Reducible spectral theory with applications to the robustness of matrices in max algebra

P.Butković‡ R.A.Cuninghame-Green‡ S.Gaubert§

September 2, 2009

Abstract

Let \(a \oplus b = \max(a, b)\) and \(a \otimes b = a + b\) for \(a, b \in \mathbb{R} := \mathbb{R} \cup \{-\infty\}\). By max-algebra we understand the analogue of linear algebra developed for the pair of operations \((\oplus, \otimes)\), extended to matrices and vectors. The symbol \(A^k\) stands for the \(k\)th max-algebraic power of a square matrix \(A\). Let us denote by \(\varepsilon\) the max-algebraic "zero" vector, all the components of which are \(-\infty\). The max-algebraic eigenvalue-eigenvector problem is the following: Given \(A \in \mathbb{R}^{n \times n}\), find all \(\lambda \in \mathbb{R}\) and \(x \in \mathbb{R}^n, x \neq \varepsilon\) such that \(A \otimes x = \lambda \otimes x\). Certain problems of scheduling lead to the following question: Given \(A \in \mathbb{R}^{n \times n}\), is there a \(k\) such that \(A^k \otimes x\) is a max-algebraic eigenvector of \(A\)? If the answer is affirmative for every \(x \neq \varepsilon\) then \(A\) is called robust. First, we give a complete account of the reducible max-algebraic spectral theory and then we apply it to characterize robust matrices.

AMS classification: 15A18
Keywords: Max-algebra; Reducible matrix; Eigenspace.

1 Introduction

Let \(a \oplus b = \max(a, b)\) and \(a \otimes b = a + b\) for \(a, b \in \mathbb{R} := \mathbb{R} \cup \{-\infty\}\). Obviously, \(-\infty\) plays the role of a neutral element for \(\oplus\). Throughout the paper we denote \(-\infty\) by \(\varepsilon\) and for convenience we also denote by the same symbol the max-algebraic "zero" vector, whose all components are \(-\infty\) or a matrix whose all components are \(-\infty\). If \(a \in \mathbb{R}\) then the symbol \(a^{-1}\) stands for \(-a\).

‡Corresponding author
1School of Mathematics, University of Birmingham, Edgbaston, Birmingham B15 2TT, United Kingdom, p.butkovic@bham.ac.uk.
2School of Mathematics, University of Birmingham, Edgbaston, Birmingham B15 2TT, United Kingdom
3INRIA and Centre de Mathématiques Appliquées, École Polytechnique, 91128 Palaiseau Cedex, France, Stephane.Gaubert@inria.fr.
By max-algebra we understand the analogue of linear algebra developed for the pair of operations $(\oplus, \otimes)$, extended to matrices and vectors. That is if $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$ are matrices of compatible sizes with entries from $\mathbb{R}$, we write $C = A \oplus B$ if $c_{ij} = a_{ij} \oplus b_{ij}$ for all $i, j$ and $C = A \otimes B$ if $c_{ij} = \sum_k a_{ik} \otimes b_{kj} = \max_k (a_{ik} + b_{kj})$ for all $i, j$. If $\alpha \in \mathbb{R}$ then $\alpha \otimes A = (\alpha \otimes a_{ij})$.

If $A$ is a square matrix then the iterated product $A \otimes A \otimes \ldots \otimes A$ in which the symbol $A$ appears $k$-times will be denoted by $A^k$. By definition $A^0 = I$ where $I$ is the matrix with diagonal entries 0 and off-diagonal entries $\varepsilon$. Obviously, $A \otimes I = I \otimes A = A$ whenever $A$ and $I$ are of compatible sizes.

The max-algebraic eigenvalue-eigenvector problem (briefly eigenproblem) is the following:

Given $A \in \mathbb{R}^{n \times n}$, find all $\lambda \in \mathbb{R}$ (eigenvalues) and $x \in \mathbb{R}^n, x \neq \varepsilon$ (eigenvectors) such that

$$A \otimes x = \lambda \otimes x.$$

This problem has been studied since the work of R.A. Cuninghame-Green [15]. One of the motivations was the analysis of the steady-state behaviour of the following multi-machine interactive production systems: Suppose that machines $M_1, \ldots, M_n$ work interactively and in stages. In each stage all machines simultaneously produce components necessary for the next stage of some or all other machines. Let $x_i(k)$ denote the starting time of the $k^{th}$ stage on machine $i$ $(i = 1, \ldots, n)$ and let $a_{ij}$ denote the duration of the operation at which machine $M_j$ prepares the component necessary for machine $M_i$ in the $(k + 1)^{st}$ stage $(i, j = 1, \ldots, n)$. Then

$$x_i(k + 1) = \max (x_i(k) + a_{i1}, \ldots, x_n(k) + a_{in}) \ (i = 1, \ldots, n; k = 0, 1, \ldots)$$

or, in max-algebraic notation

$$x(k + 1) = A \otimes x(k) \ (k = 0, 1, \ldots)$$

where $A = (a_{ij})$ is called a production matrix. More generally, systems of this kind are known to represent a class of discrete event systems [4]. We say that the system reaches a steady state if it eventually moves forward in regular steps, that is if for some $\lambda$ and $k_0$ we have $x(k + 1) = \lambda \otimes x(k)$ for all $k \geq k_0$. Obviously, a steady state is reached immediately if $x(0)$ is an eigenvector of $A$ corresponding to an eigenvalue $\lambda$, which can be interpreted as the time between consecutive events. However, if the choice of a start-time vector is restricted we may need to find out for which vectors a steady state will eventually be reached. Since $x(k) = A^k \otimes x(0)$ for every natural $k$, this question reads:

Given $A \in \mathbb{R}^{n \times n}$ and $x \in \mathbb{R}^n$ is there a natural number $k$ such that $A^k \otimes x$ is an eigenvector of $A$?

In particular, it may be of practical interest to characterize matrices for which a steady state is reached with any start-time vector, that is matrices $A \in \mathbb{R}^{n \times n}$ for which the following is true:

For every $x \in \mathbb{R}^n, x \neq \varepsilon$ there is a natural number $k$ such that $A^k \otimes x$ is an eigenvector of $A$.  

2
This property has been considered by P. Butkovič and R.A. Cuninghame-
Green [11] who called it robustness. Indeed, the system is robust when the
existence of an ultimate stationary regime is insensitive to the choice of initial
conditions. This is in accordance with the use of this term in control theory
where "robustness" generally indicates the insensitivity of certain performance
measures or qualitative properties to various types of perturbations or uncertain-
matrices in the important case of irreducible matrices (for the definition of irre-
ducible matrices see Section 2). The main aim of the current paper is to extend
these results to general (reducible) matrices.

In the language of dynamical systems, the robustness property requires every
orbit of \( x(k) = A \otimes x(k-1) \) to converge to a fixed point, modulo the addition
of a constant. Besides the motivation from discrete event systems, the study
of this property is motivated by basic questions in the theory of nonexpansive
mappings, in which the structure of the periodic orbits has received considerable
attention, see in particular [27], [29], [2], [26]. Max-algebraic linear maps are
special cases of nonexpansive mappings in Hilbert’s projective metric, and one
may try, more generally, to find conditions which guarantee that every orbit of
a nonexpansive mapping converges to a fixed point. In the present paper, we
address this problem in the special case of max-algebraic linear maps, which is of
particular interest since it may be thought of as the simplest case in which strict
contraction techniques can be applied. We note that it might be interesting to
generalize the present results to other classes of nonexpansive mappings.

The robustness problem is also of interest in relation to the power algorithm
introduced by Braker and Olsder. This algorithm computes an orbit \( A^k \otimes x \)
\( k = 0, 1, ..., \) for a given initial vector \( x \), and at each step, checks whether
\( A^k \otimes x \) is proportional in the max-algebraic sense to some \( A^m \otimes x \) with
\( m < k \). The robustness property identifies a situation in which the power algorithm
does terminate, and then the latter test can be simplified by considering only
\( m = k - 1 \).

Note that when a reducible matrix is not robust, its orbit can have a more
complex behavior, with interleaving arithmetical sequences [19], [32].

