Permuted linear system problem and permuted eigenvector problem are NP-complete

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Abstract

Using a polynomial transformation from PARTITION we prove that the following modifications of two basic linear-algebraic problems are NP-complete:

(1) Given $A \in \mathbb{Z}^{m\times n}$ and $b \in \mathbb{Z}^m$, is it possible to permute the components of b so that for the obtained vector b' the system $Ax = b'$ has a solution?

(2) Given $A \in \mathbb{Z}^{n\times n}, \lambda \in \sigma(A) \cap \mathbb{Z}$ and $x \in \mathbb{Z}^n$, is it possible to permute the components of x so that the obtained vector x' is an eigenvector of A corresponding to λ?

The second problem is polynomially solvable for positive matrices if λ is the Perron root, however the complexity remains unresolved for general non-negative matrices.

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1 Problem formulation

For a positive integer n the symbol P_n stands for the set of all permutations of the set $N = \{1, ..., n\}$. For $\pi \in P_n$ and $x = (x_1, ..., x_n)^T \in \mathbb{R}^n$ we denote by $x(\pi)$ the vector $\left(x_{\pi(1)}, ..., x_{\pi(n)}\right)^T$. In what follows the letters m, n will denote positive integers.

We analyse the following two problems.

Permuted linear system problem (PLS): Given $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$, is there a $\pi \in P_m$ such that the system

$$Ax = b(\pi)$$

has a solution?

Permuted eigenvector problem (PEV): Given $A \in \mathbb{R}^{n\times n}, \lambda \in \sigma(A)$ and $x \in \mathbb{R}^n$, is there a $\pi \in P_n$ such that $x(\pi)$ is an eigenvector of A corresponding to the eigenvalue λ?

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The aim of this paper is to prove that the integer versions of these two problems are NP-complete. In fact we will prove that both problems are NP-complete even for square matrices with entries from $T = \{0, 1, -1\}$ and (for PEV) when $\lambda = 1$. More precisely we consider the following:

Integer permuted linear system problem (IPLS): Given $A \in T^{n\times n}$ and $b \in \mathbb{Z}^n$, is there a $\pi \in P_n$ such that the system

$$Ax = b\,(\pi)$$

has a solution?

Integer permuted eigenvector problem (IPEV): Given $A \in T^{n\times n}$ and $x \in \mathbb{Z}^n$, is there a $\pi \in P_n$ such that

$$Ax(\pi) = x(\pi)?$$

In both cases the polynomial transformation will be constructed using the following basic NP-complete problem [2]:

PARTITION: Given positive integers $a_1, a_2, ..., a_n$, is there a subset $S \subseteq N$ such that

$$\sum_{j\in S} a_j = \sum_{j\in N-S} a_j? \tag{1}$$

Note that both $IPLS \in NP$ and $IPEV \in NP$ are easily verified.

Theorem 1.1 *IPLS is NP-complete.*

Theorem 1.2 *IPEV is NP-complete.*

2 Proof of Theorem 1.1

Let $a_1, a_2, ..., a_n$ be an instance of PARTITION and define

$$A = \begin{pmatrix} 1 & & & & & & 0 \\ & \ddots & & & & & \vdots \\ & & 1 & & & & 0 \\ & & & 1 & & & 0 \\ & & & & \ddots & & \vdots \\ & & & & & 1 & 0 \\ 1 & \cdots & 1 & -1 & \cdots & -1 & 0 \end{pmatrix} \tag{2}$$

(the blank entries are 0) and

$$b = (b_1, ..., b_{2n+1})^T = (a_1, a_2, ..., a_n, 0, ..., 0)^T$$

where A is of order $2n+1$, the first n entries in the last row of A are 1, and the next n entries are -1. Clearly, $r\,(A) = 2n$ and for any $\pi \in P_n$ we can easily transform the extended matrix of the system (with $b(\pi)$ standing in the last column) to the row-echelon form

$$\left(\begin{array}{ccccccc|c} 1 & & & & & & 0 & b_{\pi(1)} \\ & \ddots & & & & & 0 & b_{\pi(2)} \\ & & 1 & & & & \vdots & \vdots \\ & & & 1 & & & & \\ & & & & \ddots & & & \\ & & & & & 1 & & \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & v \end{array}\right)$$

