On the Regularity of Matrices in *min* Algebra

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ABSTRACT

We consider the concept of regularity for square matrices with entries from a linearly ordered commutative group $(G, +, \geqslant)$, the algebraic compositions being given by $x \oplus y = \min(x, y)$, $x \otimes y = x + y$. For the case where G is cyclic we derive a necessary and sufficient condition for the regularity of a matrix, which can be checked in $O(n^3)$ operations using standard algorithms.

1. INTRODUCTION

We suppose throughout that $n \geqslant 2$ is an integer and $\mathscr{G} = (G, +, \geqslant)$ is a linearly ordered, commutative group with neutral element 0. The inverse of $a \in G$ will be denoted by $-a$. G_n means the set of square matrices of order n with entries from G. N stands for the set $\{1, 2, \dots, n\}$, and P_n for the set of all permutations of N.

We say that $A = (a_{ij}) \in G_n$ has *well-distributed column minima* if there exists $\pi \in P_n$ satisfying

$$a_{i,\pi(i)} = \min_{r \in N} a_{r,\pi(i)} \qquad \text{for all} \quad i \in N. \tag{1}$$

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A matrix $B = (b_{ij}) \in G_n$ is said to be *similar to* A (notation $A \sim B$), or *directly similar to* A (notation $A \approx B$) if

$$b_{ij} = u_i + a_{ij} + v_j \tag{2}$$

or

$$b_{ij} = u_i + a_{ij} - u_j, \tag{3}$$

respectively, for all $i, j \in N$ and for some $u_1, \dots, u_n, v_1, \dots, v_n \in G$. Clearly, both $\sim$ and $\approx$ are equivalence relations.

Addition of constants to columns does not influence the property (1), and thus it is not restrictive to suppose that

$$a_{i,\pi(i)} = 0 \qquad \text{for all} \quad i \in N. \tag{4}$$

Matrices satisfying both (1) and (4) for some $\pi \in P_n$ will be called 0-*astic*. (This is a particular case of doubly 0-astic in the terminology of [3].) 0-astic matrices play a significant role in solving the well-known assignment problem (AP) for the matrix A, consisting of finding $\pi \in P_n$ satisfying

$$\sum_{i \in N} a_{i,\pi(i)} = \min_{\sigma \in P_n} \sum_{i \in N} a_{i,\sigma(i)}. \tag{5}$$

Denote

$$\operatorname{ap}(A) = \{\pi \in P_n;\, \pi \text{ satisfies } (5)\}.$$

One can check easily that $A \sim B$ implies $\operatorname{ap}(A) = \operatorname{ap}(B)$, and if A is 0-astic then $\pi \in \operatorname{ap}(A)$ whenever π satisfies (1) and (4). Based on these two ideas, the Hungarian (or Kuhn) method for solving AP ([4, 5]) actually transforms any $A \in G_n$ to a 0-astic matrix similar to A using $O(n^3)$ operations.

One can ask more specifically whether a given matrix cannot be transformed to a similar 0-astic matrix in which all the column minima are strict (i.e., each column contains a unique 0). Such a matrix will be called *strictly* 0-*astic*; naturally, it does not exist for every matrix, say for a zero matrix, or for the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{6}$$

in the additive group of integers.

Strictly 0-astic matrices have a special importance in min-algebra theory. For a fixed matrix $A = (a_{ij}) \in G_n$, denote by $T(A)$ the set of all possible cardinalities of the solution set to the system of equations

$$\min_{j=1,\ldots,n} \left(a_{ij} + x_j\right) = b_i, \qquad i = 1,\ldots,n,$$

where the b_i's range over the whole of G. It is not difficult to show that either $T(A) = \{0,1,\infty\}$ or $T(A) = \{0,\infty\}$ for every $A \in G_n$. In the former case A is called *regular*, and it is shown in [3] that a matrix is regular if and only if it is similar to a strictly 0-astic matrix. This motivates the effort to develop a method for recognizing regular matrices.

