

# Permuted max-algebraic eigenvector problem is $NP$ -complete

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## Abstract

Let  $a \oplus b = \max(a, b)$  and  $a \otimes b = a + b$  for  $a, b \in \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\}$  and extend these operations to matrices and vectors as in conventional linear algebra. The following eigenvector problem has been intensively studied in the past: Given  $A \in \overline{\mathbb{R}}^{n \times n}$  find all  $x \in \overline{\mathbb{R}}^n$ ,  $x \neq (-\infty, \dots, -\infty)^T$  (eigenvectors) such that  $A \otimes x = \lambda \otimes x$  for some  $\lambda \in \overline{\mathbb{R}}$ . The present paper deals with the permuted eigenvector problem: Given  $A \in \overline{\mathbb{R}}^{n \times n}$  and  $x \in \overline{\mathbb{R}}^n$ , is it possible to permute the components of  $x$  so that it becomes a (max-algebraic) eigenvector of  $A$ ? Using a polynomial transformation from BANDWIDTH we prove that the integer version of this problem is  $NP$ -complete.

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## 1. Definitions, problem formulation and previous results

Let  $a \oplus b = \max(a, b)$  and  $a \otimes b = a + b$  for  $a, b \in \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\}$ . Obviously,  $-\infty$  plays the role of a neutral element for  $\oplus$ . Throughout the paper we denote  $-\infty$  by  $\varepsilon$ . If  $\alpha \in \mathbb{R}$  then the symbol  $\alpha^{-1}$  stands for  $-\alpha$ . If  $a_1, \dots, a_n \in \mathbb{R}$  then the sum  $a_1 \oplus \dots \oplus a_n$  will be denoted by  $\sum_{i=1, \dots, n}^{\oplus} a_i$ .

By *max-algebra* we understand the analogue of linear algebra developed for the pair of operations  $(\oplus, \otimes)$ , extended to matrices and vectors. That is if  $A = (a_{ij})$ ,  $B = (b_{ij})$  and  $C = (c_{ij})$

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are matrices of compatible sizes with entries from  $\overline{\mathbb{R}}$ , we write  $C = A \oplus B$  if  $c_{ij} = a_{ij} \oplus b_{ij}$  for all  $i, j$  and  $C = A \otimes B$  if  $c_{ij} = \sum_k^{\oplus} a_{ik} \otimes b_{kj} = \max_k (a_{ik} + b_{kj})$  for all  $i, j$ . If  $\alpha \in \overline{\mathbb{R}}$  then  $\alpha \otimes A = (\alpha \otimes a_{ij})$ . We assume everywhere in this paper that  $n \geq 1$  is an integer.  $P_n$  will stand for the set of permutations of the set  $\{1, \dots, n\}$ . If  $A$  is an  $n \times n$  matrix then the iterated product  $A \otimes A \otimes \dots \otimes A$  in which the symbol  $A$  appears  $k$ -times will be denoted by  $A^k$  and  $\Gamma(A) = A \oplus A^2 \oplus \dots \oplus A^n$ .

Max algebra has been studied by various authors [1,3,7–10,12–14,19,20]. The name “tropical algebra” has also been used in recent years [16–18].

A square matrix  $D$  is called *diagonal*, notation  $D = \text{diag}(d_1, \dots, d_n)$ , if its diagonal entries are  $d_1, \dots, d_n \in \mathbb{R}$  and off-diagonal entries are  $\varepsilon$ . We also denote  $I = \text{diag}(0, \dots, 0)$ . Obviously,  $A \otimes I = I \otimes A = A$  whenever  $A$  and  $I$  are of compatible sizes. Any matrix arising from  $I$  by permuting its rows and/or columns is called a *permutation matrix*.

For  $\pi, \sigma \in P_n$  the symbol  $A(\pi, \sigma)$  stands for the matrix arising from  $A$  after applying  $\pi$  [ $\sigma$ ] to the set of row [column] indices of  $A$ . Similarly we define  $v(\pi)$  for a vector  $v$ . Hence  $A(\pi, \sigma) = Q \otimes A \otimes T$  and  $v(\pi) = Q \otimes v$  for some permutation matrices  $Q$  and  $T$ .

