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Discrete Applied Mathematics 105 (2000) 73–86

DISCRETE APPLIED MATHEMATICS

Simple image set of (max, +) linear mappings

Peter Butkovic

School of Mathematics and Statistics, The University of Birmingham, Edgbaston, Birmingham B15 2TT, UK

Received 7 April 1999; revised 8 March 2000; accepted 10 April 2000

Abstract

Let us denote $a \oplus b = \max(a, b)$ and $a \otimes b = a + b$ for $a, b \in \mathbf{R}$ and extend this pair of operations to matrices and vectors in the same way as in conventional linear algebra, that is if $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$ are real matrices or vectors of compatible sizes then $C = A \otimes B$ if $c_{ij} = \sum_k^{\oplus} a_{ik} \otimes b_{kj}$ for all i, j.

If A is a real $n \times n$ matrix then the mapping $x \mapsto A \otimes x$ from $\mathbf{R}^n$ to $\mathbf{R}^n$ ($n > 1$) is neither surjective nor injective. However, for some of such mappings (called strongly regular) there is a nonempty subset (called the simple image set) of the range, each element of which has a unique pre-image. We present a description of simple image sets, from which criteria for strong regularity follow. We also prove that the closure of the simple image set of a strongly regular mapping f is the image of the kth iterate of f after normalization for any $k \geqslant n - 1$ or, equivalently, the set of fixed points of f after normalization.

Keywords: Max-algebra; Strong regularity; Fixed points; Eigenproblem; Assignment problem

1. Introduction

A wide class of problems in different research areas, like graph theory, automata theory, scheduling theory, etc., can be expressed using an attractive formulation language by setting up an algebra of, say, real numbers in which the operations of addition and multiplication are replaced by the selection of the maximum of the two numbers and arithmetical addition, respectively. Monographs [1,6,16] can be used as a comprehensive guide in this field. Specifically, significant effort has been developed to build up a theory similar to that of linear algebra, for instance to study systems of linear equations, eigenvalue problems, independence, rank, regularity and dimension. It turned out that there is only a thin barrier separating these concepts and combinatorial properties of matrices [2].

E-mail address: p.butkovic@bham.ac.uk (P. Butkovic).

PII: S0166-218X(00)00212-2

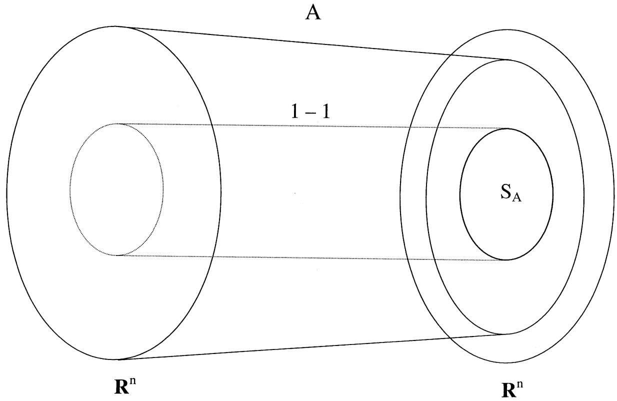

Fig. 1.

Let us denote $a \oplus b = \max(a, b)$ and $a \otimes b = a + b$ for $a, b \in \mathbf{R}$ and extend this pair of operations to matrices and vectors in the same way as in conventional linear algebra, that is if $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$ are real matrices or vectors of compatible sizes then $C = A \oplus B$ if $c_{ij} = a_{ij} \oplus b_{ij}$ for all i, j and $C = A \otimes B$ if $c_{ij} = \sum_k^{\oplus} a_{ik} \otimes b_{kj}$ for all i, j.

If A is a real $n \times n$ matrix then the mapping $x \mapsto A \otimes x$ from $\mathbf{R}^n$ to $\mathbf{R}^n$ $(n > 1)$ is neither surjective nor injective [2]. However, for some of such mappings there exist elements of the range, which have exactly one pre-image – the corresponding matrices are called *strongly regular* (Fig. 1). A practical motivation for investigating strong regularity in the area of job scheduling can be found in [2,5]. That paper also provides a survey of results, which enable us to decide in polynomial time whether a square matrix is strongly regular.

For any mapping $f : X \to Y$ we denote $S_f = \{y \in Y; (\exists! x \in X) f(x) = y\}$. The set S_f will be called the *simple image set* (corresponding to f). If $f : x \mapsto A \otimes x$ then S_f will also be denoted by S_A. Hence a matrix A is strongly regular if and only if $S_A \neq \emptyset$.

A real matrix is called *definite* [6] if all diagonal elements are zero (that is they are neutral with respect to $\otimes$) and there are no positive cycles in the associated digraph (the definition of an associated digraph can be found in Section 2). For an $n \times n$ matrix A the matrix A^* is defined as $I \oplus A \oplus A^2 \oplus \cdots$ (if the sum exists). If A is definite then $A^* = A^k$ for all $k \geqslant n - 1$ [7] and the (max, +) column space of A^*, denoted as $\mathrm{Im}(A^*)$, is the eigenspace of A [10,11], which coincides with the set of fixed points of A in the case of definite matrices.