The characterization of robustness in [11] substantially relies on the max-
algebraic spectral theory. A full solution of the eigenproblem in the case of
irreducible matrices has been presented by R.A. Cuninghame-Green [16], [17]
and M. Gondran and M. Minoux [23], see also N.N. Vorobyov [33].

The general (reducible) case considered in the present paper requires de-
tailed information about the spectral problem for reducible matrices. A general
spectral theorem for reducible matrices was presented by S. Gaubert [18] and R.
Bapat, D. Stanford and P. van den Driessche [6]. Some of the results of [6] were
stated (without proofs) in R.B. Bapat, D. Stanford and P. van den Driessche
[7]. Additional results can be found in the work of M. Akian, S. Gaubert and
C. Walsh [3] where the emphasis is on the denumerable case. A survey, again
without proofs, appeared in [1].

Since there is currently no complete account of the reducible spectral theory
in journals or books, we give in Section 3 a systematic presentation of this
theory. The ideas of the proofs in this theory are instrumental for the proofs of the results on robustness of reducible matrices, which constitute the main aim of the paper. It should be noted, however, that reducible spectral theory is of general interest (independently of the application that we consider here) due to its remarkable connections to the Perron-Frobenius theory of reducible (non-negative) matrices. It is also of importance in the analysis of discrete-event systems [4], [14], [20].

In Section 4 we give answers to some specific questions related to the finiteness of the eigenvectors. A comparison with the theory of non-negative matrices in conventional linear algebra is made (Remark 4.2). We also show how to efficiently find a basis of the eigenspace corresponding to an eigenvalue. These results are used in Section 5 to provide a characterization of robustness for reducible matrices, thus completing the solution of this question for all $A \in \mathbb{R}^{n \times n}$. This characterization is presented in the main result of the paper, Theorem 5.5.

## 2 Notation, definitions and preliminary results

Unless stated otherwise, we assume everywhere in this paper that $n \geq 1$ is an integer, $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ and $\lambda \in \mathbb{R}$. We denote by $V(A, \lambda)$ the set containing $\varepsilon$ and all eigenvectors of $A$ corresponding to $\lambda \in \mathbb{R}$, by $V(A)$ the set containing $\varepsilon$ and all eigenvectors of $A$ and by $\Lambda(A)$ the set of all eigenvalues of $A$. The sets $V^+(A, \lambda)$ and $V^+(A)$ contain finite eigenvectors of $A$ corresponding to $\lambda \in \mathbb{R}$ and all finite eigenvectors of $A$, respectively.

Note that if $A = \varepsilon$ then $\Lambda(A) = \{\varepsilon\}$ and $V(A) = \mathbb{R}^{n}$.

An ordered pair $D = (N, F)$ is called a digraph if $N$ is a non-empty set (of nodes) and $F \subseteq N \times N$ (the set of arcs). A sequence $\pi = (v_1, \ldots, v_p)$ of nodes is called a path (in $D$) if $p = 1$ or $p > 1$ and $(v_i, v_{i+1}) \in F$ for all $i = 1, \ldots, p - 1$. The node $v_1$ is called the starting node and $v_p$ the end node of $\pi$, respectively. If there is a path in $D$ with starting node $u$ and end node $v$ then we say that $v$ is reachable from $u$, notation $u \to v$. Thus $u \to u$ for any $u \in N$. As usual a digraph $D$ is called strongly connected if $u \to v$ and $v \to u$ for any nodes $u, v$ in $D$. A path $(v_1, \ldots, v_p)$ is called a cycle if $v_1 = v_p$ and $p > 1$ and it is called an elementary cycle if, moreover, $v_i \neq v_j$ for $i, j = 1, \ldots, p - 1, i \neq j$.

In the rest of the paper $N = \{1, \ldots, n\}$. The digraph associated with $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ is $D_A = (N, \{(i, j); a_{ij} > \varepsilon\})$.

The matrix $A$ is called irreducible if $D_A$ is strongly connected, reducible otherwise. Thus, every $1 \times 1$ matrix is irreducible.

If $\pi = (i_1, \ldots, i_p)$ is a path in $D_A$ then the weight of $\pi$ is $w(\pi, A) = a_{i_1i_2} + a_{i_2i_3} + \ldots + a_{i_{p-1}i_p}$ if $p > 1$ and $\varepsilon$ if $p = 1$. The symbol $\lambda(A)$ stands for the maximum cycle mean of $A$, that is if $D_A$ has at least one cycle then

$$\lambda(A) = \max_\sigma \mu(\sigma, A), \quad (1)$$
where the maximization is taken over all cycles in $D_A$ and

$$
\mu(\sigma, A) = \frac{w(\sigma, A)}{k}
$$

(2)

denotes the mean of the cycle $\sigma = (i_1, ..., i_k, i_1)$. Note that $\lambda(A)$ remains unchanged if the maximization in (1) is taken over all elementary cycles. If $D_A$ is acyclic we set $\lambda(A) = \varepsilon$. Various algorithms for finding $\lambda(A)$ exist. One of them is Karp’s [25] of computational complexity $O(nm)$ where $m$ is the number of finite entries in $A$ (or, equivalently the number of arcs in $D_A$).

We say that $A$ is definite if $\lambda(A) = 0$. It is easily seen that $V(\alpha \odot A) = V(A)$ and $\lambda(\alpha \odot A) = \alpha \otimes \lambda(A)$ for any $\alpha \in \mathbb{R}$. Hence $\lambda(A)^{-1} \odot A$ is definite whenever $\lambda(A) > \varepsilon$.

In order to construct eigenvectors explicitly, it is convenient to define the metric matrix

$$
\Gamma(A) = A \oplus A^2 \oplus ... .
$$

The matrix $\Gamma(A)$ is sometimes denoted by $A^+$, see e.g. [4].

Lemma 2.1 [16] If $\lambda(A) \leq 0$, in particular when $A$ is definite, then $\Gamma(A)$ finitely converges and is equal to $A \oplus A^2 \oplus ... \oplus A^n$. If $\lambda(A) > 0$ then the value of at least one position in $A^k$ is unbounded as $k \to \infty$ and, consequently, at least one entry of $\Gamma(A)$ is $+\infty$.

Note that the $(i, j)$ entry of $\Gamma(A)$ yields the maximum weight of a path with starting node $i$ and end node $j$ in $D_A$. The metric matrix of a matrix with $\lambda(A) \leq 0$ can be computed using the Floyd-Warshall algorithm in $O(n^3)$ time [17].

We also denote $E(A) = \{ i \in N; \exists \sigma = (i = i_1, ..., i_k, i_1) : \mu(\sigma, A) = \lambda(A) \}$. The elements of $E(A)$ are called eigen-nodes (of $A$), or critical nodes. A cycle $\sigma$ is called critical if $\mu(\sigma, A) = \lambda(A)$. The critical digraph of $A$ is the digraph $C(A)$ with the set of nodes $N$; the set of arcs is the union of the sets of arcs of all critical cycles. It is well known that all cycles in a critical digraph are critical [4]. Two nodes $i$ and $j$ in $C(A)$ are called equivalent (notation $i \sim j$) if $i$ and $j$ belong to the same critical cycle of $A$. Clearly, $\sim$ constitutes a relation of equivalence in $N$.

Note that if $\lambda(A) = \varepsilon$ then $\Lambda(A) = \{ \varepsilon \}$ and the eigenvectors of $A$ are exactly the vectors $(x_1, ..., x_n)^T \in \mathbb{R}^n$ such that $x_j = \varepsilon$ whenever the $j^{th}$ column of $A$ is not $\varepsilon$ (clearly in this case at least one column of $A$ is $\varepsilon$). We will therefore usually assume that $\lambda(A) > \varepsilon$.

The following proposition presents elementary properties relating metric matrices and critical digraphs.

Proposition 2.1 [16] Suppose that $A = (a_{ij}) \in \mathbb{R}^{n \times n}, \lambda(A) > \varepsilon$ and let $g_1, ..., g_n$ be the columns of $\Gamma((\lambda(A))^{-1} \odot A) = (g_{ij})$. Then

- $i \in E(A) \iff g_{ii} = 0$
\begin{itemize}
\item If $i, j \in E(A)$ then $g_i = \alpha \otimes g_j$ for some $\alpha \in \mathbb{R}$ if and only if $i \sim j$.
\end{itemize}

The following early version of the spectral theorem was proved in [16]. Related results can be found in [23], [12], [4], [24] and in the case of a denumerable state space in [3].