One of basic problems in max-algebra is:

**EIGENVECTOR [EV]:** Given  $A \in \overline{\mathbb{R}}^{n \times n}$  find all  $x \in \overline{\mathbb{R}}^n$ ,  $x \neq (\varepsilon, \dots, \varepsilon)^T$  (*eigenvectors*) such that  $A \otimes x = \lambda \otimes x$  for some  $\lambda \in \overline{\mathbb{R}}$  (*eigenvalue*).

EV has been studied since the 1960s and can now be efficiently solved [1,3,7,8,10,14]. It is known that an  $n \times n$  matrix may have up to  $n$  eigenvalues. The set of eigenvectors corresponding to a particular eigenvalue is a max-algebraic linear subspace, whose generators can be found using polynomial algorithms. For solution methods the reader is referred to [1,6,9,13,14], see also [12,19].

One of the motivations for studying EV was the following analysis of the behaviour of production systems [8,9]: Suppose that machines  $M_1, \dots, M_n$  work interactively and in stages. At each stage all machines simultaneously produce components necessary for the next stage of some or all other machines. Let  $x_i(r)$  denote the starting time of the  $r$ th stage on machine  $i$  ( $i = 1, \dots, n$ ) and let  $a_{ij}$  denote the duration of the operation at which machine  $M_j$  prepares the component necessary for machine  $M_i$  at the  $(r+1)$ th stage ( $i, j = 1, \dots, n$ ). Then

$$x_i(r+1) = \max(x_1(r) + a_{i1}, \dots, x_n(r) + a_{in}) \quad (i = 1, \dots, n; r = 0, 1, \dots)$$

or, in max-algebraic notation

$$x(r+1) = A \otimes x(r) \quad (r = 0, 1, \dots)$$

where  $A = (a_{ij})$  is called a *production matrix*. We say that the system is in a *steady regime* if it moves forward in regular steps, that is if for some  $\lambda$  we have  $x(r+1) = \lambda \otimes x(r)$  for all  $r \geq 0$ ,  $r$  integer. Obviously, the system is in a steady regime if and only if  $x(0)$  is an eigenvector of  $A$  corresponding to an eigenvalue  $\lambda$ . However, if a start-time vector  $x$  is given and happens not to be an eigenvector of the production matrix we may wish to renumber (if possible) the individual jobs (that is simultaneously permute the rows and columns of  $A$ ) so that  $x$  becomes an eigenvector of  $A$ . This may be quite important since in a production process starting times may be restricted to particular instances and in general an eigenvector may or may not be “hit” by repeated applications of  $A$  to a given starting vector. For instance when  $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$  is repeatedly applied to  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$  then the resulting sequence (“orbit”) is  $\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 5 \\ 4 \end{pmatrix}, \begin{pmatrix} 6 \\ 7 \end{pmatrix}, \begin{pmatrix} 9 \\ 8 \end{pmatrix}, \dots$  which will obviously never reach an eigenvector. In fact there is only a small class of matrices for which the orbit hits an eigenvector for any starting vector (see [5] where such matrices are called robust).

So we are concerned with the task of finding an assignment of the given (fixed) starting times to individual machines so that the process is immediately in a steady regime or to decide that such an assignment does not exist. This leads to the following modification of EV:

**PERMUTED EIGENVECTOR [PEV]:** Given  $A \in \mathbb{R}^{n \times n}$  and  $x \in \mathbb{R}^n$ , is there a  $\pi \in P_n$  such that  $x(\pi)$  is an eigenvector of  $A$ ?

The main aim of the present paper is to prove the  $NP$ -completeness of the following integer version of the PEV:

**INTEGER PERMUTED EIGENVECTOR [IPEV]:** Given  $A \in \mathbb{Z}^{n \times n}$  and  $x \in \mathbb{Z}^n$ , is there a  $\pi \in P_n$  such that  $x(\pi)$  is an eigenvector of  $A$ ?

Since we do not need to consider the matrices with  $-\infty$  entries we will only summarize a selection of known results on EV for finite matrices. The reader is referred to [1,3,10,14]] for information about the general case.