The aim of the paper is to present a description of the simple image set of mappings $x \mapsto A \otimes x$ for definite matrices A (Theorem 3.1). This result has two consequences. Firstly, it provides a generalized criterion for strong regularity, from which previously known criteria [2,3] can be derived and secondly, it enables us to prove that for strongly regular definite matrices A, $\mathrm{cl}(S_A) = \mathrm{Im}(A^*)$, that is the closure of the simple image set of A is the set of fixed points of A. This result is then extended to all matrices.

2. Definitions and preliminary results

Throughout the paper we assume that $\mathscr{G}=(G,\otimes,\leqslant)$ is a nontrivial linearly ordered, commutative group with neutral element e (thus G is infinite). $\mathscr{G}$ is called *dense* if the open interval $(a,b)\neq\emptyset$ for all $a,b\in G,\ a<b$ and $\mathscr{G}$ is called *sparse* if it is not dense. $\mathscr{G}$ is called *radicable* if for each $a\in G$ and each positive integer k there exists an element $b\in G$ satisfying $b^k=a$ (here b^k stands for the iterated product $b\otimes b\otimes\cdots\otimes b$, in which the letter b appears k-times). Such an element b is unique and we denote it by $\sqrt[k]{a}$. As usual, $\mathscr{G}$ is called *cyclic* if $G=\{g^k;\ k \text{ integer}\}$ for some $g\in G,\ g>e$, called the *generator* of G.

So, for instance $\mathscr{G}_1=(\mathbf{R},+,\leqslant)$ is radicable, $\mathscr{G}_2=(\mathbf{Z},+,\leqslant)$ and $\mathscr{G}_3=(\mathbf{Z}_2,+,\leqslant)$ are cyclic with generators 1 and 2, respectively, $\mathscr{G}_4=(\mathbf{R}^+,.,\leqslant)$ is radicable, $\mathscr{G}_5=(\mathbf{Q}^+,.,\leqslant)$ is dense but not radicable, $\mathscr{G}_6=(\mathbf{Z}\times\mathbf{Z},+,\leqslant^L)$ is sparse but not cyclic. Here $\mathbf{R},\mathbf{Z},\mathbf{Z}_2,\mathbf{R}^+$ and $\mathbf{Q}^+$ are the sets of reals, integers, even integers, positive reals and positive rationals, respectively. The symbols $+,.$ and $\leqslant$ stand for the conventional arithmetic operations and ordering, respectively. The ordering of $\mathscr{G}_6$ is lexicographic. $\mathscr{G}_1$ is sometimes called “the main interpretation” [6] since practical applications of max-algebra are mostly formulated in this setting. Also, the main results of Section 4 are proved for $\mathscr{G}_1$. Being motivated by this we shall call an element $a\in G$ *positive* [*negative*, *nonpositive*, *nonnegative*] if $a>e$ [$a<e,\ a\leqslant e,\ a\geqslant e$].

We extend G by a new element $-\infty$, we denote $G\cup\{-\infty\}$ by $\bar{G}$ and extend $\otimes$ and $\leqslant$ to $\bar{G}$: $a\otimes-\infty=-\infty\otimes a=-\infty$ and $-\infty<a$ for all $a\in G$ (thus, $-\infty$ will be called negative).

The operation $\oplus$ is defined by $a\oplus b=\max\{a,b\}$ for $a,b\in\bar{G}$. $(\bar{G},\oplus,\otimes)$ is a commutative semiring [16,6,1].

For any set X and positive integers m,n the symbol $X(m,n)$ will stand for the set of $m\times n$ matrices over X and X_n denotes the set of n-tuples (vectors) over X. The operations $\oplus$ and $\otimes$ are extended to matrices and vectors from $\bar{G}(n,n)$ and $\bar{G}_n$ as in conventional linear algebra; the ordering is extended componentwise. It is known that $(\bar{G}_n,\oplus,\otimes)$ is a semimodule [16,8]. For $A\in\bar{G}(n,n)$ the symbol A^k (k positive integer) stands for the product $A\otimes A\otimes\cdots\otimes A$, in which the letter A appears k-times.

The symbol $\mathrm{diag}(d_1,d_2,\ldots,d_n)$ denotes the matrix with diagonal elements equal to $d_1,d_2,\ldots,d_n$ and off-diagonal elements equal to $-\infty$. This matrix will be called *diagonal* if all $d_1,d_2,\ldots,d_n\in G$. A *permutation matrix* is a matrix that can be obtained from a diagonal matrix by permuting its rows (or columns). The letter I will stand for the matrix $\mathrm{diag}(e,e,\ldots,e)$ of appropriate order. Clearly, for any permutation matrix P there exists a permutation matrix P^{-1} such that $P\otimes P^{-1}=P^{-1}\otimes P=I$. Also, $A\otimes I=I\otimes A=A$ for any square matrix A.

We say that square matrices $A=(a_{ij})$ and $B=(b_{ij})$ of equal order are similar (notation $A\sim B$) if $A=P\otimes B\otimes Q$ for some permutation matrices P and Q.

A matrix over $\bar{G}$ is called *positive* [*negative*, *nonpositive*, *nonnegative*] if all its elements are positive [negative, nonpositive, nonnegative].