**Theorem 2.1** Suppose $A = (a_{ij}) \in \mathbb{R}^{n \times n}, A \neq \varepsilon$. Then the following hold:

1. $V^+(A) \subseteq V(A, \lambda(A))$.
2. $V^+(A) \neq \emptyset$ if and only if $\lambda(A) > \varepsilon$ and in $D_A$ we have
   \[(\forall j \in N)(\exists i \in E(A)) j \rightarrow i.\]
3. If, moreover, $V^+(A) \neq \emptyset$ then
   \[V^+(A) = \{ \sum_{i \in E(A)} \oplus \alpha_i \otimes g_i; \alpha_i \in \mathbb{R} \}\]
   where $g_1, ..., g_m$ are the columns of $\Gamma(\lambda(A)^{-1} \otimes A)$.

**Corollary 2.1** [17] $A$ irreducible $\Rightarrow V^+(A) \neq \emptyset$.

As we will see later (Proposition 3.1) in fact $V(A) = V^+(A) \cup \{ \varepsilon \} = V(A, \lambda(A))$ and thus $\Lambda(A) = \{ \lambda(A) \}$ if $A$ is irreducible. The fact that $\lambda(A)$ is the unique eigenvalue of an irreducible matrix $A$ was proved in [15] and then independently in [33]. The description of $V^+(A)$ for irreducible matrices as given in part 3 of Theorem 2.1 was also proved in [23].

A set $S \subseteq \mathbb{R}^n$ is called a (max-algebraic) subspace if $u, v \in S, \alpha, \beta \in \mathbb{R}$ imply $\alpha \otimes u \oplus \beta \otimes v \in S$. It is easily seen that $V(A, \lambda)$ (the set containing $\varepsilon$ and all eigenvectors of $A$ corresponding to $\lambda$, if any) is a subspace for all $\lambda \in \mathbb{R}$. We will therefore call $V(A, \lambda)$ the eigenspace of $A$ corresponding to the eigenvalue $\lambda$.

Let $S \subseteq \mathbb{R}^n$ be a subspace. A vector $v \in \mathbb{R}^n$ is called an extremal in $S$ if $v = u \oplus w$ for $u, v \in S$ implies $v = u \text{ or } v = w$. We say that $v_1, ..., v_m \in S$ is a basis of $S$ if

1. $v_1, ..., v_m$ are extremals in $S$ and
2. for every $v \in S$ we have $v = \sum_{i} \oplus \alpha_i \otimes v_i$ for some $\alpha_1, ..., \alpha_m \in \mathbb{R}$.

The following fundamental spectral theorem determines a basis of the eigenspace of $A$ corresponding to the eigenvalue $\lambda(A)$.

**Theorem 2.2** [3] Suppose that $A = (a_{ij}) \in \mathbb{R}^{n \times n}, \lambda(A) > \varepsilon$ and let $g_1, ..., g_m$ be the columns of $\Gamma((\lambda(A))^{-1} \otimes A)$. Then we obtain a basis of $V(A, \lambda(A))$ by taking exactly one $g_i$ for each equivalence class.
The vectors \( g_i, i \in E(A) \), are called the fundamental eigenvectors of \( A \) (FEV) [16].

Obviously

\[
V^+(A) \cup \{\varepsilon\} = \{\Gamma((\lambda(A))^{-1} \otimes A) \otimes z; z \in \mathbb{R}^n, z_j = \varepsilon \text{ for all } j \notin E(A)\}.
\]

and also, if non-empty,

\[
V^+(A) = \left\{ \sum_{i \in E^+(A)} \alpha_i \otimes g_i; \alpha_i \in \mathbb{R} \right\}
\]

where \( E^+(A) \) is any maximal set of non-equivalent eigen-nodes of \( A \).

Finally, we introduce some notation that will be used in the rest of the paper.

If

\[
1 \leq i_1 < i_2 < \ldots < i_k \leq n, K = \{i_1, \ldots, i_k\} \subseteq N
\]

then \( A[K] \) denotes the principal submatrix

\[
\begin{pmatrix}
  a_{i_1i_1} & \cdots & a_{i_1i_k} \\
  \vdots & \ddots & \vdots \\
  a_{i_1i_k} & \cdots & a_{i_ki_k}
\end{pmatrix}
\]

of the matrix \( A = (a_{ij}) \) and \( x[K] \) denotes the subvector \((x_{i_1}, \ldots, x_{i_k})^T\) of the vector \( x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n \).

If \( D = (N, E) \) is a digraph and \( K \subseteq N \) then \( D[K] \) denotes the induced subgraph of \( D \), that is

\[
D[K] = (K, E \cap (K \times K)).
\]

Obviously, \( D_{A[K]} = D[K] \).

### 3 Finding All Eigenvalues

The symbol \( A \sim B \) for matrices \( A \) and \( B \) means that \( A \) can be obtained from \( B \) by a simultaneous permutation of rows and columns. It follows that \( D_A \) can be obtained from \( D_B \) by a renumbering of the nodes if \( A \sim B \). Hence if \( A \sim B \) then \( A \) is irreducible if and only if \( B \) is irreducible.

It is obvious that if \( A \otimes x = \lambda \otimes x \) and a matrix \( B \) arises from \( A \) by a simultaneous permutation of the rows and columns then the same permutation applied to the components of \( x \) yields a vector \( y \) such that \( B \otimes y = \lambda \otimes y \). Hence:

**Lemma 3.1** If \( A \sim B \) then \( \Lambda(A) = \Lambda(B) \) and there is a bijection between \( V(A) \) and \( V(B) \).

The following lemma is of a special significance for the rest of the paper.
Lemma 3.2 Let \( A = (a_{ij}) \in \mathbb{R}^{n \times n} \) and \( \lambda \in \Lambda(A) \). If \( x \in V(A, \lambda) - V^+(A, \lambda), x \neq \varepsilon \), then \( n > 1 \),

\[
A \sim \begin{pmatrix}
A^{(11)} & \varepsilon \\
A^{(21)} & A^{(22)}
\end{pmatrix},
\]

\( \lambda = \lambda(A^{(22)}) \) and hence \( A \) is reducible.

**Proof.** Permute the rows and columns of \( A \) simultaneously so that the vector arising from \( x \) by the same permutation of its components is \( x' = \begin{pmatrix} x^{(1)} \\ x^{(2)} \end{pmatrix} \), where \( x^{(1)} = \varepsilon \in \mathbb{R}^p \) and \( x^{(2)} \in \mathbb{R}^{n-p} \) for some \( p \) (1 \( \leq \) \( p \) \( < \) \( n \)). Denote the obtained matrix by \( A' \) and let us write blockwise

\[
A' = \begin{pmatrix}
A^{(11)} & A^{(12)} \\
A^{(21)} & A^{(22)}
\end{pmatrix},
\]

where \( A^{(11)} \) is \( p \times p \). The equality \( A' \otimes x' = \lambda \otimes x' \) now yields blockwise:

\[
A^{(12)} \otimes x^{(2)} = \varepsilon \\
A^{(22)} \otimes x^{(2)} = \lambda \otimes x^{(2)}
\]

Since \( x^{(2)} \) is finite, it follows from Theorem 2.1 that \( \lambda = \lambda(A^{(22)}) \); also clearly \( A^{(12)} = \varepsilon \). ■

**Proposition 3.1** Let \( A = (a_{ij}) \in \mathbb{R}^{n \times n} \). Then \( V(A) = V^+(A) \) if and only if \( A \) is irreducible.

