



# On matrix powers in max-algebra

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Received 30 June 2006; accepted 28 September 2006

Available online 28 November 2006

Submitted by R.A. Brualdi

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## Abstract

Let  $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}$ ,  $N = \{1, \dots, n\}$  and  $D_A$  be the digraph  
 $(N, \{(i, j); a_{ij} > -\infty\})$ .

The matrix  $A$  is called *irreducible* if  $D_A$  is strongly connected, and *strongly irreducible* if every max-algebraic power of  $A$  is irreducible.  $A$  is called *robust* if for every  $x$  with at least one finite component,  $A^{(k)} \otimes x$  is an eigenvector of  $A$  for some natural number  $k$ . We study the eigenvalue–eigenvector problem for powers of irreducible matrices. This enables us to characterise robust irreducible matrices. In particular, robust strongly irreducible matrices are described in terms of eigenspaces of matrix powers.

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*AMS classification:* 15A18

*Keywords:* Max-algebra; Matrix powers; Irreducible matrix; Eigenspace

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## 1. Max-algebra basics

*Max-algebra* is an analogue of classical linear algebra, developed in the 1960s for the study of certain industrial production, data-processing and related systems [4,11]. It begins by defining  $a \oplus b = \max(a, b)$  and  $a \otimes b = a + b$  for  $a, b \in \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\}$ . The operations  $\oplus$ ,  $\otimes$  are then extended to matrices and vectors exactly as in linear algebra. That is if  $A = (a_{ij})$ ,  $B = (b_{ij})$  and  $C = (c_{ij})$  are matrices of compatible sizes with entries from  $\overline{\mathbb{R}}$ , we write  $C = A \oplus B$

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if  $c_{ij} = a_{ij} \oplus b_{ij}$  for all  $i, j$  and  $C = A \otimes B$  if  $c_{ij} = \sum_k^{\oplus} a_{ik} \otimes b_{kj} = \max_k (a_{ik} + b_{kj})$  for all  $i, j$ . If  $\alpha \in \overline{\mathbb{R}}$  then  $\alpha \otimes A = (\alpha \otimes a_{ij})$ . Here and elsewhere the symbol  $\sum^{\oplus}$  indicates as usual a repeated use of the operator  $\oplus$ .

Obviously,  $-\infty$  plays the role of a neutral element for  $\oplus$ . Throughout the paper we denote  $-\infty$  by  $\varepsilon$  and for convenience we also denote by the same symbol any vector or matrix all of whose entries are  $-\infty$ .

The iterated product  $a \otimes a \otimes \dots \otimes a$  in which the element  $a$  is used  $k$  times will be denoted by  $a^{(k)}$ . Clearly  $a^{(k)} = ka$ . The symbol  $a^{(-1)}$  or just  $a^{-1}$  will denote  $-a$  for  $a \in \mathbb{R}$ .

Similarly, if  $A$  is a square matrix then the iterated product  $A \otimes A \otimes \dots \otimes A$  in which the symbol  $A$  appears  $k$  times will be denoted by  $A^{(k)}$ . We also denote  $\Gamma(A) = A \oplus A^{(2)} \oplus \dots$  whenever this terminates finitely.

A square matrix  $D$  is called *diagonal*, notation  $D = \text{diag}(d_1, \dots, d_n)$ , if its diagonal entries are  $d_1, \dots, d_n$  and off-diagonal entries are  $\varepsilon$ . We also define  $I = \text{diag}(0, \dots, 0)$ . Obviously,  $A \otimes I = I \otimes A = A$  whenever  $A$  and  $I$  are of compatible sizes. By definition  $A^{(0)} = I$  for any square matrix  $A$ . A matrix arising from  $I$  by permuting the rows or columns is called a *permutation matrix*. If  $A$  is a square matrix and  $A \otimes B = I = B \otimes A$  for some matrix  $B$  then  $B$  is called the *inverse* of  $A$  and denoted by  $A^{-1}$ . Every matrix has at most one inverse. If  $P$  is a permutation matrix then  $P^{-1}$  exists and is again a permutation matrix.

An ordered pair  $D = (N, F)$  is called a *digraph* if  $N$  is a non-empty set (of *nodes*) and  $F \subseteq N \times N$  (the set of *arcs*). A sequence  $\pi = (v_1, \dots, v_p)$  of nodes is called a *path* (in  $D$ ) if  $p = 1$  or  $p > 1$  and  $(v_i, v_{i+1}) \in F$  for all  $i = 1, \dots, p - 1$ . The number  $p - 1$  is called the *length* of  $\pi$ . The node  $v_1$  is called the *starting node* and  $v_p$  the *endnode* of  $\pi$ , respectively. If  $p > 1$  then the arcs  $(v_i, v_{i+1})$  ( $i = 1, \dots, p - 1$ ) are said to *belong* to the path  $\pi$ . Note that a path may consist of a single node. If there is a path in  $D$  with starting node  $u$  and endnode  $v$  then we say that  $v$  is *reachable* from  $u$ , notation  $u \rightarrow v$ . Thus  $u \rightarrow u$  for any  $u \in N$ . As usual a digraph  $D$  is called *strongly connected* if  $u \rightarrow v$  and  $v \rightarrow u$  for any nodes  $u, v \in N$ . A path  $(v_1, \dots, v_p)$  is called a *cycle* if  $v_1 = v_p$  and  $p > 1$  and it is called an *elementary cycle* if, moreover,  $v_i \neq v_j$  for  $i, j = 1, \dots, p - 1, i \neq j$ .

