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A note on the parity assignment problem

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Although the basic version of the linear assignment problem (AP) can be solved very efficiently, there are variants of this problem which are much harder, some being *NP*-complete or with undecided computational complexity. One of them is the parity AP, in which an optimal permutation of a prescribed parity is sought. A related variant is the weak parity AP, in which we only need to know whether the set of optimal permutations to the AP contains permutations of both parities. In this short note, we prove that both these problems are efficiently solvable for Monge matrices, as well as for diagonally dominant symmetric matrices. We also note that the parity bottleneck AP is polynomially solvable for any matrix.

Keywords: Assignment problem; Even and odd permutations; Optimal solution; Monge matrix

1. Introduction

Classical (linear) assignment problem (AP) is the following: Given an $n \times n$ real matrix A find n entries of A, no two belonging to the same row or column, so that their sum is maximal. This problem has numerous applications and belongs to basic combinatorial optimization problems. It has been studied by many authors since the 1950's. A thorough overview of the achievements can be found in [3] and [4]. However, although the basic version of the AP can be solved very efficiently (say by the Hungarian method in $O(n^3)$ steps [14]), there are variants of this problem which are much harder, some being *NP*-complete or with undecided computational complexity. One of them is the *parity AP*: Obviously, n entries of an $n \times n$ matrix, no two belonging to the same row or column, correspond to a permutation of the set $N = \{1, \ldots, n\}$. In the classical AP, no additional conditions are set on the optimal permutation. In the parity AP, this permutation has to be of a prescribed parity. Note that if the additional requirement is that the permutation be cyclic, then the arising task is the well-known (*NP*-complete) travelling salesman problem.

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If $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ and P_n is the set of all permutations of the set N then

$$w(A,\pi) = \sum_{i\in N} a_{i,\pi(i)}$$

for any $\pi \in P_n$. We will use the same notation, if π is a permutation of a subset of N.

Let P_n^+ [P_n^-] be the set of all even [odd] permutations of the set N. Let us denote

$$ap(A) = \left\{\pi^* \in P_n;\, w(A,\pi^*) = \max_{\pi\in P_n} w(A,\pi)\right\},$$

$$ap^+(A) = \left\{\pi^* \in P_n^+;\, w(A,\pi^*) = \max_{\pi\in P_n^+} w(A,\pi)\right\},$$

$$ap^-(A) = \left\{\pi^* \in P_n^-;\, w(A,\pi^*) = \max_{\pi\in P_n^-} w(A,\pi)\right\}.$$

We can now give definitions of the AP and its variants studied in this article. Note that $ap(A) \subseteq P_n^+ \cup P_n^- = P_n$.

Assignment Problem (AP) Given $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, find a permutation $\pi^* \in ap(A)$.

Even Parity Assignment Problem ($EPAP$) Given $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, find a permutation $\pi^* \in ap^+(A)$.

Odd Parity Assignment Problem ($OPAP$) Given $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, find a permutation $\pi^* \in ap^-(A)$.

Parity Assignment Problem (PAP) Given $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, solve both EPAP and OPAP.

Weak Parity Assignment Problem ($WPAP$) Given $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, is there both $ap(A) \cap P_n^+ \neq \emptyset$ and $ap(A) \cap P_n^- \neq \emptyset$? Equivalently, is

$$\max_{\pi\in P_n^+} w(A,\pi) = \max_{\pi\in P_n^-} w(A,\pi)?$$

Note that WPAP plays an important role in recognizing regular matrices in *max-algebra*, which is a theory developed for the pair of operations $(\oplus,\otimes) = (\max,+)$ in the same way as linear algebra [8,9,12]. A matrix $A \in \mathbb{R}^{n\times n}$ with columns $a_1,\ldots,a_n$ is called *regular* (in max-algebra) if real numbers λ_j and two non-empty, disjoint subsets S and T of N, such that

$$\sum_{j\in S}{}^{\oplus} \lambda_j \otimes a_j = \sum_{j\in T}{}^{\oplus} \lambda_j \otimes a_j$$

do not exist. Gondran and Minoux [11] proved the following criterion for the regularity of a matrix, see also [6].

THEOREM 1 *$A \in \mathbb{R}^{n\times n}$ is regular if and only if every permutation yielding the maximum value for the linear AP with cost matrix A has the same parity, that is*

$$\max_{\pi\in P_n^+} w(A,\pi) \neq \max_{\pi\in P_n^-} w(A,\pi).$$

Note that max-algebraic regularity of matrices plays a crucial role in solving the minimal-dimensional realization problem for discrete-event dynamic systems [1,10].

A matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is called *normal* if $a_{ij} \leq a_{ii} = 0$ for all $i, j \in N$. We say that A is *diagonally dominant*, if $id \in ap(A)$. It is obvious that every normal matrix is diagonally dominant. A normal matrix $A \in \mathbb{R}^{n\times n}$ is called a *normal form* of a matrix $B \in \mathbb{R}^{n\times n}$ if there is a constant $z \in \mathbb{R}$ and a 1-1 mapping ω of P_n onto itself, such that

$$w(A, \pi) = z + w(B, \omega(\pi))$$

for every $\pi \in P_n$.