**Proof.** It remains to prove the "only if" part since the "if" part follows from Lemma 3.2 immediately. If \( A \) is reducible then \( n > 1 \) and \( A \sim \begin{pmatrix}
A^{(11)} & \varepsilon \\
A^{(21)} & A^{(22)}
\end{pmatrix}, \)

where \( A^{(22)} \) is irreducible. By setting \( \lambda = \lambda(A^{(22)}), x^{(2)} \in V^+(A_{22}), x = \begin{pmatrix} \varepsilon \\ x^{(2)} \end{pmatrix} \in \mathbb{R}^n \) we see that \( x \in V(A) - V^+(A), x \neq \varepsilon \). ■

Every matrix \( A = (a_{ij}) \in \mathbb{R}^{n \times n} \) can be transformed in linear time by simultaneous permutations of the rows and columns to a *Frobenius normal form* (FNF) \[30\]

\[
\begin{pmatrix}
A_{11} & \varepsilon & \cdots & \varepsilon \\
A_{21} & A_{22} & \varepsilon & \cdots \\
\vdots & \vdots & \ddots & \varepsilon \\
\varepsilon & \cdots & \cdots & A_{rr}
\end{pmatrix}
\]

(3)

where \( A_{11}, ..., A_{rr} \) are irreducible square submatrices of \( A \). If \( A \) is in an FNF then the corresponding partition of the node set \( N \) of \( D_A \) will be denoted as \( N_1, ..., N_r \) and these sets will be called *classes* (of \( A \)). It follows that each of the induced subgraphs \( D_A[N_i] \) (\( i = 1, ..., r \)) is strongly connected and an arc from
$N_i$ to $N_j$ in $D_A$ exists only if $i \geq j$. As a slight abuse of language we will also say for simplicity that $\lambda(A_{jj})$ is the eigenvalue of $N_j$.

If $A$ is in an FNF, say (3), then the condensation digraph, notation $C_A$, is the digraph $\{(N_i, N_j); (\exists k \in N_i)(\exists \ell \in N_j) a_{k\ell} > \varepsilon\}$. Observe that $C_A$ is acyclic.

Recall that the symbol $N_i \rightarrow N_j$ means that there is a directed path from a node in $N_i$ to a node in $N_j$ in $C_A$ (and therefore from each node in $N_i$ to each node in $N_j$ in $D_A$).

If there are neither outgoing nor incoming arcs from or to an induced subgraph $C_A[N_{i_1}, \ldots, N_{i_s}]$ ($1 \leq i_1 < \ldots < i_s \leq r$) and no proper subdigraph has this property then the submatrix

$$
\begin{pmatrix}
A_{i_1 i_1} & \varepsilon & \cdots & \varepsilon \\
A_{i_2 i_1} & A_{i_2 i_2} & \cdots & \varepsilon \\
\vdots & \vdots & \ddots & \vdots \\
A_{i_s i_1} & A_{i_s i_2} & \cdots & A_{i_s i_s}
\end{pmatrix}
$$

is called an isolated superblock (or just superblock). The nodes of $C_A$ with no incoming arcs are called the initial classes, those with no outgoing arcs are called the final classes. Note that an isolated superblock may have several initial and final classes.

For instance the condensation digraph for the matrix

$$
\begin{pmatrix}
A_{11} & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
* & A_{22} & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
* & * & A_{33} & \varepsilon & \varepsilon & \varepsilon \\
* & \varepsilon & \varepsilon & A_{44} & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & A_{55} & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & * & A_{66}
\end{pmatrix}
$$

(4)

can be seen in Figure 1 (note that here and elsewhere the symbols * indicate submatrices different from $\varepsilon$). It consists of two superblocks and six classes including three initial and two final ones.
Lemma 3.3 If \( x \in V(A), N_i \rightarrow N_j \) and \( x[N_j] \neq \varepsilon \) then \( x[N_i] \) is finite. In particular, \( x[N_j] \) is finite.

Proof. Suppose that \( x \in V(A, \lambda) \) for some \( \lambda \in \mathbb{R} \). Fix \( s \in N_j \) such that \( x_s > \varepsilon \). Since \( N_i \rightarrow N_j \) we have that for every \( r \in N_i \) there is a positive integer \( q \) such that \( b_{rs} > \varepsilon \) where \( B = A^q = (b_{ij}) \). Since \( x \in V(B, \lambda^q) \) we also have \( \lambda^q \otimes x_r > b_{rs} \otimes x_s > \varepsilon \). Hence \( x_r > \varepsilon \). 

The following key result appeared in the thesis [18] and [7]. The latter work refers to the report [6] for a proof. Related results can be found in [5].

Theorem 3.1 (Spectral Theorem) Let (3) be an FNF of a matrix \( A = (a_{ij}) \in \mathbb{R}^{n \times n} \). Then

\[
\lambda(A) = \{ \lambda(A_{jj}); \lambda(A_{jj}) = \max_{N_i \rightarrow N_j} \lambda(A_{ii}) \}.
\]

Proof. Note first that

\[
\lambda(A) = \max_{i=1, \ldots, r} \lambda(A_{ii})
\] (5)

for a matrix \( A \) in FNF (3).

First we prove the inclusion \( \supseteq \). Suppose \( \lambda(A_{jj}) = \max \{ \lambda(A_{ii}); N_i \rightarrow N_j \} \) for some \( j \in R = \{ 1, \ldots, r \} \). Denote \( S_2 = \{ i \in R; N_i \rightarrow N_j \} \), \( S_1 = R - S_2 \), \( M_p = \bigcup_{i \in S_p} N_i \) \((p = 1, 2) \). Then \( \lambda(A_{jj}) = \lambda(A[M_2]) \) and \( A \sim \begin{pmatrix} A[M_1] & \varepsilon \\ \ast & A[M_2] \end{pmatrix} \).

If \( \lambda(A_{jj}) = \varepsilon \) then at least one column, say the \( \ell \)th column in \( A[M_2] \) is \( \varepsilon \). We set \( x_{\ell} \) to any real number and \( x_j = \varepsilon \) for \( j \neq \ell \). Then \( x \in V(A, \lambda(A_{jj})) \).

If \( \lambda(A_{jj}) > \varepsilon \) then \( A[M_2] \) has a finite eigenvector by Theorem 2.1, say \( \tilde{x} \). Set \( x[M_2] = \tilde{x} \) and \( x[M_1] = \varepsilon \). Then \( x = \begin{pmatrix} x[M_1] \\ x[M_2] \end{pmatrix} \in V(A, \lambda(A_{jj})) \).

Now we prove \( \subseteq \). Suppose that \( x \in V(A, \lambda), x \neq \varepsilon \), for some \( \lambda \in \mathbb{R} \).

If \( \lambda = \varepsilon \) then \( A \) has an \( \varepsilon \) column, say the \( k \)th column, thus \( a_{kk} = \varepsilon \). Hence the \( 1 \times 1 \) submatrix \( (a_{kk}) \) is a diagonal block in an FNF of \( A \). In the corresponding decomposition of \( N \) one of the sets, say \( N_k \), is \( \{ k \} \). The set \( \{ i; N_i \rightarrow N_j \} = \{ j \} \) and the theorem statement follows.

If \( \lambda > \varepsilon \) and \( x \in V^+(A) \) then \( \lambda = \lambda(A) \) (cf. Theorem 2.1) and the statement now follows from (5).

If \( \lambda > \varepsilon \) and \( x \notin V^+(A) \) then similarly as in the proof of Lemma 3.2 permute the rows and columns of \( A \) simultaneously so that \( x = \begin{pmatrix} x^{(1)} \\ x^{(2)} \end{pmatrix} \), where \( x^{(1)} = \varepsilon \in \mathbb{R}^p \) and \( x^{(2)} \in \mathbb{R}^{n-p} \) for some \( p \) \((1 \leq p \leq n) \). Hence \( A \sim \begin{pmatrix} A^{(11)} & \varepsilon \\ A^{(21)} & A^{(22)} \end{pmatrix} \) and we can assume without loss of generality that both \( A^{(11)} \) and \( A^{(22)} \) are in
an FNF and therefore also \( \begin{pmatrix} A^{(11)} & \varepsilon \\ A^{(21)} & A^{(22)} \end{pmatrix} \) is in an FNF. Let

\[
A^{(11)} = \begin{pmatrix}
A_{i_1i_1} & \varepsilon & \ldots & \varepsilon \\
A_{i_2i_1} & A_{i_2i_2} & \ldots & \varepsilon \\
\vdots & \vdots & \ddots & \vdots \\
A_{i_si_1} & A_{i_si_2} & \ldots & A_{i_si_s}
\end{pmatrix}
\]

\[
A^{(22)} = \begin{pmatrix}
A_{i_s+1i_{s+1}} & \varepsilon & \ldots & \varepsilon \\
A_{i_s+2i_{s+1}} & A_{i_s+2i_{s+2}} & \ldots & \varepsilon \\
\vdots & \vdots & \ddots & \vdots \\
A_{i_qi_{s+1}} & A_{i_qi_{s+2}} & \ldots & A_{i_qi_q}
\end{pmatrix}
\]

We have \( \lambda = \lambda(A^{(22)}) = \lambda(A_{jj}) = \max_{i=s+1, \ldots, q} \lambda(A_{ii}) \) where \( j \in \{s+1, \ldots, q \} \).