An ordered pair  $D = (N, F)$  is called a *digraph* if  $N$  is a non-empty set (of *nodes*) and  $F \subseteq N \times N$  (the set of *arcs*). A sequence  $\pi = (v_1, \dots, v_p)$  of nodes is called a *path* (in  $D$ ) if  $p = 1$  or  $p > 1$  and  $(v_i, v_{i+1}) \in F$  for all  $i = 1, \dots, p-1$ . A path  $(v_1, \dots, v_p)$  is called a *cycle* if  $v_1 = v_p$  and  $p > 1$ . If  $\pi = (i_1, \dots, i_p)$  is a path in  $D$  then the *weight* of  $\pi$  is  $w(\pi, A) = a_{i_1 i_2} + a_{i_2 i_3} + \dots + a_{i_{p-1} i_p}$  if  $p > 1$  and  $\varepsilon$  if  $p = 1$ . We will only consider digraphs with at least one cycle. The symbol  $\lambda(A)$  stands for the *maximum cycle mean* of  $A$ , that is

$$\lambda(A) = \max_{\sigma} \mu(\sigma, A),$$

where the maximisation is taken over all cycles in  $D$  and

$$\mu(\sigma, A) = \frac{w(\sigma, A)}{k}$$

denotes the *mean* of the cycle  $\sigma = (i_1, \dots, i_k, i_1)$ . Note that  $\lambda(A)$  can be found in  $O(n^3)$  time using Karp's algorithm [10,14,15].

In the rest of the paper  $N = \{1, \dots, n\}$ . The *digraph associated* with  $A = (a_{ij}) \in \mathbb{R}^{n \times n}$  is

$$D_A = (N, N \times N).$$

We now present some results on EV for finite matrices. Here and elsewhere we denote the set of all eigenvectors of  $A$  by  $V(A)$ .

**Theorem 1.1** (Cuninghame-Green [9]). *Let  $A = (a_{ij}) \in \mathbb{R}^{n \times n}$  and  $k$  be the number of columns of  $\Gamma((\lambda(A))^{-1} \otimes A)$  with zero diagonal entries. Then*

1.  $\lambda(A)$  is the unique eigenvalue of  $A$ ,
2.  $k > 0$  and  $V(A) = \{\Gamma_0 \otimes x; x \in \mathbb{R}^k\}$  where  $\Gamma_0$  is the  $n \times k$  matrix consisting of the columns of  $\Gamma((\lambda(A))^{-1} \otimes A)$  with zero diagonal entries.

In this paper we follow terminology introduced in [9]. Being motivated by Theorem 1.1, the columns of  $\Gamma((\lambda(A))^{-1} \otimes A)$  with zero diagonal entries are called the *fundamental eigenvectors* of  $A$  (FEV). Two vectors  $x, y \in \mathbb{R}^n$  are called *equivalent* ( $x \sim y$ ) if  $x = \alpha \otimes y$  for some  $\alpha \in \mathbb{R}$ .

**Corollary 1.1.**  $V(A) = \{\Gamma'_0 \otimes x; x \in \mathbb{R}^d\}$  where  $d$  is the maximal number of non-equivalent fundamental eigenvectors of  $A$  and  $\Gamma'_0$  is any matrix consisting of  $d$  non-equivalent fundamental eigenvectors of  $A$ .

It is known that none of the FEVs can be expressed as a linear combination of other (non-equivalent) FEVs [1,9]. The number  $d$  in Corollary 1.1 is called the *dimension* of the eigenspace for  $A$  and will be denoted by  $d(A)$ .

**Remark 1.1.** The PEV can easily be solved when  $d = 1$  since  $V(A)$  is then the set of multiples of a single FEV, say  $g$ . Then deciding for a given  $x \in \mathbb{R}^n$  whether  $x(\pi) \in V(A)$  for some  $\pi \in P_n$  reduces to the ordering of the components of  $x$  in the same way as in  $g$  and checking whether  $x(\pi)$  is a multiple of  $g$ . This approach may help when solving the PEV for small values of  $d$  but becomes combinatorially too involved when  $d$  increases and is practically unusable for non-trivial values of  $d$ .