Every $n \times n$ matrix B can be transformed in $O(n^3)$ steps to one of its normal forms by adding constants to the rows and columns of B and using suitable row and/or column permutations (see the Hungarian method [3] and [14]). Hence, there is an easily identifiable 1-1 mapping between $ap(A)$ and $ap(B)$. Since swapping two rows or columns changes the sign of all permutations, there is also such a 1-1 mapping either between $ap^+(A)$ and $ap^+(B)$ and between $ap^-(A)$ and $ap^-(B)$ or between $ap^+(A)$ and $ap^-(B)$ and between $ap^-(A)$ and $ap^+(B)$. Therefore, the Hungarian method solves AP and one of OPAP and EPAP for any matrix in $O(n^3)$ steps. It also implies that when solving any of the earlier mentioned problems for general matrices we may assume without loss of generality that they are in a normal form. Since for a normal matrix id is an optimal solution to both AP and EPAP, PAP for a matrix B reduces to OPAP for a normal form of B. Yet, to the author's knowledge no polynomial method is known for PAP in general.

Note that a diagonally dominant matrix can be transformed to a normal form by adding constants to the rows and/or columns and no permutations of the rows or columns are needed. These constants can be found in a straightforward way, without using the Hungarian method or other method for solving the AP [7]. In this case $ap^+(A) = ap^+(B)$ and $ap^-(A) = ap^-(B)$ and since $id \in ap^+(A)$, PAP reduces to OPAP.

$A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is called *Monge* if

$$a_{ij} + a_{kl} \geq a_{il} + a_{kj}$$

for every i, j, k, l, such that $1 \leq i \leq k \leq n$ and $1 \leq j \leq l \leq n$.

It is well known [2] that every Monge matrix is diagonally dominant. It is easy to see that adding constants to the rows and columns does not change the Monge property. Hence, PAP for Monge matrices reduces to OPAP for matrices which are both Monge and normal.

As usual, $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is called *symmetric*, if $a_{ij} = a_{ji}$ for every $i, j \in N$. Every diagonally dominant symmetric matrix has a symmetric normal form [5]. Hence, PAP for diagonally dominant symmetric matrices reduces to OPAP for matrices which are symmetric and normal.

Note that for any matrix a solution to PAP solves WPAP and that the positive answer to WPAP solves PAP.

It is also easily seen that after transforming a matrix B to its normal form A, WPAP reduces to the question: Is $w(A, \pi) = 0$ for some $\pi \in P_n^-$? The parity of a cyclic permutation is odd, if and only if the cycle it represents contains an even number of nodes. Hence, a necessary condition for the affirmative answer to WPAP (for a normal matrix) is the existence of an even cycle in the digraph $D_A = (N, \{(i, j); a_{ij} = 0\})$.

Conversely, if there is an even cycle in D_A then it can be extended (if necessary) by loops to an odd permutation π of N satisfying $w(A, \pi) = 0$. Thus, we have [6]:

THEOREM 2 *Let A be a normal matrix. Then*

$$\max_{\pi \in P_n^+} w(A, \pi) = \max_{\pi \in P_n^-} w(A, \pi)$$

if and only if the digraph D_A contains an even cycle.

The question whether a given digraph contains an even cycle is known as the *even cycle problem* or, sometimes, as *Pólya's permanent problem*. Its polynomial solvability was unclear for some 30 years, until the late 1990's when a polynomial method was discovered [13,16].

2. PAP for special matrices

For $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ we denote

$$ap^*(A) = \left\{ \pi^* \in P_n - \{id\};\ w(A, \pi^*) = \max_{\pi \in P_n - \{id\}} w(A, \pi) \right\}.$$

Note that by *id* we understand the identity on an appropriate set. For instance, in the next statement it is the identity on $N - \{k, l\}$. Also note that in the statement and proof of the next theorem $k + 1$ is meant mod n. In what follows, the symbol $\circ$ will denote the product (composition) of two permutations.

THEOREM 3 *If $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ is Monge and normal then there exists a permutation $\pi \in ap^*(A)$ and $k \in N$, such that*

$$\pi = (k, k + 1) \circ id.$$

Proof Let $\pi \in ap^*(A)$, $\pi = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_s$ where $\pi_1, \pi_2, \ldots, \pi_s$ are permutations of subsets of N. Then there is a $t \in \{1, \ldots, s\}$, such that

$$w(A, \pi_i) = 0 \quad \text{for all } i = 1, \ldots, s; \quad i \neq t.$$

Hence, without loss of generality

$$\pi_i = (i) \quad \text{for all } i = 1, \ldots, s; \quad i \neq t.$$

Let $\pi_t = (j_1, j_2, \ldots, j_p)$ and suppose that $p > 2$. For convenience of this discussion and without loss of generality, suppose that $j_2 = \max\{j_1, j_2, \ldots, j_p\}$. Then

