Permuted linear system problem and permuted eigenvector problem are \( NP \)-complete

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Abstract

Using a polynomial transformation from \textsc{Partition} we prove that the following modifications of two basic linear-algebraic problems are \( NP \)-complete:

(1) Given \( A \in \mathbb{R}^{m \times n} \) and \( b \in \mathbb{R}^m \), is it possible to permute the components of \( b \) so that for the arising vector \( b' \) the system \( Ax = b' \) has a solution?

(2) Given \( A \in \mathbb{R}^{n \times n} \), \( \lambda \in \sigma(A) \) and \( x \in \mathbb{R}^n \), is it possible to permute the components of \( x \) so that the arising vector \( x' \) is an eigenvector of \( A \) corresponding to \( \lambda \)?

The second problem is polynomially solvable for positive matrices if \( \lambda \) is the Perron root, however the complexity remains unresolved for general non-negative matrices.

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1 Problem formulation

For a positive integer \( n \) the symbol \( P_n \) stands for the set of all permutations of the set \( N = \{1, \ldots, n\} \). For \( \pi \in P_n \) and \( x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n \) we denote by \( x(\pi) \) the vector \( (x_{\pi(1)}, \ldots, x_{\pi(n)})^T \). In what follows the letters \( m, n \) will denote positive integers.

We analyse the following two problems.

\textbf{Permuted linear system problem (PLS):} Given \( A \in \mathbb{R}^{m \times n} \) and \( b \in \mathbb{R}^m \), is there a \( \pi \in P_m \) such that the system

\[ Ax = b(\pi) \]

has a solution?

\textbf{Permuted eigenvector problem (PEV):} Given \( A \in \mathbb{R}^{n \times n} \), \( \lambda \in \sigma(A) \) and \( x \in \mathbb{R}^n \), is there a \( \pi \in P_n \) such that \( x(\pi) \) is an eigenvector of \( A \) corresponding to the eigenvalue \( \lambda \)?

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The aim of this paper is to prove that the integer versions of these two problems are NP-complete. In fact we will prove that both problems are NP-complete even for square matrices with entries from $T = \{0, 1, -1\}$ and (for PEV) when $\lambda = 1$. More precisely we consider the following:

**Integer permuted linear system problem (IPLS):** Given $A \in T^{n \times n}$ and $b \in \mathbb{Z}^n$, is there a $\pi \in P_n$ such that the system

$$Ax = b(\pi)$$

has a solution?

**Integer permuted eigenvector problem (IPEV):** Given $A \in T^{n \times n}$ and $x \in \mathbb{Z}^n$, is there a $\pi \in P_n$ such that

$$Ax(\pi) = x(\pi)?$$

In both cases the polynomial transformation will be constructed using the following basic NP-complete problem [2]:

**PARTITION**: Given positive integers $a_1, a_2, ..., a_n$, is there a subset $S \subseteq N$ such that

$$\sum_{j \in S} a_j = \sum_{j \in N - S} a_j?$$

(1)

Note that both IPLS $\in$ NP and IPEV $\in$ NP are easily verified.

**Theorem 1.1** IPLS is NP-complete.

**Theorem 1.2** IPEV is NP-complete.

## 2 Proof of Theorem 1.1

Let $a_1, a_2, ..., a_n$ be an instance of PARTITION and define

$$A = \begin{pmatrix}
1 & & & 0 \\
& \ddots & & \\
& & 1 & 0 \\
1 & & & 0 \\
& & & & & & \ddots & \\
& & & & & & 1 & 0 \\
1 & \cdots & -1 & -1 & \cdots & -1 & 0
\end{pmatrix}$$

(2)

(the blank entries are 0) and

$$b = (b_1, ..., b_{2n+1})^T = (a_1, a_2, ..., a_n, 0, ..., 0)^T$$

where $A$ is of order $2n + 1$. Clearly $r(A) = 2n$ and for any $\pi \in P_n$ we can easily transform the extended matrix of the system (with $b(\pi)$ standing in the last
column) to the row-echelon form

\[
\begin{pmatrix}
1 & 0 & b_{\pi(1)} \\
\vdots & \ddots & \vdots \\
1 & \cdots & 1 \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & v
\end{pmatrix}
\]

where

\[v = - \sum_{1 \leq \pi(i) \leq n} b_i + \sum_{n+1 \leq \pi(i) \leq 2n+1} b_i\]

or, equivalently

\[v = - \sum_{i \in S} a_i + \sum_{i \in N \setminus S} a_i\]

where \(S = \{ i \in N; \pi(i) \leq n \}\). Clearly, \(Ax = b\) has a solution if and only if \(v = 0\), that is if and only if

\[\sum_{i \in S} a_i = \sum_{i \in N \setminus S} a_i.\]

Note that the expressions on either side are non-vacant as all \(a_i\) are positive. Hence the set \(S = \{ i \in N; \pi(i) \leq n \}\) is a solution to PARTITION once \(\pi\) is a solution to IPLS for \(A, b\) and conversely, if (1) holds for some \(S\) then for \(\pi\) we can take any permutation from \(P_{2n+1}\) such that \(\pi(i) \leq n\) for all \(i \in S\). The construction of the instance of IPLS is clearly \(O(n^2)\), thus the transformation is polynomial.

3 Proof of Theorem 1.2

Let \(a_1, a_2, \ldots, a_n\) be an instance of PARTITION. Let \(A\) be the matrix defined by (2) and

\[x = (x_1, \ldots, x_{2n+1})^T = (a_1, a_2, \ldots, a_n, 0, \ldots, 0)^T.\]

Note that \(1 \in \sigma(A)\). Let \(\pi \in P_n\), then \(Ax(\pi) = x(\pi)\) comprises \(2n\) trivial identities of the form \(x_{\pi(i)} = x_{\pi(i)}\) and the following equation:

\[\sum_{1 \leq \pi(i) \leq n} x_i - \sum_{n+1 \leq \pi(i) \leq 2n+1} x_i = 0.\]

We deduce that \(Ax(\pi) = x(\pi)\) if and only if

\[\sum_{i \in S} a_i = \sum_{i \in N \setminus S} a_i.\]
where $S = \{ i \in N; \pi (i) \leq n \}$. Hence the set $S = \{ i \in N; \pi (i) \leq n \}$ is a solution to PARTITION once $\pi$ is a solution to IPEV for $A$ and conversely, if (1) for some $S$ then for $\pi$ we can take any permutation from $P_{2n+1}$ such that $\pi (i) \leq n$ for all $i \in S$. The construction of the instance of IPEV is clearly $O(n^2)$, thus the transformation is polynomial.

4 Some related question

As usual some special cases are easily solvable.

If $A \in \mathbb{R}^{n \times n}$ and $r (A) = n$ then the answer to PLS is always yes, thus PLS is solvable in $O(1)$ time in this case. Similarly if $r (A) = 1$ then the column space of $A$ is formed by multiples of a single vector $v$ (any non-zero column of $A$ can be used). PLS can be solved by ranking the components of $v$ and $b$ in the same way. A simple check then verifies whether they are multiples of each other.

For similar reasons PEV is easily solvable if the eigenspace corresponding to the given eigenvalue is one-dimensional, in particular for eigenvectors corresponding to the Perron root of positive matrices. For non-negative matrices this question remains open since the eigenspace corresponding to the Perron root may be multi-dimensional and the matrix used in the proof of Theorem 1.2 is not non-negative.

Note that the max-algebraic versions of both PLS and PEV have also been proved to be $NP$-complete [1].

References