$A$  is called *definite* if  $\lambda(A) = 0$ . It is easily seen that  $V(\alpha \otimes A) = V(A)$  and  $\lambda(\alpha \otimes A) = \alpha \otimes \lambda(A)$  for any  $\alpha \in \mathbb{R}$ . Hence  $(\lambda(A))^{-1} \otimes A$  is definite for every  $A$  and  $V((\lambda(A))^{-1} \otimes A) = V(A)$ . Therefore with respect to solving the PEV we may assume without loss of generality that  $A$  is definite.

We denote  $E(A) = \{i \in N; \exists \sigma = (i = i_1, \dots, i_k, i_1) : \mu(\sigma, A) = \lambda(A)\}$ . The elements of  $E(A)$  are called *eigen-nodes* (of  $A$ ), or *critical nodes*. A cycle  $\sigma$  is called *critical* if  $\mu(\sigma, A) = \lambda(A)$ . The *critical digraph* of  $A$  is the digraph  $C(A)$  with the set of nodes  $\{1, \dots, n\}$ ; the set of arcs is the union of the sets of arcs of all critical cycles. It is well known that all cycles in a critical digraph are critical [1].

**Theorem 1.2** ([9]). Suppose that  $A \in \mathbb{R}^{n \times n}$ ,  $\Gamma((\lambda(A))^{-1} \otimes A) = (g_{ij})$  and let  $g_1, \dots, g_n$  be the columns of  $\Gamma((\lambda(A))^{-1} \otimes A)$ . Then

- $i \in E(A)$  if and only if  $g_{ii} = 0$ .
- If  $i, j \in E(A)$  then  $g_i \sim g_j$  if and only if  $i$  and  $j$  belong to the same critical cycle of  $A$ .

**Corollary 1.2.**  $V(A) = \{\sum_{i \in E^*(A)}^{\oplus} x_i \otimes g_i; x_i \in \mathbb{R}\}$  where  $E^*(A)$  is any maximal set of indices of non-equivalent FEVs of  $A$  and  $d(A) = |E^*(A)|$  is the number of non-trivial strongly connected components of  $C(A)$ .

Let us consider linear systems of the form

$$A \otimes x = b$$

where  $A = (a_{ij}) \in \mathbb{R}^{m \times n}$  and  $b \in \mathbb{R}^m$ . Denote  $M = \{1, \dots, m\}$  and

$$S = \{x \in \mathbb{R}^n; A \otimes x = b\},$$

$$\bar{x}_j = -\max_i (a_{ij} - b_i) \text{ for all } j \in N,$$

$$\bar{x} = (\bar{x}_1, \dots, \bar{x}_n)^T,$$

$$M_j = \{k \in M; a_{kj} - b_k = \max_i (a_{ij} - b_i)\} \text{ for all } j \in N.$$

**Theorem 1.3** ([3,19,20]). Let  $x \in \mathbb{R}^n$ . Then  $x \in S$  if and only if

- $x \leq \bar{x}$  and
- $\bigcup_{j \in N_x} M_j = M$

where  $N_x = \{j \in N; x_j = \bar{x}_j\}$ .

It follows immediately that  $A \otimes x = b$  has no solution if  $\bar{x}$  is not a solution. Therefore  $\bar{x}$  is called the *principal solution* [9]. More precisely we have

**Corollary 1.3.** *The following three statements are equivalent:*

- (a)  $S \neq \emptyset$ ,
- (b)  $\bar{x} \in S$ ,
- (c)  $\bigcup_{j \in N} M_j = M$ .

**Corollary 1.4.**  $S = \{\bar{x}\}$  if and only if

- (i)  $\bigcup_{j \in N} M_j = M$  and
- (ii)  $\bigcup_{j \in N'} M_j \neq M$  for any  $N' \subseteq N$ ,  $N' \neq N$ .