It remains to say that if \( N_i \to N_j \) then \( i \in \{s+1, \ldots, q \} \).

Note that significant correlation exists between the max-algebraic spectral theory and that for non-negative matrices in linear algebra [31], [8], see also [30]. For instance the Frobenius normal form and accessibility between classes play a key role in both theories. The maximum cycle mean corresponds to the Perron root for irreducible (nonnegative) matrices and finite eigenvectors in max-algebra correspond to positive eigenvectors in the non-negative spectral theory. However there are also differences, see Remark 4.2 after Theorem 4.2 below.

Let \( A \) be in the FNF (3). If

\[ \lambda(A_{jj}) = \max_{N_i \to N_j} \lambda(A_{ii}) \]

then \( A_{jj} \) (and also \( N_j \) or just \( j \)) will be called spectral. Thus \( \lambda(A_{jj}) \in \Lambda(A) \) if \( j \) is spectral but not necessarily the other way round.

**Corollary 3.1** All initial classes of \( CA \) are spectral.

**Proof.** Initial classes have no predecessors and so the condition of the Theorem is satisfied. ■

**Corollary 3.2** \( \lambda(A) \in \Lambda(A) \) for every matrix \( A \).

**Proof.** If \( A \) is in an FNF, say (3), then \( \lambda(A) = \max_{i=1, \ldots, r} \lambda(A_{ii}) = \lambda(A_{jj}) \) for some \( j \) and so the condition of the Theorem is satisfied. ■

**Corollary 3.3** \( 1 \leq |\Lambda(A)| \leq n \) for every \( A \in \mathbb{R}^{n \times n} \).

**Proof.** Follows from the previous corollary and from the fact that the number of classes of \( A \) is at most \( n \).

**Corollary 3.4** \( V(A) = V(A, \lambda(A)) \) if and only if all initial classes have the same eigenvalue \( \lambda(A) \).
Proof. The eigenvalues of all initial classes are in $\Lambda(A)$ since all initial classes are spectral, hence all must be equal to $\lambda(A)$ if $\Lambda(A) = \{\lambda(A)\}$. On the other hand, if all initial classes have the same eigenvalue $\lambda(A)$, and $\lambda$ is the eigenvalue of any spectral class then

$$\lambda \geq \lambda(A) = \max_i \lambda(A_{ii})$$

since there is a path from some initial class to this class and thus $\lambda = \lambda(A)$.  

Figure 2 shows a condensation digraph with 14 classes including two initial classes and four final ones. The numbers indicate the eigenvalues of the corresponding classes. The six bold classes are spectral, the others are not.

4 Finding All Eigenvectors

Note that the unique eigenvalue of every class (that is of a diagonal block of an FNF) can be found in $O(n^3)$ time by applying Karp’s algorithm (see Section 1) to each block. The condition for identifying all spectral submatrices in an FNF provided in Theorem 3.1 enables us to find them in $O(r^2) \leq O(n^2)$ time by applying standard reachability algorithms to $C_A$. 
Let \(A \in \mathbb{R}^{n \times n}\) be in the FNF (3), \(N_1, \ldots, N_r\) be the classes of \(A\) and \(R = \{1, \ldots, r\}\). Suppose \(\lambda \in \Lambda(A)\), \(\lambda > \varepsilon\) and denote \(I(\lambda) = \{i \in R; \lambda(N_i) = \lambda, N_i\) spectral\}. Similarly as in Section 2 we denote by \(g_1, \ldots, g_n\) the columns of \(\Gamma(\lambda^{-1} \otimes A) = (g_{ij})\). Note that \(\lambda(\lambda^{-1} \otimes A) = \lambda^{-1} \otimes \lambda(A)\) may be positive since \(\lambda \leq \lambda(A)\) and thus \(\Gamma(\lambda^{-1} \otimes A)\) may now include entries equal to \(+\infty\) (see Lemma 2.1). Let us denote
\[
E(\lambda) = \bigcup_{i \in I(\lambda)} E(A_{ii}) = \{j \in N; g_{jj} = 0, j \in i \in I(\lambda)\}.
\]

Two nodes \(i\) and \(j\) in \(E(\lambda)\) are called \(\lambda -\) equivalent (notation \(i \sim \lambda j\)) if \(i\) and \(j\) belong to the same cycle of cycle mean \(\lambda\).

**Theorem 4.1** Suppose \(A \in \mathbb{R}^{n \times n}\) and \(\lambda \in \Lambda(A), \lambda > \varepsilon\). Then \(g_j \in \mathbb{R}^n\) for all \(j \in E(\lambda)\) and a basis of \(V(A, \lambda)\) can be obtained by taking one \(g_j\) for each \(\sim \lambda\) equivalence class.

**Proof.** Let us denote \(M = \bigcup_{i \in I(\lambda)} N_i\). By Lemma 3.1 we may assume without loss of generality that \(A\) is of the form
\[
\begin{pmatrix}
\varepsilon & & \\
& \ddots & \\
& & \varepsilon
\end{pmatrix}
\begin{pmatrix}
A[M]
\end{pmatrix}
\]

Hence
\[
\begin{pmatrix}
\varepsilon \\
& \ddots \\
& & \varepsilon
\end{pmatrix}
\begin{pmatrix}
C
\end{pmatrix}
\]

where \(C = \Gamma((\lambda(A[M]))^{-1} \otimes A[M])\), and the statement now follows by Theorems 2.1 and 2.2 since \(\lambda = \lambda(A[M])\) and thus \(\sim \lambda\) equivalence for \(A\) is identical with \(\sim\) equivalence for \(A[M]\). 

**Corollary 4.1** A basis of \(V(A, \lambda)\) for \(\lambda \in \Lambda(A)\) can be found using \(O(n^3)\) operations and we have
\[
V(A, \lambda) = \{\Gamma(\lambda^{-1} \otimes A) \otimes z; z \in \mathbb{R}^n, z_j = \varepsilon \text{ for all } j \notin E(\lambda)\}.
\]

Note that if the set \(I(\lambda)\) consists of only one index then it follows from the proofs of Lemma 3.2 and Theorem 3.1 that \(V(A, \lambda)\) can alternatively be found as follows: If \(I(\lambda) = \{j\}\) then define \(M_2 = \bigcup_{N_i \rightarrow N_j} N_i, M_1 = N - M_2\).

Hence
\[
V(A, \lambda) = \{x; x[M_1] = \varepsilon, x[M_2] \in V^+(A[M_2])\}.
\]

**Theorem 4.2** \(V^+(A) \neq \emptyset\) if and only if \(\lambda(A)\) is the eigenvalue of all final classes.
Proof. The set \( M_1 \) in the above construction must be empty to obtain a finite eigenvector, hence a class in \( S \) must be reachable from every class of its superblock. This is only possible if \( S \) is the set of all final classes since no class is reachable from a final class (other than the final class itself). Conversely, if all final classes have the same eigenvalue \( \lambda(A) \) then for \( \lambda = \lambda(A) \) the set \( S \) contains all the final classes, they are reachable from all classes of their superblocks, and consequently \( M_1 = \emptyset \), yielding a finite eigenvector.

Corollary 4.2 \( V^+(A) = \emptyset \) if and only if a final class has eigenvalue less than \( \lambda(A) \).

Remark 4.1 Note that a final class with eigenvalue less than \( \lambda(A) \) may not be spectral and so \( \Lambda(A) = \{ \lambda(A) \} \) is possible even if \( V^+(A) = \emptyset \). For instance in the case of

\[
A = \begin{pmatrix}
1 & \varepsilon & \varepsilon \\
\varepsilon & 0 & \varepsilon \\
0 & 0 & 1
\end{pmatrix}
\]

we have \( \Lambda(A) = \{ 1 \} \), but \( V^+(A) = \emptyset \).