We first derive a basic condition for regularity. If A is strictly 0-astic and $\pi \in P_n$ satisfies (1), then

$$\sum_{i \in N} a_{i,\pi(i)} < \sum_{i \in N} a_{i,\sigma(i)}$$

holds for all $\sigma \in P_n$, $\sigma \neq \pi$, and hence $|\mathrm{ap}(A)| = 1$. Thus, we have proved:

THEOREM 1. *A necessary condition for A to be regular is that the* AP *for A has a unique solution.*

It follows from the results in [2] that the regularity condition in Theorem 1 is also sufficient whenever $\geqslant$ is dense. This is not the case e.g. for the additive group of integers, and the matrix (6) is a corresponding counterexample. However, in Section 4 we will present a condition which is necessary and sufficient for the regularity of a matrix over the integers. It enables also to find the similar strictly 0-astic matrix in $O(n^3)$ operations.

A (strictly) 0-astic matrix will be called (*strictly*) *normal* if all its diagonal elements are 0.

It is quite easy to see that the problem of checking the regularity of matrices can be reduced to the same question for normal matrices. To see this, notice that the Hungarian method applied to $A = (a_{ij}) \in G_n$ produces as a result $\alpha_1,\ldots,\alpha_n, \beta_1,\ldots,\beta_n \in G$ and $\pi \in P_n$ such that the matrix $B = (b_{ij}) \in G_n$ defined by

$$b_{ij} = \alpha_i + a_{ij} + \beta_j \qquad \text{for all} \quad i,j \in N$$

is 0-astic and $\pi \in \mathrm{ap}(B)$ [= ap(A)]. Applying π^{-1} to the columns of B, we obtain the matrix $C = (c_{ij})$ for which id $\in \mathrm{ap}(C)$, i.e., C is normal. Since neither the transformation (2) nor the permutation of columns changes regularity, we have reduced our original problem for A to the same problem for a normal matrix C. Moreover, if we find $u_1, \ldots, u_n, v_1, \ldots, v_n \in G$ such that $F = (f_{ij})$ determined by

$$f_{ij} = u_i + c_{ij} + v_j \qquad \text{for all} \quad i, j \in N$$

is strictly 0-astic, then $H = (h_{ij})$ given by

$$h_{ij} = u_i + b_{ij} + v_{\pi^{-1}(j)} \qquad \text{for all} \quad i, j \in N$$

is strictly 0-astic and similar to B. Hence

$$h_{ij} = \alpha'_i + a_{ij} + \beta'_j \qquad \text{for all} \quad i, j \in N,$$

where $\alpha'_i = \alpha_i + u_i$, $\beta'_j = \beta_j + v_{\pi^{-1}(j)}$ for all $i, j \in N$.

2. UNIQUE SOLUBILITY OF THE ASSIGNMENT PROBLEM

A cyclic permutation of a subset of N will be called a *cycle*. The set of all cycles will be denoted by C_n. Clearly, each $\sigma \in C_n$ coresponds naturally to a graph-theoretical cycle (which may be a self-loop) in a complete digraph with n nodes, and it will be convenient to understand the notation and terminology of cycles in either sense, according to context.

Given any $A = (a_{ij}) \in G_n$ and $\sigma \in C_n$, $\sigma = (i_1, \ldots, i_k)$, we define $w_A(\sigma)$, the *weight* of σ, as

$$a_{i_1 i_2} + a_{i_2 i_3} + \cdots + a_{i_k i_1}.$$

The number k, the *length* of σ, will be denoted by $l(\sigma)$.

We show how to check whether $|\mathrm{ap}(A)| = 1$ for a normal matrix A. If $A = (a_{ij}) \in G_n$ is nonnegative, we denote by $D_A = (N, E)$ the digraph with node set N and edge set E defined by the formula

$$(i, j) \in E \quad \text{if and only if} \quad a_{ij} = 0 \text{ and } i \neq j.$$

THEOREM 2. *Let $A \in G_n$ be normal. Then a necessary and sufficient condition for $|\mathrm{ap}(A)| = 1$ is that D_A is acyclic.*

Proof. If $(i_1, \dots, i_k)$ is an (elementary) cycle in D_A, then

$$a_{i_1 i_2} = a_{i_2 i_3} = \cdots = a_{i_k i_1} = 0,$$

$k \geqslant 2$, and $i_r \neq i_s$ for $r \neq s$. Hence, setting

$$\pi(i_r) = i_{r+1} \quad \text{for} \quad r = 1, 2, \dots, k-1,$$

$$\pi(i_k) = i_i,$$

$$\pi(i) = i \quad \text{for} \quad i \in N - \{i_1, \dots, i_k\},$$

we have $\pi \in \mathrm{ap}(A)$ and $\pi \neq \mathrm{id}$.