where

$$\begin{aligned} v &= -\sum_{1\le j\le n} b_{\pi(j)} + \sum_{n+1\le j\le 2n+1} b_{\pi(j)} = \\ &= -\sum_{1\le \pi^{-1}(i)\le n} b_i + \sum_{n+1\le \pi^{-1}(i)\le 2n+1} b_i \end{aligned}$$

or, equivalently

$$v = -\sum_{i\in S} a_i + \sum_{i\in N-S} a_i$$

where $S = \left\{i \in N; \pi^{-1}(i) \le n\right\}$. Clearly, $Ax = b$ has a solution if and only if $v = 0$, that is if and only if

$$\sum_{i\in S} a_i = \sum_{i\in N-S} a_i.$$

Note that the expressions on either side are non-vacant as all a_i are positive. Hence the set $S = \left\{i \in N; \pi^{-1}(i) \le n\right\}$ is a solution to PARTITION once π is a solution to IPLS for A, b.

Conversely, if (1) holds for some S then for π we can take any permutation from P_{2n+1} such that $\pi^{-1}(i) \le n$ for all $i \in S$ and $\pi^{-1}(i) \ge n+1$ for all $i \in N - S$.

The construction of the instance of IPLS is clearly $O(n^2)$, thus the transformation is polynomial.

3 Proof of Theorem 1.2

Let $a_1, a_2, ..., a_n$ be an instance of PARTITION. Let A be the matrix defined by (2) and

$$x = (x_1, ..., x_{2n+1})^T = (a_1, a_2, ..., a_n, 0, ..., 0)^T.$$

Note that $1 \in \sigma(A)$. Let $\pi \in P_n$, then $Ax(\pi) = x(\pi)$ comprises $2n$ trivial identities of the form $x_{\pi(i)} = x_{\pi(i)}$ and the following equation:

$$\sum_{1\le \pi^{-1}(i)\le n} x_i - \sum_{n+1\le \pi^{-1}(i)\le 2n+1} x_i = 0.$$

We deduce that $Ax(\pi) = x(\pi)$ if and only if

$$\sum_{i \in S} a_i = \sum_{i \in N-S} a_i$$

where $S = \left\{i \in N; \pi^{-1}(i) \leq n\right\}$. Hence the set $S = \left\{i \in N; \pi^{-1}(i) \leq n\right\}$ is a solution to PARTITION once π is a solution to IPEV for A.

Conversely, if (1) holds for some S then for π we can take any permutation from P_{2n+1} such that $\pi^{-1}(i) \leq n$ for all $i \in S$ and $\pi^{-1}(i) \geq n+1$ for all $i \in N - S$.

The construction of the instance of IPEV is clearly $O(n^2)$, thus the transformation is polynomial.

4 Some related question

As usual some special cases are easily solvable.

If $A \in \mathbb{R}^{n \times n}$ and $r(A) = n$ then the answer to PLS is always yes, thus PLS is solvable in $O(1)$ time in this case. Similarly if $r(A) = 1$ then the column space of A is formed by multiples of a single vector z (any non-zero column of A can be used). Therefore, PLS can be solved by ranking the components of z and b in the same way. A simple check then verifies whether they are multiples of each other.

For similar reasons PEV is easily solvable if the eigenspace corresponding to the given eigenvalue is one-dimensional, in particular for eigenvectors corresponding to the Perron root of positive matrices. For non-negative matrices this question remains open since the eigenspace corresponding to the Perron root may be multi-dimensional and the matrix used in the proof of Theorem 1.2 is not non-negative.

Note that the max-algebraic versions of both PLS and PEV have also been proved to be NP-complete [1].

References

[1] P. Butkovic, Permuted max-algebraic (tropical) eigenvector problem is NP-complete, Linear Algebra and its Applications 428 (2008) 1874-1882.

[2] M.R. Garey, D.S. Johnson, Computers and Intractability, A Guide to the Theory of NP-completeness, Bell Labs (1979).