A matrix over $\bar{G}$ is called *normal* [*strictly normal*], if it is nonpositive [if all off-diagonal elements are negative] and all diagonal elements are zero (that is e).

The *permanent* of $A=(a_{ij})\in\bar{G}(n,n)$ is defined as an analogue of the classical one

$$\mathrm{per}(A)=\sum_{\pi\in P_n}^{\oplus} w(A,\pi),$$

where P_n stands for the set of all permutations of the set $N=\{1,\dots,n\}$ and

$$w(A,\pi)=\prod_{i\in N}^{\otimes} a_{i,\pi(i)}.$$

For $\mathscr{G}_1=(\mathbf{R},+,\leqslant)$ we have $\mathrm{per}(A)=\max_{\pi\in P_n}\sum_{i\in N}a_{i,\pi(i)}$, which is the optimal value of the classical (linear) assignment problem. Therefore, the problem of finding per(A) is also referred to as the *algebraic assignment problem*. Efficient solution methods for this problem are known [16]. If there is only one permutation π satisfying $w(A,\pi)=\mathrm{per}(A)$ then we say that A has *strong permanent* [4]. It is easily seen that if $A\sim B$ and A has strong permanent then B has strong permanent, too.

Given $A=(a_{ij})\in G(n,n)$ and $b=(b_1,\dots,b_n)^{\mathrm{T}}\in G_n$, the system of equations

$$\sum_{j}^{\oplus} a_{ij}\otimes x_j=b_i \quad (i=1,\dots,n)$$

can also be written as

$$A\otimes x=b. \tag{2.1}$$

Max-algebraic linear systems have been studied in many papers (e.g. [1,6–8,11,15,16]). They are more easily solvable than the classical ones and as the next theorem indicates they have interesting combinatorial aspects.

Theorem 2.1 (Zimmermann [15], Butkovic [2]). *System* (2.1) *has a unique solution if and only if the matrix* $C=(c_{ij})=(a_{ij}\otimes b_i^{-1})$ *has exactly one column maximum in every row and in every column, that is there is a permutation* $\pi\in P_n$ *such that* $c_{\pi(j),j}>c_{ij}$ *for every* j *and every* $i\neq\pi(j)$.

Given a matrix $A\in G(n,n)$, we say that A is *strongly regular* if there exists $b\in G_n$ such that (2.1) has exactly one solution. Note that like in conventional linear algebra the number of solutions of (2.1) is always either 0 or 1 or ∞ but unlike in the conventional case for every A there are a vector b for which it is unsolvable and another right-hand side vector for which it has an infinite number of solutions [2].

Corollary 1 of Theorem 2.1. *If* $A\sim B$ *then* A *is strongly regular if and only if* B *is strongly regular.*

Proof. Straightforward using definitions. □

Note that the Hungarian method [16] for solving the assignment problem transforms any matrix A to a normal matrix $B \sim A$. By Corollary 1 of Theorem 2.1 the problem of strong regularity for A is the same as for B.

Corollary 2 of Theorem 2.1. *If A is strongly regular then A has strong permanent.*

Proof. The unique column maxima determine a permutation $\pi \in P_n$ such that $w(A, \pi)$ is bigger than the weight of any other permutation from P_n. $\square$

The converse statement to that of Corollary 2 is not true in some linearly ordered commutative groups, say for

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

in $\mathscr{G}_2 = (\mathbf{Z}, +)$ it is not possible to add constants from $\mathbf{Z}$ to the rows of A so that the arising matrix has exactly one column maximum in every row and in every column. However, the following holds.

Theorem 2.2 (Butkovic and Hevery [4]). *If $\mathscr{G}$ is dense then $A \in G(n, n)$ is strongly regular if and only if A has strong permanent.*

Note that the proof of this theorem as presented in [4] is rather complicated but (as it will be shown) it easily follows from the main result of Section 3, Theorem 3.1. If $A = (a_{ij}) \in \bar{G}(n, n)$ then D_A denotes the complete arc-weighted digraph (called the *associated digraph*) whose set of nodes is $\{1, 2, \ldots, n\}$ and the weight of the arc (i, j) is a_{ij} $(i, j = 1, \ldots, n)$. If $\sigma = (i_0, i_1, i_2, \ldots, i_t)$ is a cycle in D_A (thus $i_0 = i_t$) then the weight of σ is $w(\sigma) = a_{i_0 i_1} \otimes a_{i_1 i_2} \otimes \cdots \otimes a_{i_{t-1} i_t}$. If, moreover, $\mathscr{G}$ is radicable then the *average weight* of σ is $\sqrt[t]{w(\sigma)}$. If $w(\sigma) < e$ then σ is called *negative*; if $w(\sigma) = e$ then σ is called a *zero cycle*. The number t will also be called the *length* of σ and denoted by $\ell(\sigma)$. Also, we say that σ is *elementary* if all $i_0, i_1, \ldots, i_{t-1}$ are distinct (thus, $\ell(\sigma) \leqslant n$ if σ is elementary).