Remark 4.2 In the Perron-Frobenius theory of non-negative matrices, a class of a non-negative matrix is called basic if its spectral radius coincides with the spectral radius of the matrix. A classical result shows that a non-negative matrix has a positive eigenvector if and only if its basic and final classes coincide. In the max-algebraic setting we may define a class to be basic when its eigenvalue is \( \lambda(A) \). Then, Theorem 4.2 shows that the existence of a finite eigenvector only requires all final classes to be basic; unlike in the Perron-Frobenius theory, there may be non-final basic classes. For instance the non-negative matrix

\[
A = \begin{pmatrix}
1 & 0 \\
1 & 1
\end{pmatrix}
\]

has two basic classes \{1\} and \{2\} and only one final class, namely \{1\}, thus it does not have a positive eigenvector. However, its max-algebraic counterpart

\[
A = \begin{pmatrix}
0 & -\infty \\
0 & 0
\end{pmatrix}
\]

which satisfies the condition of Theorem 4.2 has a finite eigenvector (for instance \((0,0)^T\)). This fundamental discrepancy is due to the idempotency of \( \oplus \) in max-algebra.

5 Robustness of matrices

Let \( A = (a_{ij}) \in \mathbb{R}^{n \times n} \). The set of vectors \( x \in \mathbb{R}^n \) such that for some \( r \), \( A^r \otimes x \) is an eigenvector of \( A \) corresponding to a finite eigenvalue, will be called the
attraction space (of $A$). Obviously, if $A^r \otimes x$ is an eigenvector for some $r$ then $A^k \otimes x$ is an eigenvector for every $k \geq r$. Also, the attraction space of any matrix contains all eigenvectors of this matrix.

It may happen that the attraction space of $A$ contains only eigenvectors of $A$; for instance when $A$ is the irreducible matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$: here $\lambda(A) = 0$ and by Theorem 2.1

$$V(A) - \{\varepsilon\} = \{\alpha \otimes (0,0)^T; \alpha \in \mathbb{R}\}.$$ 

Since

$$A \otimes \begin{pmatrix} a \\ b \end{pmatrix} = (\max(a - 1, b), \max(a, b - 1))^T,$$

we have that $A \otimes \begin{pmatrix} a \\ b \end{pmatrix}$ is an eigenvector of $A$ if and only if $a = b$, that is $A \otimes x$ is an eigenvector of $A$ if and only if $x$ is an eigenvector of $A$. Hence the attraction space is $V(A) - \{\varepsilon\}$.

The attraction space of $A$ may be different from both $V(A) - \{\varepsilon\}$ and $\mathbb{R}^n - \{\varepsilon\}$. Consider the irreducible matrix $A = \begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ -1 & -1 & 0 \end{pmatrix}$.

Here $\lambda(A) = 0$ and $x = (-2, -2, 0)^T$ is not an eigenvector of $A$ but $A \otimes x = (-1, -1, 0)^T$ is, showing that the attraction space contains also vectors other than eigenvectors. At the same time if $y = (0, -1, 0)^T$ then $A^k \otimes y$ is $y$ for $k$ even and $(-1, 0, 0)^T$ for $k$ odd, showing that $y$ is not in the attraction space.

A matrix $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ is called robust if the attraction space of $A$ contains all vectors in $\mathbb{R}^n$ except $\varepsilon$. Hence $A$ is robust if and only if $A^k \otimes x$ is an eigenvector of $A$ for any $x \in \mathbb{R}^n, x \not= \varepsilon$ and large enough $k$. Alternatively, for every $x \in \mathbb{R}^n, x \not= \varepsilon$ we have $A^{k+1} \otimes x = \lambda \otimes A^k \otimes x, A^k \otimes x \not= \varepsilon$, for some positive integer $k$ and $\lambda \in \Lambda(A)$. The importance of robustness has been explained in Section 1.

Clearly, if $A \sim B$ then $A$ is robust if and only if $B$ is robust. Therefore we may without loss of generality investigate robustness of matrices arising from a given matrix by a simultaneous permutation of the rows and columns.

Now we present some characterizations of robust matrices. First we observe that matrices with an $\varepsilon$ column are not robust. Following the terminology introduced in [16] we say that $A$ is column $\mathbb{R}$-astic if it has no $\varepsilon$ column. Note that every node of a non-trivial strongly connected digraph has at least one incoming arc and so every irreducible $n \times n$ matrix ($n > 1$) is column $\mathbb{R}$-astic (but not conversely).

**Lemma 5.1** If $A \in \mathbb{R}^{n \times n}$ is column $\mathbb{R}$-astic and $x \not= \varepsilon$ then $A^k \otimes x \not= \varepsilon$ for every $k$. Hence if $A \in \mathbb{R}^{n \times n}$ is column $\mathbb{R}$-astic then $A^k$ is column $\mathbb{R}$-astic for every $k$. This is true in particular when $A$ is irreducible and $n > 1$. 

15
Proof. Immediate from definition. ■

We say that $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ is ultimately periodic of period $p$ if there is a natural number $p$ such that the following holds for some $\lambda \in \mathbb{R}$ and $k_0$ natural:

$$A^{k+p} = \lambda^p \otimes A^k \text{ for all } k \geq k_0.$$ 

If $p$ is the smallest natural number with this property then we call $p$ the period of $A$ and denote it as $p(A)$. If $A$ is not ultimately periodic then we set $p(A) = +\infty$. It is easily seen that $\lambda = \lambda(A)$ and every column of $A^k$ is in $V(A^p, \lambda^p)$ if $p = p(A) < +\infty$ and $A$ is irreducible. Robustness of irreducible matrices was studied in [11] and we now mention some results of that paper before we proceed with the reducible case. Note that if $A$ is the $1 \times 1$ matrix $(\varepsilon)$ then $A$ is irreducible, $p(A) = 1$ but $A$ is not robust. This is an exceptional case that has to be excluded in the statements that follow.

**Theorem 5.1** [11] Let $A \in \mathbb{R}^{n \times n}$ be irreducible, $A \neq \varepsilon$. Then $A$ is robust if and only if $p(A) = 1$.

**Corollary 5.1** [11] Let $A \in \mathbb{R}^{n \times n}$ be irreducible, $A \neq \varepsilon$. If $p(A) = 1$, $x \neq \varepsilon$ then $A^k \otimes x$ is finite for all sufficiently big $k$.

It was shown in [11] how the next statement follows from the results in [10], Theorem 3.4.5.

**Theorem 5.2** Let $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ be irreducible. Then $A^k$ is irreducible for every $k = 1, 2, \ldots$ if and only if the lengths of all cycles in $D_A$ are co-prime.

Previous results are closely related to the famous "Cyclicity Theorem", Theorem 5.3 below. For this we need to introduce a few more concepts: Let $D'$ be a maximal strongly connected subdigraph of a digraph $D$. Then $D'$ is called a strongly connected component of $D$ and the greatest common divisor of all directed cycles in $D'$ is called the cyclicity of $D'$, notation $\sigma(D')$. By definition $\sigma(D') = 1$ if $D'$ consists only of a single node. The cyclicity of $D$ is the least common multiple of cyclicities of all strongly connected components of $D$.

**Theorem 5.3** Every irreducible matrix $A$ is ultimately periodic and $p(A) = \sigma(C(A))$.

Note that the "if" statement of Theorem 5.1 follows immediately from Theorem 5.3.

The first part of Theorem 5.3 was proved for finite matrices in [16]. A proof of the whole statement was presented in [12], see also [13] for an overview without proofs. A proof in a more general setting covering the case of finite matrices is given in [28]. The irreducible case is also proved in [3], [24], [4] and [21]. Note that a different generalization to the reducible case is studied in [22].

**Corollary 5.2** Let $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ be irreducible and robust. Then $A^k$ is irreducible for every $k = 1, 2, \ldots$. 

Proof. If the lengths of all critical cycles in $D_A$ are co-prime then also the lengths of all cycles are co-prime. The rest follows from Theorem 5.2.