**Corollary 1.5.** *If  $m = n$  then  $S = \{\bar{x}\}$  if and only if there is a  $\pi \in P_n$  such that  $M_{\pi(j)} = \{j\}$  for all  $j \in N$ . Equivalently  $a_{i,\pi(j)} - b_i < a_{j,\pi(j)} - b_j$  for all  $i, j \in N$ ,  $i \neq j$ .*

$A = (a_{ij}) \in \mathbb{R}^{n \times n}$  is called *strongly regular* [SR] [3,9] if  $A \otimes x = b$  has a unique solution for at least one  $b \in \mathbb{R}^n$ . If  $\pi \in P_n$  then  $w(A, \pi)$  denotes  $\sum_{i \in N} a_{i,\pi(i)}$ . The task of finding a  $\pi \in P_n$  maximising  $w(A, \pi)$  is the *classical assignment problem* for  $A$ . We therefore denote

$$ap(A) = \{\sigma \in P_n; w(A, \sigma) = \max_{\pi \in P_n} w(A, \pi)\}.$$

The next theorem will be essential for our main result.

**Theorem 1.4** ([3,4]).  $A = (a_{ij}) \in \mathbb{R}^{n \times n}$  is strongly regular if and only if  $|ap(A)| = 1$ .

## 2. The proof of $NP$ -completeness

If  $A \in \mathbb{R}^{n \times n}$  then the set  $\{b \in \mathbb{R}^n; A \otimes x = b \text{ has a unique solution}\}$  is called the *simple image set* (of the mapping  $x \mapsto A \otimes x$ ), notation  $sim(A)$ . Hence “ $A$  is strongly regular” means  $sim(A) \neq \emptyset$ .  $A$  is called *normalised* if  $\lambda(A) = 0$  and all diagonal entries of  $A$  are zero. (Note that normalised matrices are called “definite” in [2] but in the present paper this word is used with a different meaning.) We will prove that IPEV is  $NP$ -complete for the class of normalised, strongly regular matrices and the  $NP$ -completeness for all integer matrices then immediately follows.

Recall that  $d(A)$  stands for the number of non-trivial strongly connected components of the critical digraph of  $A$ . Obviously,  $d(A) = d(\alpha \otimes A)$  for any  $\alpha \in \mathbb{R}$  and so when looking for  $d(A)$  we may restrict our attention to definite matrices.

**Theorem 2.1.** *Let  $A \in \mathbb{R}^{n \times n}$  be definite. Then  $d(A) = n$  if and only if  $A$  is normalised and strongly regular.*

**Proof.** If  $A$  is normalised and strongly regular then all non-trivial cycles in  $D_A$  have negative weight and thus the  $n$  loops are the only critical cycles, implying  $d(A) = n$ .

If  $d(A) = n$  then there are  $n$  pairwise disjoint critical cycles in  $D_A$ , thus the loops are the only critical cycles. Hence all diagonal entries are equal and since  $A$  is definite their value is zero. This means  $A$  is normalised. If  $A$  was not strongly regular then there would be a non-trivial cycle of weight zero or more in  $D_A$  which contradicts the fact that the loops are the only critical cycles.  $\square$

It has been known for some time that there is a strong link between the simple image set of a strongly regular matrix and its eigenspace (note that  $\text{cl}(S)$  denotes here the topological closure of the set  $S$ ):

**Theorem 2.2** ([2]). *If  $A \in \mathbb{R}^{n \times n}$  is normalised and strongly regular then*

$$V(A) = \text{cl}(\text{sim}(A)).$$

**Corollary 2.1.** *If  $A \in \mathbb{R}^{n \times n}$  is normalised and strongly regular then*

1.  $A \otimes b = b$  for every  $b \in \text{sim}(A)$ .
2. For every  $b \in V(A)$  there is a sequence  $\{b^{(k)}\}_{k=0}^{\infty} \subseteq \text{sim}(A)$  such that  $b^{(k)} \rightarrow b$ .