The *eigenproblem* is formulated as follows: Given $A \in G(n, n)$, find $x \in G_n$ and $\lambda \in G$ satisfying

$$A \otimes x = \lambda \otimes x.$$

The following result is fundamental [6,16,1]:

Theorem 2.3 (Cuninghame–Green [6]). *Every square matrix has at most one eigenvalue. If $\mathscr{G}$ is radicable then every square matrix A has exactly one eigenvalue (denoted as $\lambda(A)$ in what follows). This unique eigenvalue is equal to the maximal average weight of cycles in D_A.*

The set of all eigenvectors of the matrix A will be called the *eigenspace* and denoted by $\mathrm{Sp}(A)$. There are a number of efficient algorithms for finding the eigenvalue and generators of the eigenspace in the case of $\mathscr{G}_1$ [12,6,1]. The statement of Theorem 2.3 remains valid if $A \in \bar{G}(n,n)$ but all off-diagonal elements are from G ($-\infty$ may only appear on the diagonal) [6]. In order to avoid unnecessary notation, from now on $G(n,n)$ will denote the set of $n \times n$ matrices whose off-diagonal elements are from G. Note also that Theorem 2.3 does not hold in general for matrices from $\bar{G}(n,n)$ and in fact these may have more than one eigenvalue [8]; this question is significantly related to irreducibility of matrices.

Theorem 2.4 (Cuninghame–Green [6]). *Let $\mathscr{G}$ be radicable, $A \in G(n,n)$ and $\alpha \in G$. Then*

- $\lambda(\alpha \otimes A) = \alpha \otimes \lambda(A)$ *and*
- $\lambda(A) \leqslant e$ *if A is nonpositive (in particular, if A is normal).*

Proof. Straightforward using definitions only. □

In the rest of the paper $\tilde{A}$ will denote the matrix arising from A after replacing all diagonal elements by $-\infty$.

Theorem 2.5 (Butkovic [2]). *If $\mathscr{G}$ is radicable and $A \in G(n,n)$ is normal then A is strongly regular if and only if $\lambda(\tilde{A}) \neq e$ (i.e. $\lambda(\tilde{A}) < e$).*

Corollary of Theorem 2.5. *If $\mathscr{G}$ is radicable and $A \in G(n,n)$ is normal then A is strongly regular if and only if every cycle of length 2 or more in D_A is negative.*

Proof. Immediately follows from the theorem. □

The following is a duality statement; it will be very helpful later in this study for showing the links between the main results of this paper and properties established in the previous publications.

Theorem 2.6 (Cuninghame–Green [6], Gaubert [8]). *Let $\mathscr{G}$ be radicable and $A \in G(n,n)$. Then $\lambda(A) = \min\{\lambda; (\exists v) A \otimes v \leqslant \lambda \otimes v\}$.*

If $A \in \bar{G}(n,n)$ and k is a positive integer, then $A^{[k]}$ denotes the matrix $I \oplus A \oplus A^2 \oplus \cdots \oplus A^k$. The following result is an expression of some elementary principles in combinatorial optimization related to the shortest/longest path problem. A formal proof can be found, for instance, in [7].

Theorem 2.7. *If $A \in \bar{G}(n,n)$ and there are no positive cycles in D_A then $A^{[k]}$ is the same matrix for all $k \geqslant n-1$ (this matrix will be denoted in the rest of the paper by A^*). If, moreover, all diagonal elements of A are equal to e then $A^* = A^k$ for all $k \geqslant n-1$. This is in particular true for normal matrices.*

Sketch of the proof. An (i,j) entry of A^k is $\max_{r_1,\dots,r_{k-1}} a_{ir_1} \otimes a_{r_1r_2} \otimes \cdots \otimes a_{r_{k-1}j}$ and can thus be interpreted as the length of the longest path from node i to node j using $k-1$ intermediate nodes. The same entry in $A^{[k]}$ is therefore the length of the longest path from node i to node j using at most $k-1$ intermediate nodes. If $i \neq j$ then we cannot increase the maximal length by allowing more than $n-2$ intermediate nodes because this would lead to a repetition of some of the nodes, that is to a cycle whose contribution can only be zero or negative. If $i=j$ then in all matrices A^k the (i,j) entry is the length of a cycle, thus nonpositive and so in all $A^{[k]}$ it is zero due to the matrix I. If all diagonal elements of A are equal to e then $I \leqslant A$, implying $A \leqslant A^2 \leqslant \cdots$, that is $A^{[k]} = A^k$ for all k and thus $A^* = A^k$ for all $k \geqslant n-1$. □

Matrices from $G(n,n)$ whose associated digraph does not contain a positive cycle and all their diagonal elements are equal to e are called *definite*. Thus by Theorem 2.7, $A^* = A^{n-1} = A^n$ holds for all definite matrices A. Hence $A \otimes A^* = A^*$ and so $A \otimes v = v$ for every column v of A^*, which using Theorem 2.3 implies that e is the unique eigenvalue of every definite matrix A and all columns of A^* are the eigenvectors of A.