Suppose now that $\pi \in \mathrm{ap}(A)$, $\pi \neq \mathrm{id}$. Then π can be decomposed into the product of cyclic permutations at least one of which, say σ, has length $l \geqslant 2$. σ is a permutation of some subset, say S, of the set N. Hence $a_{i,\sigma(i)} = 0$ for all $i \in S$, due to the linearity of $\geqslant$. Consequently, taking a (fixed) index $i \in S$, we have that

$$\left(i, \sigma(i), \sigma^2(i), \dots, \sigma^{l-1}(i)\right)$$

is a cycle in D_A. ■

It is well known that a necessary and sufficient condition for a digraph D to be acyclic is that the nodes of D can be numbered so that (i, j) is an edge only if $i < j$. This property can be checked by an $O(n^2)$ algorithm presented e.g. in [4]. A renumbering of the nodes in D_A corresponds to a reordering of the rows and the same reordering of the columns of A and, naturally, does not change the set of diagonal elements of A. We get the following

COROLLARY of Theorem 2. *Let $A \in G_n$ be normal. Then a necessary and sufficient condition for $|\mathrm{ap}(A)| = 1$ is that A can be transformed by a permutation of the rows and the same permutation of the columns to a normal matrix $A' = (a'_{ij})$ in which $a'_{ij} = 0$ only if $i \leqslant j$. This condition can be verified in $O(n^2)$ operations.*

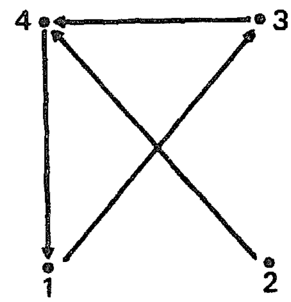

FIG. 1.

EXAMPLE 1. Consider the normal matrix

$$A = \begin{pmatrix} 0 & 4 & 0 & 5 \\ 3 & 0 & 2 & 0 \\ 4 & 5 & 0 & 0 \\ 0 & 6 & 1 & 0 \end{pmatrix}$$

in the additive group of integers. Since (cf. Figure 1) D_A contains the cycle $(1, 3, 4)$, we deduce (using Theorems 1 and 2) that A is not regular.

EXAMPLE 2. Consider the matrix

$$A = \begin{pmatrix} 12 & 8 & 14 & 4 \\ 2 & 2 & 4 & 6 \\ 8 & 0 & 4 & 2 \\ 10 & 4 & 6 & 4 \end{pmatrix}$$

in the additive group of integers. The Hungarian method will find (say) $\alpha = (-2, 0, 0, -2)^T$, $\beta = (-2, 0, -4, -2)$ such that

$$B = \left(b_{ij}\right) = \left(\alpha_i + a_{ij} + \beta_j\right) = \begin{pmatrix} 8 & 6 & 8 & 0 \\ 0 & 2 & 0 & 4 \\ 6 & 0 & 0 & 0 \\ 6 & 2 & 0 & 0 \end{pmatrix}$$

is 0-astic and $\pi = (4, 3, 2, 1) \in \mathrm{ap}(B)$. Applying π^{-1} to the columns of B, we get a normal matrix

$$C = \begin{pmatrix} 0 & 8 & 6 & 8 \\ 4 & 0 & 2 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 6 & 2 & 0 \end{pmatrix}.$$

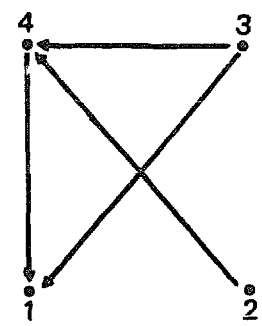

FIG. 2.