In the rest of the paper  $N = \{1, \dots, n\}$ . The digraph associated with  $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}$  is

$$D_A = (N, \{(i, j); a_{ij} > \varepsilon\}).$$

The matrix  $A$  is called *irreducible* if  $D_A$  is strongly connected, *reducible* otherwise. Thus, every  $1 \times 1$  matrix is irreducible.

Notice that an irreducible matrix of order 2 or more has no  $\varepsilon$  columns and thus we have:

**Lemma 1.1.** *If  $A \in \overline{\mathbb{R}}^{n \times n}$  ( $n \geq 2$ ) is irreducible and  $x \in \overline{\mathbb{R}}^n - \{\varepsilon\}$  then also  $A \otimes x \in \overline{\mathbb{R}}^n - \{\varepsilon\}$ .*

The max-algebraic *eigenvalue–eigenvector problem* (briefly *eigenproblem*) is the following:

Given  $A \in \overline{\mathbb{R}}^{n \times n}$ , find  $\lambda \in \overline{\mathbb{R}}, x \in \overline{\mathbb{R}}^n, x \neq \varepsilon$  such that  $A \otimes x = \lambda \otimes x$ .

This problem has been studied since the 1960s [4]. One of the motivations was the following analysis of the steady-state behaviour of production systems: Suppose that machines  $M_1, \dots, M_n$  work interactively and in repetitive stages. At each stage all machines simultaneously produce components necessary for the next stage of some or all other machines. Let  $x_i(r)$  denote the starting time of the  $r$ th stage on machine  $i$  ( $i = 1, \dots, n$ ) and let  $a_{ij}$  denote the duration of the operation at which machine  $M_j$  prepares the component necessary for machine  $M_i$  in the  $(r + 1)$ st stage ( $i, j = 1, \dots, n$ ). Then

$$x_i(r + 1) = \max(x_1(r) + a_{i1}, \dots, x_n(r) + a_{in}) \quad (i = 1, \dots, n; r = 0, 1, \dots)$$

or, in max-algebraic notation

$$x(r + 1) = A \otimes x(r) \quad (r = 0, 1, \dots),$$

where  $A = (a_{ij})$  is called a *production matrix*. We say that the system reaches a *steady state* if it eventually moves forward in regular steps, that is if for some  $\lambda$  and  $r_0$  we have  $x(r + 1) = \lambda \otimes x(r)$  for all  $r \geq r_0$ . Obviously, a steady state is reached immediately if  $x(0)$  is an eigenvector of  $A$  corresponding to an eigenvalue  $\lambda$ . However, if the choice of a start-time vector is restricted we may need to find out for which vectors a steady state will be reached. A particular task is to characterise those production matrices for which a steady state is reached with any start-time vector. Such matrices are called *robust* and it is the primary objective of the present paper to provide a characterisation of robust irreducible matrices.

If  $\pi = (i_1, \dots, i_p)$  is a path in  $D_A$  then the *weight* of  $\pi$  is  $w(\pi, A) = a_{i_1 i_2} + a_{i_2 i_3} + \dots + a_{i_{p-1} i_p}$  if  $p > 1$  and  $\varepsilon$  if  $p = 1$ . The symbol  $\lambda(A)$  stands for the *maximum cycle mean* of  $A$ , that is if  $D_A$  has at least one cycle then

$$\lambda(A) = \max_{\sigma} \mu(\sigma, A), \tag{1}$$

where the maximisation is taken over all cycles in  $D_A$  and

$$\mu(\sigma, A) = \frac{w(\sigma, A)}{k} \tag{2}$$

denotes the *mean* of the cycle  $\sigma = (i_1, \dots, i_k, i_1)$ . Note that  $\lambda(A)$  remains unchanged if the maximisation in (1) is taken over all elementary cycles. If  $D_A$  is acyclic we set  $\lambda(A) = \varepsilon$ . Various algorithms for finding  $\lambda(A)$  exist. One of them is Karp’s [10] of computational complexity  $O(n^3)$ .

$A$  is called *definite* if  $\lambda(A) = 0$ . It is easily seen that  $\lambda(\alpha \otimes A) = \alpha \otimes \lambda(A)$  for any  $\alpha \in \mathbb{R}$ . Hence  $\lambda(A)^{-1} \otimes A$  is definite whenever  $\lambda(A) > \varepsilon$ .