We now continue by studying robustness of reducible matrices. Theorem 5.1 can straightforwardly be generalized to a class of reducible matrices:

**Theorem 5.4** Let $A \in \mathbb{R}^{n \times n}$ be column $\mathbb{R}$-astic and $|\Lambda(A)| = 1$ (that is $\Lambda(A) = \{\lambda(A)\}$). Then $A$ is robust if and only if $p(A) = 1$.

**Proof.** Let $p(A) = 1$, $x \in \mathbb{R}^n - \{\varepsilon\}$ and $k \geq k_0$. Then $A^k \otimes x \in \mathbb{R}^n - \{\varepsilon\}$ by Lemma 5.1, $A^{k+1} \otimes x = \lambda \otimes A^k \otimes x$ and so $A^k \otimes x \in V(A, \lambda)$ and $\lambda = \lambda(A)$. Hence $A$ is robust and all columns of $A^k$ are eigenvectors of $A$.

Now let $A$ be robust and all columns of $A^{k_0}$ be eigenvectors of $A$ corresponding to the unique eigenvalue $\lambda(A)$. Then $A \otimes A^{k_0} = \lambda(A) \otimes A^{k_0}$ and thus $A \otimes A^k = \lambda(A) \otimes A^k$ for all $k \geq k_0$. So $p(A) = 1$.

We will now characterize robust reducible matrices in general - we start with two lemmas.

**Lemma 5.2** If $A \in \mathbb{R}^{n \times n}$ is robust then $\varepsilon \notin \Lambda(A)$.

**Proof.** If $\varepsilon \in \Lambda(A)$ then by Lemma 5.1 some column, say $k$th is $\varepsilon$. Take $x \in \mathbb{R}^n$ so that $x_k = 0$ and $x_j = \varepsilon$ for $j \neq k$. Then $A^k \otimes x = \varepsilon$ for every $k$ and thus $A^k \otimes x$ is never an eigenvector.

A class of $A$ is called trivial if it contains only one index, say $k$, and $a_{kk} = \varepsilon$.

**Lemma 5.3** If every non-trivial class of $A \in \mathbb{R}^{n \times n}$ has eigenvalue 0 and period 1 then $A^{k+1} = A^k$ for some $k$.

**Proof.** We prove the statement by induction on the number of classes.

If $A$ has only one class then either this class is trivial or $A$ is irreducible. In both cases the statement follows immediately.

If $A$ has at least two classes then by Lemma 3.1 we can assume without loss of generality:

$$A = \begin{pmatrix} A_{11} & \varepsilon \\ A_{21} & A_{22} \end{pmatrix}$$

and thus

$$A^k = \begin{pmatrix} A_{11}^k & \varepsilon \\ B_k & A_{22}^k \end{pmatrix}$$

where

$$B_k = \sum_{i+j=k-1} \oplus A_{i2}^l \otimes A_{21} \otimes A_{11}^j.$$

By the induction hypothesis there are $k_1$ and $k_2$ such that

$$A_{11}^{k_1+1} = A_{11}^{k_1} \text{ and } A_{22}^{k_2+1} = A_{22}^{k_2}.$$
It is sufficient now to prove that

$$B_k = \sum \left\{ A_{i2}^j \otimes A_{21} \otimes A_{11}^i; i \leq k_2, j \leq k_1, i = k_2 \text{ or } j = k_1 \right\}$$

(6)

holds for all $k \geq k_1 + k_2 + 1$.

For all $i, j$ we have

$$A_{i2}^j \otimes A_{21} \otimes A_{11}^i = A_{i2}^{j'} \otimes A_{21} \otimes A_{11}^{i'}$$

where $i' = \min(i, k_2), j' = \min(j, k_1)$. If $i + j + 1 = k \geq k_1 + k_2 + 1$ then either $i \geq k_2$ or $j \geq k_1$. Hence either $i' = k_2$ or $j' = k_1$ and therefore $\leq$ in (5) follows.

For $\geq$ let $i = k_2$ (say) and $j \leq k_1$. Since $k \geq k_1 + k_2 + 1 \geq j + i + 1$, we have $k - j - 1 \geq i = k_2$ and thus

$$A_{i2}^j \otimes A_{21} \otimes A_{11}^i = A_{i2}^{k-j-1} \otimes A_{21} \otimes A_{11}^i \leq B_k.$$

Recall that if $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ is in the FNF (3) and $N_1, \ldots, N_r$ are the classes of $A$ then we have denoted $R = \{1, \ldots, r\}$. If $i \in R$ then we now also denote $T_i = \{k \in R; N_k \rightarrow N_i\}$ and $M_i = \bigcup_{j \in T_i} N_j$.

We are now ready to present the main result of this paper.

**Theorem 5.5** Let $A \in \mathbb{R}^{n \times n}$ be column $\mathbb{R}$-astic and in the FNF (3). Let $N_1, \ldots, N_r$ be the classes of $A$ and $R = \{1, \ldots, r\}$. Then $A$ is robust if and only if the following hold:

1. All non-trivial classes $N_1, \ldots, N_r$ are spectral.
2. If $i, j \in R, N_i, N_j$ are non-trivial, $i \notin T_j$ and $j \notin T_i$ then $\lambda(N_i) = \lambda(N_j)$.
3. $p(A_{jj}) = 1$ for all $j \in R$.

**Proof.** If $r = 1$ then $A$ is irreducible and the statement follows by Theorem 5.1. We will therefore assume $r \geq 2$ in this proof.

Let $A$ be robust.

1. Let $i \in R, A_{ii} \neq \varepsilon$ and $x \in \mathbb{R}^n$ be defined by taking any $x_s \in \mathbb{R}$ for $s \in M_i$ and $x_s = \varepsilon$ for $s \notin M_i$. Then $A_{i2}^k \otimes x = \lambda \otimes A_{i2}^k \otimes x$ for some $k$ and $\lambda \in \Lambda(A)$. Let $z = A_{i2}^k \otimes x$. Then $z[M_i]$ is finite since $A[M_i]$ has no $\varepsilon$ row and $A[M_i] \otimes z[M_i] = (A \otimes z)[M_i] = \lambda \otimes z[M_i]$ and thus $z[M_i] \in V^+(A[M_i])$. By Lemma 5.2 $\lambda > \varepsilon$ and so by Theorem 2.1 then $\lambda(N_i) \leq \lambda(N_i)$ for all $t \in T_i$. Hence $N_i$ is spectral.
2. Suppose $i, j \in R, N_i, N_j$ are non-trivial and $i \notin T_j, j \notin T_i$. Let $x \in \mathbb{R}^n$ be defined by taking any $x[N_i] \in V^+(A[N_i]), x[N_j] \in V^+(A[N_j])$ and $x_s = \varepsilon$ for $s \in N_i \cup N_j$. Then $A^{k+1} \otimes x = \lambda \otimes A^k \otimes x$ for some $k$ and $\lambda \in \Lambda(A)$. Denote $z = A^k \otimes x$. Then $z[N_j]$ is finite. Since $i \notin T_j$ we have $a_{uv} = \varepsilon$ for all $u \in N_i$ and $v \in N_j$. Hence

$$\lambda \otimes z[N_j] = (A \otimes z)[N_j] = A[N_j] \otimes z[N_j]$$

and so by Theorem 2.1 $\lambda(N_j) = \lambda$. Similarly it is proved that $\lambda(N_i) = \lambda$.

3. Let $j \in R$ and $A[N_j] \neq \varepsilon$ (otherwise the statement follows trivially). Let $x \in \mathbb{R}^n$ be any vector such that $x \neq \varepsilon$ and $x_s = \varepsilon$ for $s \notin N_j$. Then $A^{k+1} \otimes x = \lambda \otimes A^k \otimes x$ for some $k$ and $\lambda \in \Lambda(A)$. Let $z = A^k \otimes x$. Since $z[N_j] = (A[N_j])^k \otimes x[N_j]$ we may assume without loss of generality that $z[N_j] \neq \varepsilon$. At the same time $A[N_j] \otimes z[N_j] = (A \otimes z)[N_j] = \lambda \otimes z[N_j]$ and thus $z[N_j] \in V(A[N_j])$. Hence $A[N_j]$ is irreducible and robust. Thus by Theorem 5.1 $p(A[N_j]) = p(A_{jj}) = 1$.