Since $(\bar{G}_n, \oplus, \otimes)$ is a semimodule, we have $A \otimes (\alpha \otimes u \oplus \beta \otimes v) = \lambda \otimes (\alpha \otimes u \oplus \beta \otimes v)$ for any matrix A from $G(n,n)$ with eigenvalue λ, eigenvectors u and v and scalars α, β and so any (max-algebraic) linear combination of eigenvectors is an eigenvector. Also, any vector of the form $A^* \otimes x$ is a linear combination of the columns of A^*, and therefore an eigenvector as well. Thus, if we denote $\mathrm{Im}(B) = \{B \otimes x; x \in G_n\}$ for any $B \in G(n,n)$ we have that $\mathrm{Im}(A^*)$ is a subset of $\mathrm{Sp}(A)$. Conversely, if $A \otimes x = x$ then $A^* \otimes x = A^n \otimes x = x$, so that actually $\mathrm{Im}(A^*) = \mathrm{Sp}(A)$. Hence, if we denote by $\mathscr{F}(A)$ the set of fixed points of A, we can summarize our observations [13].

Theorem 2.8. *If $A \in G(n,n)$ is definite then* $\mathrm{Im}(A^*) = \mathrm{Sp}(A) = \mathscr{F}(A)$.

Theorem 2.9. *If $A \in G(n,n)$ is definite then $A^* = \tilde{A}^*$.*

Proof. $A = \tilde{A} \oplus I$, thus, using Theorem 2.7 we get

$$A^* = (\tilde{A} \oplus I)^{n-1} = I \oplus \tilde{A} \oplus \tilde{A}^2 \oplus \cdots \oplus \tilde{A}^{n-1} = \tilde{A}^*. \qquad \square$$

We will also need the following result which is due to Gavalec [9].

Theorem 2.10. *If $\mathscr{G}$ is dense, $a > e$ and k is a positive integer then there is an element $b > e$ such that $b^k < a$.*

Proof. By density there is an element c such that $a > c > e$ and an element d such that $\min(c, a \otimes c^{-1}) > d > e$. Then $d^2 = d \otimes d < c \otimes a \otimes c^{-1} = a$. The rest follows by induction on k. □

3. The simple image set: generalization of criteria for strong regularity

Let $A \in G(n,n)$. Recall that in the notation introduced in Section 1 the symbol S_A stands for the set $\{b; (\exists! x) A \otimes x = b\}$. Hence "$A$ is strongly regular" means $S_A \neq \emptyset$. The main result presented in Theorem 3.1 below is a description of the simple image set of the mapping $x \mapsto A \otimes x$ from G_n to G_n, yielding a universal (i.e. independent of the properties of the underlying group) criterion for strong regularity of definite matrices. The same question for general matrices is discussed at the end of Section 4.

If $A \in G(n,n)$ then $ap(A)$ will denote the set of optimal permutations to the assignment problem for the matrix A, i.e. $ap(A) = \{\pi \in P_n; w(A,\pi) = \text{per}(A)\}$.

Lemma 3.1. *If $A, B \in G(n,n)$ and $A \sim B$ then $ap(A) = ap(B)$.*

Proof. Straightforward using definitions only. $\square$

Theorem 3.1. *Let $A \in G(n,n)$ be a definite matrix. Then $S_A = \{v; \tilde{A} \otimes v \leqslant g \otimes v$ for some $g < e\}$ and, in particular, A is strongly regular $\Leftrightarrow (\exists g < e)\ \{v; \tilde{A} \otimes v \leqslant g \otimes v\} \neq \emptyset$.*

Proof. Let $A = (a_{ij})$ be definite and $\pi \in P_n$. If π is a product of cycles $\pi_1, \ldots, \pi_k$ $(k \geqslant 1)$ then $w(A,\pi) = w(A,\pi_1) \otimes \cdots \otimes w(A,\pi_k) \leqslant e \otimes \cdots \otimes e = e = w(A, id)$.

We have

$$\begin{aligned} b \in S_{\mathrm{A}} &\Leftrightarrow A \otimes x = b \text{ has a unique solution [definition]} \\ &\Leftrightarrow B = (b_i^{-1} \otimes a_{ij}) \text{ has the column maxima only on the diagonal} \end{aligned}$$

[By Theorem 2.1 the column maxima in B determine a permutation σ such that $a_{i,\sigma(i)} > a_{r,\sigma(i)}$ for all i and r, $r \neq i$. If $\sigma \neq id$ then $w(B,\sigma) > w(B, id)$ which is a contradiction since by Lemma 3.1 $B \sim A$ and thus $id \in ap(B)$.]

$$\begin{aligned} &\Leftrightarrow (\exists c)\ (b_i^{-1} \otimes a_{ij} \otimes c_j) \text{ is strictly normal} \\ &\qquad \text{[for } c_j \text{ take the inverse of the column } j \text{ maximum]} \\ &\Leftrightarrow (b_i^{-1} \otimes a_{ij} \otimes b_j) \text{ is strictly normal} \quad [b_i^{-1} \otimes a_{ii} \otimes c_i = e \Rightarrow b_i = c_i] \\ &\Leftrightarrow (\forall i \neq j)\ (b_i^{-1} \otimes a_{ij} \otimes b_j < e) \\ &\Leftrightarrow (\exists g)\ (\forall i,j)(b_i^{-1} \otimes \tilde{a}_{ij} \otimes b_j \leqslant g < e) \\ &\Leftrightarrow (\exists g < e)\ (\forall i,j)(\tilde{a}_{ij} \otimes b_j \leqslant g \otimes b_i) \\ &\Leftrightarrow (\exists g < e)\ (\forall i) \left(\max_j (\tilde{a}_{ij} \otimes b_j) \leqslant g \otimes b_i \right) \\ &\Leftrightarrow (\exists g < \mathrm{e})\ (\tilde{A} \otimes b \leqslant g \otimes b). \qquad \square \end{aligned}$$

The previously published criterion for strong regularity in radicable groups, Theorem 2.5, now directly follows from Theorem 3.1 and can be formulated in a slightly stronger form.