Since D_C is acyclic (see Figure 2), we conclude that

$$|\mathrm{ap}(A)| = |\mathrm{ap}(C)| = 1.$$

3. SOME PRELIMINARY RESULTS

PROPOSITION 3.1. *Let $A = (a_{ij}) \in G_n$ be normal. Then A is similar to a strictly 0-astic matrix (i.e., A is regular) if and only if A is directly similar to a strictly normal matrix.*

Proof. The "if" part is trivial. For the converse, let A be normal. Then $\mathrm{id} \in \mathrm{ap}(A)$, and if, moreover, A is similar to a strictly 0-astic matrix, then $\mathrm{ap}(A) = \{\mathrm{id}\}$ by Theorem 1.

Suppose that $B = (b_{ij})$ is strictly 0-astic, satisfying (2) and $b_{i,\pi(i)} = 0$ for all $i \in N$ and some $\pi \in P_n$. Hence $\pi \in \mathrm{ap}(B) = \mathrm{ap}(A) = \{\mathrm{id}\}$, implying that B is strictly normal. Thus we have by (2)

$$0 = u_i + 0 + v_i \qquad \text{for all} \quad i \in N$$

and so $B \approx A$. ∎

Let ∞ be an element adjoined to G, and let us introduce the following rules for ∞:

$$a \leqslant \infty$$

and

$$a + \infty = \infty + a = \infty,$$

for all $a \in G'$, where $G' = G \cup \{\infty\}$. By G'_n we denote the set of $n \times n$ matrices with entries from G'. Given $A = (a_{ij}) \in G'_n$, the symbol $\tilde{A}$ will stand for the matrix $(\tilde{a}_{ij})$ such that

$$\tilde{a}_{ii} = \infty \qquad \text{for all} \quad i \in N,$$

$$\tilde{a}_{ij} = a_{ij} \qquad \text{for all} \quad i, j \in N, \quad i \neq j.$$

We now define a particular algebraic structure on G' and G'_n, following well-known principles set out in [3] and elsewhere. Specifically, for $A = (a_{ij})$, $B = (b_{ij}) \in G'_n$, we denote

(i) by $A \otimes B$ the matrix

$$\left(\bigoplus_{k \in N} \left(a_{ik} \otimes b_{kj} \right) \right)$$

and by $A \oplus B$ the matrix

$$\left(a_{ij} \oplus b_{ij} \right),$$

where between elements of G', $\oplus$ stands for minimum and $\otimes$ for $+$;

(ii) by A^s (s integer) the (necessarily associative) iterated product

$$\underbrace{A \otimes A \otimes \cdots \otimes A}_{s \text{ times}};$$

(iii) by $\Gamma(A)$ the matrix $A \oplus A^2 \oplus \cdots \oplus A^n$;

(iv) $a + A = (a + a_{ij})$ for $a \in G'$.

It is well known that the elements of A^s express the weights of lightest paths consisting of s arcs between any two nodes of the complete n-node digraph Δ_A the arcs of which are weighted by the elements of A, and that the Floyd-Warshall algorithm (see e.g. [5]) applied to A gives as a result $\Gamma(A)$ in $O(n^3)$ operations.

LEMMA. *If Δ_A does not contain cycles with negative weight, then*

$$A^s \geqslant \Gamma(A) \qquad \text{for all} \quad s \geqslant 1. \tag{7}$$

Proof. This follows directly from the results in [3]. ∎

It follows from this lemma that $\Gamma(A)$ expresses the weights of the lightest paths in Δ_A of an arbitrary length whenever Δ_A does not contain cycles with negative weight.

4. THE CYCLIC-GROUP CASE

If $k \geqslant 1$ is an integer and $a \in G$, then we define

$$ka = \underbrace{a + a + \cdots + a}_{k \text{ times}}, \qquad 0a = 0, \quad \text{and} \quad (-k)a = -ka.$$

In the rest of the paper we suppose that $\mathscr{G}$ is a nontrivial cyclic group, i.e., there exists $g \in G$, $g > 0$ such that

$$G = \{kg; k \text{ integer}\}.$$

PROPOSITION 4.1. *If* $A, B \in G_n$ *are directly similar*, B *is strictly normal, and* $\sigma \in C_n$, $l(\sigma) \geqslant 2$, *then*

$$w_A(\sigma) \geqslant l(\sigma) g.$$

Proof. Suppose that

$$a_{ij} = u_i + b_{ij} - u_j$$

for all $i, j \in N$ and some $u_1, \ldots, u_n \in G$. Then

$$\begin{aligned} &a_{i_1 i_2} + \cdots + a_{i_{k-1} i_k} + a_{i_k i_1} \\ &\qquad = u_{i_1} + b_{i_1 i_2} - u_{i_2} + u_{i_2} + b_{i_2 i_3} - u_{i_3} + \cdots + u_{i_k} + b_{i_k i_1} - u_{i_1} \\ &\qquad = b_{i_1 i_2} + \cdots + b_{i_k i_1} \geqslant kg \qquad \text{for any} \quad (i_1, \ldots, i_k) \in C_n. \qquad \blacksquare \end{aligned}$$