**Lemma 2.1.** *Let  $A = (a_{ij}) \in \mathbb{R}^{n \times n}$  be normalised and strongly regular. Then  $b \in \text{sim}(A)$  if and only if in the matrix  $A' = (a_{ij} - b_i)$  the (unique) column maximum in every column is attained on the diagonal. Equivalently*

$$a_{ij} - b_i < -b_j \quad \text{for every } i, j \in N, i \neq j.$$

**Proof.** By Corollary 1.5 there is a  $\pi \in P_n$  such that

$$a_{i, \pi(j)} - b_i < a_{j, \pi(j)} - b_j \quad \text{for all } i, j \in N, i \neq j.$$

This implies that  $\pi \in \text{ap}(A)$  thus  $\pi = id$  and the result follows.  $\square$

**Corollary 2.2.** *Let  $A = (a_{ij}) \in \mathbb{R}^{n \times n}$  be normalised and strongly regular and  $b \in \mathbb{R}^n$ . Then  $b \in V(A)$  if and only if in the matrix  $A' = (a_{ij} - b_i)$  a column maximum in every column is attained on the diagonal. Equivalently*

$$a_{ij} - b_i \leq -b_j \quad \text{for every } i, j \in N.$$

**Proof.** By Corollary 2.1  $b \in V(A)$  if and only if there is a sequence  $\{b^{(k)}\}_{k=0}^{\infty} \subseteq \text{sim}(A)$  such that  $b^{(k)} \rightarrow b$ . By Lemma 2.1 this is equivalent to

$$a_{ij} - b_i^{(k)} < -b_j^{(k)} \quad \text{for every } i, j \in N, i \neq j \quad \text{and } k = 0, 1, \dots$$

The result now follows by taking  $k \rightarrow \infty$ .  $\square$

**Corollary 2.3.** *Let  $A = (a_{ij}) \in \mathbb{Z}^{n \times n}$  be normalised and strongly regular and  $b \in \mathbb{Z}^n$ . Then  $b \in V(A)$  if and only if in the matrix  $A' = (a_{ij} - b_i)$  a column maximum in every column is attained on the diagonal. Equivalently*

$$a_{ij} - b_i \leq -b_j \quad \text{for every } i, j \in N.$$

**Proof.** Follows immediately from Corollary 2.2.  $\square$

**Theorem 2.3.** *Let  $A = (a_{ij}) \in \mathbb{Z}^{n \times n}$  be normalised and strongly regular;  $b \in \mathbb{Z}^n$  and  $\pi \in P_n$ . Then  $b(\pi) \in V(A)$  if and only if  $C = (c_{ij}) = A(\pi, \pi)$  satisfies*

$$c_{ij} \leq b_i - b_j \quad \text{for every } i, j \in N.$$

**Proof**

$$A \otimes b(\pi) = b(\pi)$$

is equivalent to

$$A \otimes (Q \otimes b) = Q \otimes b$$

for some permutation matrix  $Q$ . This is equivalent to

$$(Q^{-1} \otimes A \otimes Q) \otimes b = b.$$

Hence  $b \in V(C)$  for  $C = Q^{-1} \otimes A \otimes Q$ . Since  $Q^{-1} \otimes A \otimes Q = A(\pi, \pi)$  the rest now follows from Corollary 2.3.  $\square$

The following problem [BW] is known to be  $NP$ -complete ([11], Problem GT40 BANDWIDTH):

**Problem 2.4.** Given an undirected graph  $G = (N, E)$  and a positive integer  $K \leq n$ , is there a  $\pi \in P_n$  such that  $|\pi(u) - \pi(v)| \leq K$  for all  $uv \in E$ ?

In matrix terminology BW is:

**Problem 2.5.** Given an  $n \times n$  symmetric 0–1 matrix  $M = (m_{ij})$  with zero diagonal, and a positive integer  $K \leq n$ , is there a  $\pi \in P_n$  such that  $|\pi(i) - \pi(j)| \leq K$  whenever  $m_{ij} = 1$ ?

Equivalently:

**Problem 2.6.** Given an  $n \times n$  symmetric 0–1 matrix  $M = (m_{ij})$  with zero diagonal, and a positive integer  $K \leq n$ , is there a  $\pi \in P_n$  such that  $|i - j| \leq K$  whenever  $m_{\pi(i), \pi(j)} = 1$ ?

**Theorem 2.7.** *IPEV is  $NP$ -complete for the class of normalised, strongly regular matrices (and hence also for all integer matrices).*

**Proof.** IPEV is in  $NP$  since  $\pi$  (certificate) can be described polynomially and

$$A \otimes b(\pi) = b(\pi)$$

can be checked in a polynomial number of steps ( $O(n^2)$ ).