Corollary 1 of Theorems 3.1 and 2.6. *If $\mathscr{G}$ is radicable and $A \in G(n,n)$ is definite then A is strongly regular if and only if $\lambda(\tilde{A}) \neq e$ (i.e. $\lambda(\tilde{A}) < e$).*

We also have:

Corollary 2 of Theorems 3.1 and 2.6. *Let $\mathscr{G}$ be radicable and $A \in G(n,n)$ be a definite matrix. If A is strongly regular then the eigenspace of $\tilde{A}$ is a subset of S_A.*

The proof of the criterion for strong regularity in dense groups now easily follows from Theorem 3.1. We shall show this shortly but first we formulate an auxiliary statement.

Theorem 3.2. *Let $A \in G(n,n)$ be a definite matrix. Then A has strong permanent $\Leftrightarrow$ every cycle in $\tilde{A}$ is negative.*

Proof. Every permutation different from identity is a product of cycles, at least one of which has length two or more. Hence A has SP $\Leftrightarrow$ all cycles in D_A of the length two or more have negative weight $\Leftrightarrow$ all cycles in $D_{\tilde{A}}$ have negative weight. □

Theorem 3.3. *Let $\mathscr{G}$ be dense and $A \in G(n,n)$. Then A is strongly regular if and only if A has strong permanent.*

Proof. Due to Corollary 2 of Theorem 2.1 it remains to prove the "if" part. As it has already been mentioned after Corollary 1 of Theorem 2.1, the Hungarian method for solving the assignment problem transforms every matrix into a normal matrix similar to the first one. Since this transformation affects neither strong regularity (Corollary 1 of Theorem 2.1) nor the property of having strong permanent (which is trivial), we may assume without loss of generality that A is normal (and thus definite). By Theorem 3.2 we can assume that every cycle in $D_{\tilde{A}}$ has negative weight. By Theorem 2.10 there exists $g \in G$, $g < e$, such that

$$g^n \geqslant \max\{w(\sigma,\tilde{A});\ \sigma \text{ is an elementary cycle on a subset of } \{1,2,\ldots,n\}\}.$$

Let B be the matrix $g^{-1} \otimes \tilde{A}$ and σ be an elementary cycle. Then $w(\sigma,B) = g^{-\ell(\sigma)} \otimes w(\sigma,\tilde{A}) \leqslant g^{-n} \otimes w(\sigma,\tilde{A}) \leqslant e$. Clearly, there is no positive cycle in D_B.

Hence, using Theorem 2.7 we have:

$B \otimes B^* = B \otimes (I \oplus B \oplus \cdots \oplus B^{n-1}) = B \oplus B^2 \oplus \cdots \oplus B^n \leqslant I \oplus B \oplus \cdots \oplus B^n = B^*$ and so if v is any column of B^* then $B \otimes v \leqslant v$ or, equivalently, $g^{-1} \otimes \tilde{A} \otimes v \leqslant v$. Now, the statement of the theorem follows from Theorem 3.1. □

For $A \in G(n,n)$ we denote $V_A(g) = \{v; A \otimes v \leqslant g \otimes v\}$. The index A will be omitted if it is clear from the context which matrix is considered. Obviously, for any A

$$g_1 < g_2 \quad \Rightarrow \quad V_A(g_1) \subseteq V_A(g_2). \tag{3.1}$$

Every sparse group $\mathscr{G}$ has a smallest positive element [2, Proposition 1.3], say $\alpha(\mathscr{G})$, or briefly α. Clearly, α is the generator of $\mathscr{G}$ if $\mathscr{G}$ is cyclic and α^{-1} is the greatest negative element.

Theorem 3.4. *Let $\mathscr{G}$ be a sparse group and $A \in G(n,n)$ be a definite matrix. Then A is strongly regular if and only if $V_A(\alpha^{-1}) \neq \emptyset$.*

Proof. The statement follows from Theorem 3.1 and from (3.1) in a straightforward way. □

4. The simple image set and the image of A^*

This section deals with the description of S_A when $\mathscr{G} = \mathscr{G}_1$. So A is a real $n \times n$ matrix and $e = 0$. We denote by $u^{(j)}$ the jth unit vector in the conventional sense. There are many concepts of linear independence for semimodules [6,14] but in this section we will use the concepts of linear independence and dimension as defined in conventional linear algebra.

Theorem 4.1. *Let $A \in \mathbf{R}(n,n)$. If A is strongly regular then S_A (and hence also $\mathrm{Im}(A)$) contains n linearly independent vectors.*

Proof. Take any $b \in S_A$. For any $i \in \{1,2,\dots,n\}$ and $\varepsilon > 0$ sufficiently small, $b + \varepsilon u^{(i)} \in S_A \subseteq \mathrm{Im}(A)$ because by subtracting ε from row i, the set of entries where the column maxima are attained remains unchanged (see Theorem 2.1). □

One of the consequences of Theorem 3.1 is the following.