It is known that any sum of the form $a_{j_1 j_2} + a_{j_2 j_3} + \cdots + a_{j_r j_1}$, where $j_1, \ldots, j_r \in N$, can be decomposed into a sum

$$w_A(\sigma_1) + w_A(\sigma_2) + \cdots + w_A(\sigma_t)$$

where $\sigma_1, \ldots, \sigma_t \in C_n$ and $l(\sigma_1) + \cdots + l(\sigma_t) = r$.

PROPOSITION 4.2. *If $A, B \in G_n$ are directly similar, B is strictly normal, and $\Gamma(-g + \tilde{A}) = (g_{ij})$, then $g_{ii} \geqslant 0$ for all $i \in N$.*

Proof. It follows from the definitions that

$$g_{ii} = \min_{k \in N} d_{ii}^{(k)}$$

for all $i \in N$, where $d_{ij} = -g + \tilde{a}_{ij}$ for all $i, j \in N$,and $D = (d_{ij})$, $D^k = (d_{ij}^{(k)})$. Hence it suffices to show that $d_{ii}^{(k)} \geqslant 0$ for all $i, k \in N$. Taking arbitrary $i \in N$, we have either $d_{ii}^{(k)} = \infty \geqslant 0$ (which holds only for $k = 1$), or

$$\begin{aligned} \infty > d_{ii}^{(k)} &= \min\{d_{ij_1} + \cdots - d_{j_{k-1}i};\, j_1, \dots, j_k \in N\} \\ &= d_{ir_1} + \cdots + d_{r_{k-1}i} \end{aligned}$$

for some $r_1, \dots, r_{k-1} \in N$, implying that the last sum does not contain any diagonal element. Hence it can be decomposed into the sum of weights of cycles $\sigma_1, \dots, \sigma_t$ of length 2 or more, i.e., by Proposition 4.1 we have

$$\begin{aligned} d_{ii}^{(k)} &= -kg + a_{ir_1} + \cdots + a_{r_{k-1}i} \\ &= -kg + w_A(\sigma_1) + \cdots + w_A(\sigma_t) \\ &\geqslant -kg + l(\sigma_1)g + \cdots + l(\sigma_t)g \\ &= 0, \end{aligned}$$

where the last equality follows from the relation

$$l(\sigma_1) + \cdots + l(\sigma_t) = k. \qquad \blacksquare$$

THEOREM 3. *Let $A \in G_n$ be normal. Then A is regular if and only if all the diagonal elements of $\Gamma(-g + \tilde{A})$ are nonnegative.*

Proof. The "only if" part is an immediate corollary of Propositions 3.1 and 4.2.

To prove the "if" statement it suffices by Proposition 3.1 to show that $A \approx B$, B strictly normal. Denote $-g + \tilde{A}$ by $D = (d_{ij})$. Hence, $w_D(\sigma) \geqslant 0$ for every $\sigma \in C_n$, and it follows now from the earlier lemma that $D^{n+1} \geqslant \Gamma(D)$. Consequently,

$$D \otimes \Gamma(D) = D^2 \oplus \cdots \oplus D^{n+1} \geqslant \Gamma(D). \tag{8}$$

