)$ is said to be in normal form if it has only elements with nonpositive modulus and at least one permutation contains only elements with zero modulus.

We say that matrices $A, B \in S(n,n)$ are directly similar if there exist two diagonal matrices $P, Q \in S(n,n)$ with unbalanced diagonal entries such that $A = P \otimes B \otimes Q$.

**Proposition 2.1.** *If $A \in S(n,n)$ and $\det(A) \neq \varepsilon$ then $A$ is directly similar to a matrix in normal form.*

**Proof.** This result is a by-product of the Hungarian method for solving the assignment problem. □

**Example.** The application of the Hungarian method to the matrix

$$\begin{pmatrix} 3 & \Theta 5 \\ -1 & 2^{\bullet} \end{pmatrix}$$

yields

$$\boxed{\otimes\Theta\,(-5)}$$

$$\begin{pmatrix} 3 & \Theta 5 \\ -1 & 2^{\bullet} \end{pmatrix} \to \begin{pmatrix} 0 & 0 \\ -4 & -3^{\bullet} \end{pmatrix} \to \begin{pmatrix} 0 & 0 \\ -1 & 0^{\bullet} \end{pmatrix}$$

$$\boxed{\otimes(-3)} \qquad\qquad \boxed{\otimes 3}$$

From this we infer that $B = \operatorname{diag}(0,3) \otimes A \otimes \operatorname{diag}(-3, \Theta(-5))$ is in normal form.

A $(0,1,-1)$ matrix is called SNS [8], if at least one term of its standard determinant expansion is nonzero and all nonzero terms have the same sign. Clearly, the sign-nonsingularity of a matrix is not affected by a permutation of its columns.

If $A = (a_{ij}) \in S(n,n)$ is a matrix whose entries all have nonpositive modulus, then $\tilde{A} = (\tilde{a}_{ij})$ is a $(0,1,-1)$ matrix defined as follows:

$$\tilde{a}_{ij} = \begin{cases} 0 & \text{if } a_{ij} < 0 \text{ or } a_{ij} \text{ is balanced}, \\ 1 & \text{if } a_{ij} = 0, \\ -1 & \text{if } a_{ij} = \ominus 0. \end{cases}$$

**Theorem 2.1.** *Let $A \in S(n, n)$ be in normal form.*
(a) *If $A$ has unbalanced determinant then $\tilde{A}$ is SNS.*
(b) *If $\tilde{A}$ is SNS and no balanced zero in $A$ belongs to any of the permutations of zero weight then $A$ has unbalanced determinant.*

**Proof.**
(a) At least one permutation contains only elements of zero modulus and no such permutation contains a balanced zero (because $0^\bullet \otimes 0 = 0^\bullet$ and $0^\bullet \oplus 0 = 0^\bullet$). Hence at least one permutation contains only nonzero elements in $\tilde{A}$. Since $\det(A)$ is either 0 or $\ominus 0$ and $0 \oplus \ominus 0 = 0^\bullet$, all permutations containing only elements of zero modulus must have the same weight, which must be either 0 or $\ominus 0$. Thus all nonzero terms in the standard determinant expansion of $\tilde{A}$ have the same sign.
(b) Some permutation in $\tilde{A}$ contains only nonzero elements. This permutation has in $A$ weight either 0 or $\ominus 0$. For every nonzero term in the standard determinant expansion of $\tilde{A}$ there is a signed zero term in $\det(A)$. By hypothesis there is no balanced zero term in $\det(A)$. All zero terms in det(A) have the same sign because each of them corresponds to a nonzero term in the standard determinant expansion of $\tilde{A}$. Therefore $\det(A)$ is either 0 or $\ominus 0$, that is $\det(A)$ is unbalanced. □

**Corollary.** *Let $A, B \in \bar{R}(n, n)$, $C = A \ominus B$ be normal and $a_{ij} \neq b_{ij}$ for all $i, j$. Then a necessary condition that the system $A \otimes x = B \otimes x$ have a nontrivial solution is that $\tilde{C}$ is not SNS.*

**Proof.** By Corollary of Proposition 1.1 and Theorem 1.3, $C$ has balanced determinant. $C$ has no balanced zero since $a_{ij} \neq b_{ij}$ for all $i, j$. Therefore by Theorem 2.1(b) $\tilde{C}$ is not SNS. □

**Theorem 2.2.** *Let $A \in \bar{R}(n, n)$ be a normal matrix. Then a necessary and sufficient condition that the columns of $A$ be linearly dependent is that $\tilde{A}$ is not SNS.*

**Proof.** If the columns of $A$ are linearly dependent then, after an appropriate permutation of columns, $A$ can be written block-wise as $(P|Q)$ and there are nontrivial (that is different from $-\infty$) $s, t$ such that $P \otimes s = Q \otimes t$. Hence the system $(P|-\infty) \otimes x = (-\infty|Q) \otimes x$ has a nontrivial solution. By Corollary of Proposition 1.1 and Theorem 1.3 the matrix $C = (P|-\infty) \ominus (-\infty|Q) = (P|\ominus Q)$ has balanced determinant. By Lemma 2.2 then also $(P|Q)$ has balanced determinant. This matrix differs from $A$ only by the order of the columns. There are no balanced elements in $A$ other than $\varepsilon$ and so by Theorem 2.1(b), $\tilde{A}$ is not SNS.

Conversely, if $\tilde{A}$ is not SNS then by Theorem 2.1(a) the matrix $A$ has balanced determinant and hence by Theorem 1.1$'$ it has linearly dependent columns. □

**Corollary of Theorem 1.2 and Theorem 2.2.** *SNS and EVEN CYCLE are polynomially equivalent.*

We note that this corollary has previously been proved in a different way in [3].

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