Theorem 4.2. *If $A \in \mathbf{R}(n,n)$ is definite and strongly regular then $\mathrm{cl}(S_A) = \mathrm{Im}(A^*)$ ($= \mathrm{Im}(A^k)$ for all $k \geqslant n-1$). Equivalently, the closure of the simple image set of a definite, strongly regular matrix is the set of its fixed points.*

The proof of Theorem 4.2 is based on the following three lemmas.

Lemma 4.1. *If $B \in \mathbf{R}(n,n)$ and $\lambda(B) \leqslant 0$ then $\mathrm{Im}(B^*) = \{v; B \otimes v \leqslant v\} = V_B(0)$.*

Proof. B has no positive cycles. If $v = B^* \otimes w$ then (see Theorem 2.7) $B \otimes v = B \otimes B^* \otimes w = (B \oplus \cdots \oplus B^n) \otimes w \leqslant (I \oplus B \oplus \cdots \oplus B^n) \otimes w = B^* \otimes w = v$. If $B \otimes v \leqslant v$ then $v \geqslant B \otimes v \geqslant B^2 \otimes v \geqslant \cdots \geqslant B^{n-1} \otimes v$, hence $v \geqslant B^* \otimes v$. On the other hand, $B^* \geqslant I$ hence $B^* \otimes v \geqslant v$, yielding that $B^* \otimes v = v$ and so $v \in \mathrm{Im}(B^*)$. □

Lemma 4.2. *If $C \in \mathbf{R}(n,n)$ and $\lambda(C) \leqslant \gamma < 0$ then $V_C(\gamma) = \mathrm{Im}(\gamma^{-1} \otimes C)^*$.*

Proof. Follows immediately from Lemma 4.1 by setting $B=\gamma^{-1}\otimes C$ and using $\lambda(B)=\gamma^{-1}\otimes\lambda(C)$ (see Theorem 2.4). $\square$

Lemma 4.3. *If $A\in\mathbf{R}(n,n)$ is definite and strongly regular then*

$$\operatorname{Im}(\tilde{A}^{*})=\operatorname{cl}\left(\bigcup_{\lambda(\tilde{A})\leqslant g<0}\operatorname{Im}(g^{-1}\otimes\tilde{A})^{*}\right).$$

Proof. If $\lambda(\tilde{A})\leqslant g<0$ then by Lemma 4.2, (3.1) and Lemma 4.1 we have $\operatorname{Im}(g^{-1}\otimes\tilde{A})^{*}=V(g)\subseteq V(0)=\operatorname{Im}(\tilde{A}^{*})$. The inclusion $\supseteq$ in the statement of Lemma 4.3 now follows since $\operatorname{Im}(\tilde{A}^{*})$ is closed (see Lemma 4.1).

Suppose now

$$v=\tilde{A}^{*}\otimes x=x\oplus\tilde{A}\otimes x\oplus\cdots\oplus\tilde{A}^{n-1}\otimes x.$$

Set $g_k=(1/k)\lambda(\tilde{A})$ for $k\geqslant 1$, integer and $v_{g_k}=x\oplus(g_k^{-1}\otimes\tilde{A})\otimes x\oplus\cdots\oplus(g_k^{-1}\otimes\tilde{A})^{n-1}\otimes x$.

Then $v_{g_k}=(g_k^{-1}\otimes\tilde{A})^{*}\otimes x\in\operatorname{Im}(g_k^{-1}\otimes\tilde{A})^{*}$ and $\lambda(\tilde{A})\leqslant g_k<0$ for all k.

Hence, $v_{g_k}\in\bigcup_{\lambda(\tilde{A})\leqslant g<0}\operatorname{Im}(g^{-1}\otimes\tilde{A})^{*}$ for all k and $v_{g_k}\to v$ $(k\to\infty)$. $\square$

Proof of Theorem 4.2. First note that $\lambda(\tilde{A})<0$ by Corollary 1 of Theorems 3.1 and 2.6. It follows from Theorems 3.1, 2.9 and Lemma 4.1 that

$$S_A\subseteq\{x;\tilde{A}\otimes x\leqslant x\}=\operatorname{Im}(\tilde{A}^{*})=\operatorname{Im}(A^{*}).\ \operatorname{Im}(A^{*})\text{ is closed and thus }\operatorname{cl}(S_A)\subseteq\operatorname{Im}(A^{*}).$$

Take any $z\in\operatorname{Im}(g^{-1}\otimes\tilde{A})^{*}$, $\lambda(\tilde{A})\leqslant g<0$. Due to Theorem 2.9 and Lemma 4.3 it is now sufficient to prove that $z\in S_A$. By Lemma 4.2

$$\operatorname{Im}(g^{-1}\otimes\tilde{A})^{*}=\{v;\tilde{A}\otimes v\leqslant g\otimes v\}.$$

Hence, $\tilde{A}\otimes z\leqslant g\otimes z$ and the rest follows from Theorem 3.1. $\square$