Denote $\Gamma(D) = (\gamma_{ij})$. If $i, j \in N$, $i \neq j$, then $\gamma_{ii} \leqslant d_{ik} + d_{ki} < \infty$ for any $k \neq i$ and $\gamma_{ij} \leqslant d_{ij} = -g + a_{ij} < \infty$. Hence all elements of $\Gamma(D)$ are from G. Take any column $v = (v_1, \ldots, v_n)^T$ (say) of $\Gamma(D)$, and consider vectors as one-column matrices. Then we have by (8)

$$D \otimes v \geqslant v$$

or, equivalently,

$$-v_i + d_{ij} + v_j \geqslant 0$$

for all $i, j \in N$. Hence for all $i, j \in N$, $i \neq j$, we have

$$-v_i + a_{ij} + v_j \geqslant g > 0$$

and

$$-v_i + a_{ii} + v_i = a_{ii} = 0.$$

The matrix $B = (b_{ij})$ with

$$b_{ij} = -v_i + a_{ij} + v_j \tag{9}$$

is strictly normal, and $A \approx B$. ■

Now we summarize the method for checking regularity of $A \in G_n$ in the cyclic case, and for finding a strictly 0-astic matrix similar to A (if any):

1. Find a solution to the AP for A by the Hungarian method [as the result we have a 0-astic matrix $B = (b_{ij})$ and $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ such that $b_{ij} = \alpha_i + a_{ij} + \beta_j$ for all $i, j \in N$].
2. Permute the columns of B so that for the obtained matrix C we have $\mathrm{id} \in \mathrm{ap}(C)$ (comment: C is normal).
3. Check whether $\mathrm{ap}(C) = \{\mathrm{id}\}$ by the algorithm for testing the acyclicity of D_C. If not, then stop (A is not regular).
4. Put $D = -g + \tilde{C}$, and compute $\Gamma(D) = (g_{ij})$ by the Floyd-Warshall algorithm. If $g_{ii} \geqslant 0$ for all $i \in N$, then go to step 5, else stop (A is not regular).
5. Take any column $v = (v_1, \ldots, v_n)^T$ of $\Gamma(D)$, and put $\alpha_i' = \alpha_i - v_i$, $\beta_j' = \beta_j + v_{\pi^{-1}(j)}$ for all $i, j \in N$. The matrix $A' = (\alpha_i' + a_{ij} + \beta_j')$ is the wanted strictly 0-astic matrix similar to A. Stop.

The worst-case performance bounds for the number of operations in steps 1–5 are $O(n^3)$, $O(n^2)$, $O(n^2)$, $O(n^3)$, $O(n^2)$. Hence our method is $O(n^3)$. Note that step 3 can be omitted; however, it is $O(n^2)$ and it saves $O(n^3)$ operations in the negative case.

EXAMPLE 2 (Continued). Since $g = 1$, we have

$$D = -g + \tilde{C} = \begin{pmatrix} \infty & 7 & 5 & 7 \\ 3 & \infty & 1 & -1 \\ -1 & 5 & \infty & -1 \\ -1 & 5 & 1 & \infty \end{pmatrix}$$

and

$$\Gamma(D) = \begin{pmatrix} 3 & 7 & 5 & 4 \\ -2 & 4 & 0 & -1 \\ -2 & 4 & 0 & -1 \\ -1 & 5 & 1 & 0 \end{pmatrix}.$$

Hence D (and A) are regular. Take $v = (3, -2, -2, -1)^T$ (say). Hence, denoting $v_{\pi^{-1}} = (v_{\pi^{-1}(1)}, \dots, v_{\pi^{-1}(n)})^T$, we put

$$\alpha' = \alpha - v = (-5, 2, 2, -1)^T,$$

$$\beta' = \beta + v_{\pi^{-1}} = (-4, -2, -5, 1)^T.$$

Actually,

$$A' = (\alpha'_i + a_{ij} + \beta'_j) = \begin{pmatrix} 3 & 1 & 4 & 0 \\ 0 & 2 & 1 & 9 \\ 6 & 0 & 1 & 5 \\ 5 & 1 & 0 & 4 \end{pmatrix}$$

is a strictly 0-astic matrix similar to A.

On the other hand, A is not regular in the additive group of even integers, because here $g = 2$ and

$$D = -g + \tilde{C} = (d_{ij}) = \begin{pmatrix} \infty & 6 & 4 & 6 \\ 2 & \infty & 0 & -2 \\ -2 & 4 & \infty & -2 \\ -2 & 4 & 0 & \infty \end{pmatrix}.$$

Thus $d_{34} + d_{43} = -2$, implying that g_{33} (as well as g_{44}) is negative.

In conclusion, we remark on some interesting similarities between the present paper and [1], although of course the algebraic structure and the application field are quite different.

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