Suppose now that conditions 1.-3. are satisfied. We prove then that $A$ is robust by induction on the number of classes of $A$. As already observed at the beginning of this proof the case $r = 1$ follows from Theorem 5.1. Suppose now that $r \geq 2$ and let $x \in \mathbb{R}^n, x \neq \varepsilon$. Let

$$U = \{i \in N; (\exists j) i \rightarrow j, x_j \neq \varepsilon\}.$$

We have

$$(A^k \otimes x)[U] = (A[U])^k \otimes x[U]$$

and

$$(A^k \otimes x)_i = \varepsilon$$

for $i \notin U$. Therefore we may assume without loss of generality that $U = N$. Let $M$ be a final class in $\mathcal{C}_A$, clearly $x[M] \neq \varepsilon$ by the definition of $U$. Let us denote

$$S = \{i \in N; (\exists j \in M) (i \rightarrow j)\}$$

$$S' = N - S.$$

By Lemma 3.1 we may assume without loss of generality that

$$A = \begin{pmatrix} A_{11} & \varepsilon & \varepsilon \\ A_{21} & A_{22} & A_{23} \\ \varepsilon & \varepsilon & A_{33} \end{pmatrix}$$

where the individual blocks correspond (in this order) to the sets $M, S \setminus M$ and $S'$ respectively. Let us define $x^k = A^k \otimes x$ for all integers $k \geq 0$. We also set

$$x^k_1 = x^k[M]$$

$$x^k_2 = x^k[S \setminus M]$$

$$x^k_3 = x^k[S']$$

19
Obviously,

\[
\begin{align*}
    x_1^{k+1} &= A_{11} \otimes x_1^k \\
    x_2^{k+1} &= A_{21} \otimes x_1^k \oplus A_{22} \otimes x_2^k \oplus A_{23} \otimes x_3^k \\
    x_3^{k+1} &= A_{33} \otimes x_3^k
\end{align*}
\]

Assume first that \( M \) is non-trivial. Then \( \lambda(A_{11}) \neq \varepsilon \) and by taking (if necessary) \((\lambda(A_{11}))^{-1} \otimes A\) instead of \(A\), we may assume without loss of generality that \( \lambda(A_{11}) = 0 \). By assumption 3 and Theorem 5.3 we have \( A_{11}^{k_1+1} = A_{11}^{k_1} \) for some \( k_1 \). By assumption 2 every class of \( A_{33} \) has eigenvalue 0. Since each of these classes has also period 1 by assumption 3, it follows from Lemma 5.3 that \( A_{33}^{k_3+1} = A_{33}^{k_3} \) for some \( k_3 \). We may also assume without loss of generality that

\[
\begin{align*}
    x_0^0 &= x_1^0 = x_2^0 = ... \\
    x_3^0 &= x_1^3 = x_2^3 = ...
\end{align*}
\]

Therefore

\[
    x_2^{k+1} = A_{21} \otimes x_1^0 \oplus A_{22} \otimes x_2^k \oplus A_{23} \otimes x_3^0.
\]

Let \( v = A_{21} \otimes x_1^0 \oplus A_{23} \otimes x_3^0 \). We deduce that

\[
    x_2^k = A_{22}^k \otimes x_2^0 \oplus (A_{22}^{k-1} \oplus ... \oplus A_{22}^0) \otimes v
\]

(7)

for all \( k \). Moreover, \( \lambda(A_{22}) \leq \lambda(A_{11}) = 0 \) since \( M \) is spectral by assumption 1. Hence

\[
    A_{22}^{k-1} \oplus ... \oplus A_{22}^0 = \Gamma(A_{22})
\]

for all \( k \geq n \). Note that \( x_0^0 \) is finite as an eigenvector of the irreducible matrix \( A_{11} \). Also, since every node in \( S \) has access to \( M \), the vector \( \Gamma(A_{22}) \otimes x_1^0 \) is finite and hence also \( \Gamma(A_{22}) \otimes v \) is finite. If \( \lambda(A_{22}) < 0 \) then \( A_{22}^k \otimes x_2^0 \to -\infty \) as \( k \to \infty \) and we deduce that \( x_2^k = \Gamma(A_{22}) \otimes v \) for all \( k \) big enough. If \( \lambda(A_{22}) = 0 \) then

\[
    A_{22}^{k_2+1} = A_{22}^{k_2}
\]

by the induction hypothesis and thus

\[
    x_2^k = A_{22}^{k_2} \otimes x_2^0 \oplus \Gamma(A_{22}) \otimes v
\]

for all \( k \geq \max(k_1, k_2, k_3) \).

It remains to consider the case when \( A_{11} \) is trivial. Then \( x_1^k = \varepsilon \) for all \( k \geq 1 \) and we have

\[
    \begin{pmatrix}
        x_2^{k+1} \\
        x_3^{k+1}
    \end{pmatrix} = \begin{pmatrix}
        A_{22} & A_{23} \\
        \varepsilon & A_{33}
    \end{pmatrix} \otimes \begin{pmatrix}
        x_2^k \\
        x_3^k
    \end{pmatrix}
\]

for all \( k \geq 1 \). We apply the induction hypothesis to the matrix

\[
    \begin{pmatrix}
        A_{22} & A_{23} \\
        \varepsilon & A_{33}
    \end{pmatrix}
\]

and deduce that \( x^{k+1} = x^k \) for \( k \) sufficiently big. This completes the proof. \( \blacksquare \)
Example 5.1 Let $A = \begin{pmatrix} 2 & \varepsilon & \varepsilon \\ \varepsilon & 1 & \varepsilon \\ 0 & 0 & 0 \end{pmatrix}$, thus $r = 3$, $\Lambda(A) = \{0, 1, 2\}$, $N_j = \{j\}, j = 1, 2, 3$. If $x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, then $A^k \otimes x, k = 1, 2, 3, 4$ are:

$$
\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 6 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 8 \\ 4 \\ 6 \end{pmatrix}, \ldots
$$

which obviously will never reach an eigenvector. The reason is that $1 \notin T_2, 2 \notin T_1$ but $\lambda(N_1) \neq \lambda(N_2)$.

Example 5.2 Let $A = \begin{pmatrix} 2 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon \\ 0 & 0 & 0 \end{pmatrix}$, thus $r = 3$, $\Lambda(A) = \{0, 2\}$, $N_j = \{j\}, j = 1, 2, 3$. This matrix is robust since both non-trivial classes ($N_1$ and $N_3$) are spectral, $p(A_{ii}) = 1$ ($i = 1, 2, 3$) and there are no non-trivial classes $N_i, N_j$ such that $i \notin T_j$ and $j \notin T_i$. Indeed, if $x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, then $A^k \otimes x, k = 1, 2, 3, 4$ are:

$$
\begin{pmatrix} 2 \\ \varepsilon \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ \varepsilon \\ 2 \end{pmatrix}, \begin{pmatrix} 6 \\ \varepsilon \\ 4 \end{pmatrix}, \begin{pmatrix} 8 \\ \varepsilon \\ 6 \end{pmatrix}, \ldots
$$

hence an eigenvector is reached in the first step.

6 Conclusions

The primary objective of this paper was to study robustness of matrices in max-algebra. The importance of robust matrices is given by the fact that if the production matrix of a multi-machine interactive production system is robust then an ultimate stationary regime is always reached, independently of the choice of initial conditions. In addition, the problem is of an intrinsic mathematical interest, and it might be interesting to extend the present study to other classes of nonlinear maps.

In this paper (Chapters 3 and 4) we have first presented fundamental results on the eigenvector-eigenvalue theory for reducible matrices in max-algebra including a comparison with the classical Perron-Frobenius theory of non-negative matrices.

In Chapter 5 we have used these results and developed the theory of robustness of reducible matrices. The principal result of the paper, Theorem 5.5, efficiently characterises robust matrices. It is followed by two numerical examples.
Acknowledgement. It is acknowledged that research of the first author was supported by the EPSRC Grant RRAH12809 and the third author was partly supported by the joint CNRS-RFBR grant 05-01-0280. We are also grateful to two referees for their comments and suggestions which improved the presentation of the paper.

References