$$a_{j_1 j_3} = a_{j_1 j_3} + a_{j_2 j_2} \geq a_{j_1 j_2} + a_{j_2 j_3}.$$

Hence, if $\pi_t' = (j_1, j_3, \ldots, j_p)$ then $w(\pi_t', A) \geq w(\pi_t, A)$ and thus, $w(\pi_t', A) = w(\pi_t, A)$, yielding that $\pi' = \pi_t' \circ id$ is in $ap^*(A)$. By repeating this argument $p - 2$ times we find an element of $ap^*(A)$ which is the product of a cycle of length 2 and *id*. Let us denote this permutation again by π, so that $\pi = (k, l) \circ id$ for some $k, l \in N$. Let without loss of generality $k < l$. If $l = k + 1$ then the theorem statement follows. If $l > k + 1$ then since A is Monge, we have

$$a_{k,k+1} + a_{k+1,l} \geq a_{kl} + a_{k+1,k+1} = a_{kl}.$$

Hence,

$$a_{k,k+1} \geq a_{kl}.$$

By a similar argument, it is easily proved that $a_{k+1,k} \geq a_{lk}$. Therefore, $a_{k,k+1} + a_{k+1,k} \geq a_{kl} + a_{lk}$, which implies that the permutation $(k, k+1) \circ id \in ap^*(A)$. ■

The permutation π in the statement of Theorem 3 is odd and therefore is a solution to OPAP. This enables also to solve PAP and WPAP. Hence, all these problems can be solved by checking

$$a_{k,k+1} + a_{k+1,k} = a_{k,k} + a_{k+1,k+1}$$

for all $k = 1, \ldots, n-1$. This requires $O(n)$ operations.

THEOREM 4 *If $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ is symmetric and normal then there exists a permutation $\pi \in ap^*(A)$ and $k, l \in N$, such that $\pi = (k, l) \circ id$.*

Proof Take any $\sigma \in ap^*(A)$ and let $\max\{a_{i,\sigma(i)}; i \in N\} = a_{kl}$. Set $\pi = (k, l) \circ id$. Then $w(A, \pi) = a_{kl} + a_{lk} = 2a_{kl} \geq w(A, \sigma)$. Thus, $\pi \in ap^*(A)$ and the theorem statement follows. ■

The permutation π in Theorem 4 is odd and therefore is a solution to OPAP. It can be found by maximizing $2a_{kl} + \sum_{i \neq k,l} a_{ii}$. Hence, using the pre-computation of all $\binom{n}{2} = O(n^2)$ sums $\sum_{i \neq k,l} a_{ii}$, the OPAP (and therefore also PAP and WPAP) for symmetric, diagonally dominant matrices can be solved in $O(n^2)$ steps.

3. Parity bottleneck assignment problem

The bottleneck assignment problem (BAP) differs from the classical one by the definition of the weight of a permutation:

$$w(A, \pi) = \min_{i \in N} a_{i,\pi(i)}.$$

If w^* is an optimal value to BAP, that is

$$w^* = \max_{\pi \in P_n} w(A, \pi) = \max_{\pi \in P_n} \min_{i \in N} a_{i,\pi(i)}$$

then there is a permutation $\pi \in P_n$ such $a_{i,\pi(i)} \geq w$ for $w = w^*$ but not for any $w > w^*$. Since

$$w^* \in \{a_{ij}; i, j \in N\},$$

there are no more than n^2 possible values for w^*. Therefore, BAP can be solved by examining for all these values w whether

$$a_{i,\pi(i)} \geq w \quad (i = 1, \ldots, n)$$

holds for some $\pi \in P_n$. This condition easily translates to the question of the existence of a perfect matching in the bipartite graph $B_w = (U \cup V, E_w)$ with $U = \{u_1, \ldots, u_n\}$, $V = \{v_1, \ldots, v_n\}$, $E_w = \{(u_i, v_j); a_{ij} \geq w\}$. There is an $O(n^{2.5}/\sqrt{\log n})$ algorithm for solving the bipartite maximum matching problem and thus an $O(n^{4.5}/\sqrt{\log n})$ algorithm for solving BAP immediately follows. The search of up to n^2 candidates for the optimal value can be simplified by first sorting all n^2 entries in $O(n \log n)$ time and by using a binary search which then only needs $O(\log n^2) = O(\log n)$ steps, yielding total complexity improved to $O(n^{2.5}\sqrt{\log n})$. In fact, a more sophisticated approach [15] leads to an algorithm of complexity $O(n^{2.5})$.

The definitions of OPAP, EPAP, PAP and WPAP are formally repeated for BAP, yielding OPBAP, EPBAP, PBAP and WPBAP. For the same reasons as before we may concentrate on OPBAP for normal matrices. It may be interesting to notice here that unlike for the classical AP, in the case of BAP all these problems are polynomially solvable, as now we only have to recognize the biggest value of w for which

$$a_{i,\pi(i)} \geq w \quad (i = 1, \ldots, n)$$

holds for some $\pi \in P_n^-$. If $c(n)$ is the computational complexity of a (polynomial) algorithm for solving the even cycle problem then this can be done (using pre-ordering of the entries and binary search) in $O(c(n)\log n)$ operations (we assume here that $c(n) \geq n^3$).

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