We now show that Problem 2.6 polynomially transforms to IPEV. Let  $M = (m_{ij})$  and a positive integer  $K \leq n$  be an instance of Problem 2.6. Let  $A = (a_{ij}) \in \mathbb{Z}^{n \times n}$  be defined as follows:

$$\begin{aligned} a_{ij} &= -K & \text{if } m_{ij} &= 1 \\ a_{ij} &= -n & \text{if } m_{ij} &= 0, i \neq j \\ a_{ij} &= 0 & \text{if } i &= j \end{aligned}$$

Obviously  $A$  is a normalised, strongly regular matrix. Let  $b = (1, \dots, n)^T$ . Then by Theorem 2.3 the answer to IPEV for  $A$  and  $b$  is “yes” if and only if there is a  $\pi \in P_n$  such that

$$a_{\pi(i), \pi(j)} \leq i - j \quad \text{for all } i, j \in N.$$

This is equivalent to

$$-K \leq i - j \quad \text{if } m_{\pi(i)\pi(j)} = 1,$$

that is by symmetry of  $M$

$$K \geq |i - j| \quad \text{if } m_{\pi(i)\pi(j)} = 1,$$

which is equivalent to the answer “yes” for Problem 2.6.  $\square$

Theorem 2.7 enables us to easily derive  $NP$ -completeness of the permuted version of another basic problem. Let us consider the following:

**LINEAR SYSTEM [LS]:** Given  $A = (a_{ij}) \in \mathbb{R}^{m \times n}$  and  $b \in \mathbb{R}^m$ , is there an  $x \in \mathbb{R}^n$  such that

$$A \otimes x = b?$$

**PERMUTED LINEAR SYSTEM [PLS]:** Given  $A = (a_{ij}) \in \mathbb{R}^{m \times n}$  and  $b \in \mathbb{R}^m$ , is there a  $\pi \in P_m$  and an  $x \in \mathbb{R}^n$  such that

$$A \otimes x = b(\pi)?$$

**INTEGER PERMUTED LINEAR SYSTEM [IPLS]:** Given  $A = (a_{ij}) \in \mathbb{Z}^{m \times n}$  and  $b \in \mathbb{Z}^m$ , is there a  $\pi \in P_m$  and an  $x \in \mathbb{Z}^n$  such that

$$A \otimes x = b(\pi)?$$

Note that if  $A$  and  $b$  are integer then the principal solution is also integer. Therefore in the formulation of the IPLS it does not matter whether  $x \in \mathbb{Z}^n$  or  $x \in \mathbb{R}^n$ .

**Theorem 2.8.** *IPLS is  $NP$ -complete.*

**Proof.** IPLS is clearly in  $NP$ . Now it suffices to polynomially transform IPEV to IPLS. Let  $A \in \mathbb{Z}^{n \times n}$  and  $z \in \mathbb{Z}^n$  be an instance of the IPEV. We may assume without loss of generality that  $A$  is definite. By Theorem 1.1  $z(\pi) \in V(A)$  for some  $\pi \in P_n$  if and only if

$$\Gamma_0 \otimes x = z(\pi) \tag{1}$$

has a solution  $x \in \mathbb{R}^k$  where  $\Gamma_0$  is the matrix consisting of the columns of  $\Gamma(A)$  with zero diagonal entries. Since  $\Gamma(A)$  is integer,  $x$  may be assumed to be integer too and so (1) is an instance of IPLS. It remains to say that  $\Gamma(A)$  and therefore also  $\Gamma_0$  can be found in  $O(n^3)$  time using the Floyd–Warshall algorithm.  $\square$

### 3. Conclusions

We have studied the permuted max-algebraic eigenvector problem and we have proved that for a class of integer matrices this problem is  $NP$ -complete by a polynomial transformation from BANDWIDTH. We deduce that this problem is  $NP$ -complete for all integer matrices. As a by-product a similar result has been proved for the permuted linear system problem in max-algebra. These results stimulate to look for heuristic methods for solving PEV and PLS. In the cases when the answer to the above mentioned permutation problems is negative a polynomial method may be useful for finding a permutation that is best with respect to a specified criterion.

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