Corollary of Theorems 4.1 and 4.2. *Let $A\in\mathbf{R}(n,n)$ be definite. Then A is strongly regular $\Leftrightarrow\operatorname{Im}(A^{*})$ has dimension n.*

Proof. If A is not strongly regular then D_A contains a non-trivial zero cycle (Corollary 1 of Theorems 3.1 and 2.6). It follows from the results of [6, Theorem 23-6] that at least two columns of A^{*} are a scalar ($\otimes$) multiple of the same vector. Therefore, $\operatorname{Im}(A^{*})$ is generated by less than n vectors and has thus dimension $n-1$ or less. The converse implication follows immediately from Theorems 4.1 and 4.2. $\square$

It follows from the previous discussion that the relationship between a simple image set and $\operatorname{Im}(A^{*})$ is as indicated in Fig. 2 (when $S_A\neq\emptyset$) and in Fig. 3 (when $S_A=\emptyset$).

Finally, we are ready to describe the simple image set for general matrices from $\mathbf{R}(n,n)$. Columns of any matrix can be permuted so that the identical permutation becomes an optimal solution to the assignment problem and if the columns are then multiplied ($\otimes$) by the inverse values to the diagonal elements, the resulting matrix is

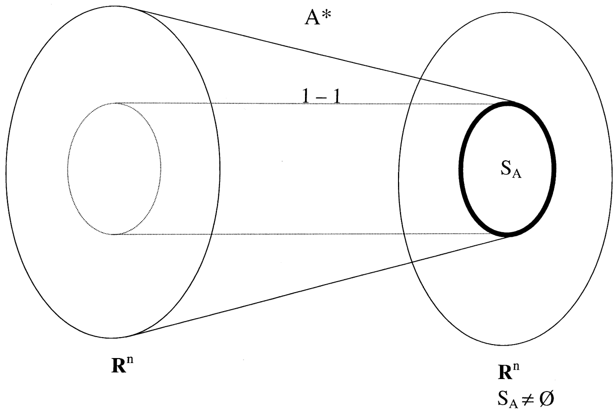

Fig. 2.

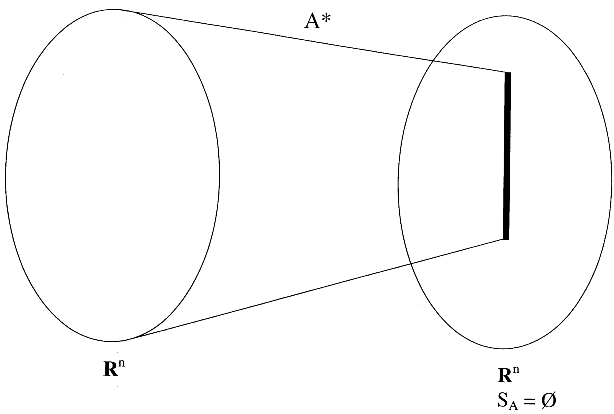

Fig. 3.

definite. Hence, for any $A \in \mathbf{R}(n,n)$ there is a permutation matrix Q such that $A \otimes Q = B$ is definite. We say that B has been obtained from A by *normalization*.

Theorem 4.3. *If $A, B \in \mathbf{R}(n,n)$, $A \otimes Q = B$ and Q is a permutation matrix then $S_A = S_B$.*

Proof. Straightforward, using the fact that $z \neq z' \Rightarrow Q \otimes z \neq Q \otimes z'$. $\square$

Corollary of Theorems 4.2 and 4.3. *If $A \in \mathbf{R}(n,n)$ is strongly regular, Q is a permutation matrix and $A \otimes Q$ is definite then $\mathrm{cl}(S_A) = \mathrm{Im}(A \otimes Q)^k$ for all $k \geqslant n-1$. Equivalently, the closure of the simple image set of a strongly regular matrix A is the set of fixed points of the matrix obtained from A by normalization.*

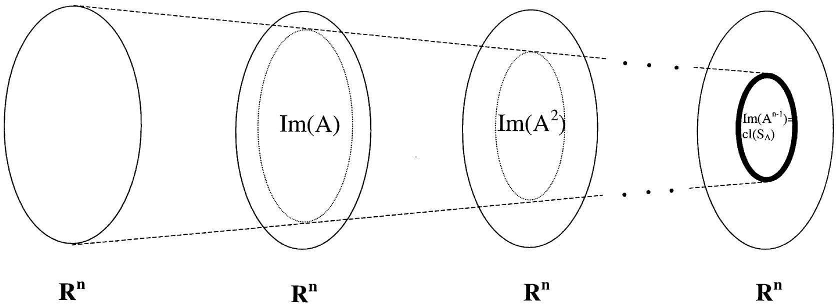

Fig. 4.

Fig. 4 illustrates the remarkable property of strongly regular (max, +) linear mappings from $\mathbf{R}^n$ to $\mathbf{R}^n$ with definite matrices: the range of the kth iterate ($k \geqslant n-1$) of the mapping is the closure of the simple image set of that mapping.

Acknowledgements

The author wishes to acknowledge with gratitude a valuable contribution of Professor Stephane Gaubert to the proof of Theorem 4.2.

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