The notation  $A = (c_1, \dots, c_n)$  means that  $c_1, \dots, c_n$  are the column vectors of  $A$ . If  $A$  is definite then  $\Gamma(A) = A \oplus A^2 \oplus \dots \oplus A^n$  [5]. In this case  $\Gamma(A) = (g_{ij})$  is the matrix of the weights of the heaviest paths in  $D_A$  and so, specifically, if  $\Gamma(A) = (\gamma_1, \dots, \gamma_n)$  then  $\gamma_i$  is the vector of the weights of the heaviest paths with endnode  $i$  ( $i = 1, \dots, n$ ).  $\Gamma(A)$  is called a *metric matrix*; it can be found using the Floyd–Warshall algorithm using  $O(n^3)$  operations [6].

We also denote  $E(A) = \{i \in N; \exists \sigma = (i = i_1, \dots, i_k, i_1) : \mu(\sigma, A) = \lambda(A)\}$ . The elements of  $E(A)$  are called *eigen-nodes* (of  $A$ ), or *critical nodes*. A cycle  $\sigma$  is called *critical* if  $\mu(\sigma, A) = \lambda(A)$ . The *critical digraph* of  $A$  is the digraph  $C(A)$  with the set of nodes  $E(A)$ ; the set of arcs is the union of the sets of arcs of all critical cycles. All cycles in a critical digraph are critical (this follows from Lemma 2.3 and Theorem 2.1 below).

Note that  $\lambda(A) > \varepsilon$  if  $n \geq 2$  and  $A$  is irreducible.

**Theorem 1.1** [5]. *If  $A \in \overline{\mathbb{R}}^{n \times n}$  ( $n \geq 2$ ) is an irreducible matrix then  $A$  has a unique eigenvalue equal to  $\lambda(A)$ , all eigenvectors of  $A$  are finite and the set of all eigenvectors is*

$$\left\{ \sum_{i \in E(A)}^{\oplus} \alpha_i \otimes \gamma_i; \alpha_i \in \mathbb{R} \right\},$$

where  $\Gamma(\lambda(A)^{-1} \otimes A) = (\gamma_1, \dots, \gamma_n)$ .

We will denote the set of all eigenvectors of  $A$  by  $V(A)$ . A set  $S \subseteq \overline{\mathbb{R}}^n$  is called a (max-algebraic) *subspace* if  $x \oplus y \in S$  and  $\alpha \otimes x \in S$  for any  $x, y \in S$  and  $\alpha \in \mathbb{R}$ . It follows from Theorem 1.1 that  $V(A)$  is a subspace for any irreducible matrix  $A$  and in this case we call  $V(A)$  the *eigenspace* (of  $A$ ). Note that the set  $V(A)$  is in general not a subspace for reducible  $n \times n$  matrices which may have up to  $n$  eigenvalues [7].

Being motivated by Theorem 1.1,  $\gamma_i, i \in E(A)$  are called the *fundamental eigenvectors* of the irreducible matrix  $A$  (FEV).

The following are also known [5]:

- $i \in E(A) \iff g_{ii} = 0$  (here  $\Gamma(\lambda(A)^{-1} \otimes A) = (g_{ij})$ ).
- If  $i, j \in E(A)$  then  $\gamma_i = \alpha \otimes \gamma_j$  if and only if  $i$  and  $j$  belong to the same critical cycle of  $A$ .

If  $i, j \in E(A)$  and  $\gamma_i = \alpha \otimes \gamma_j$  then  $\gamma_i$  and  $\gamma_j$  are called *equivalent*. Consequently,

$$V(A) = \left\{ \sum_{i \in E^*(A)}^{\oplus} \alpha_i \otimes \gamma_i; \alpha_i \in \mathbb{R} \right\}, \tag{3}$$

where  $E^*(A)$  is any maximal set of non-equivalent FEV of  $A$ . The number of maximal sets of equivalent FEVs (or, equivalently the number of strongly connected components of  $C(A)$ ) is called the *dimension* of the eigenspace and will be denoted by  $\dim(A)$ , or just  $d$ . The sets of nodes of the strongly connected components of  $C(A)$  will be denoted by  $E_1, E_2, \dots, E_d$ . Hence  $E(A) = \cup_{j=1}^d E_j$ .

The sets  $E_1, E_2, \dots, E_d$  will be called the *equivalence classes* of the set  $E$ .

## 2. Eigenspaces of powers of irreducible matrices

In this section we aim at providing information about the eigenspaces of the powers of irreducible matrices. Since the powers of an irreducible matrix may in general not be irreducible we need to assume this explicitly. A matrix  $A \in \overline{\mathbb{R}}^{n \times n}$  is called *strongly irreducible* if  $A^{(k)}$  is irreducible for every  $k = 1, \dots$ . Note that every finite matrix is strongly irreducible.

We start by describing some matrix operations that do not essentially change the eigenproblem. For simplicity we will assume that the matrices considered are irreducible although some statements are true for any matrices.

**Lemma 2.1.** *Let  $A, B \in \overline{\mathbb{R}}^{n \times n}$ ,  $A$  irreducible and  $B = \alpha \otimes A$ , where  $\alpha \in \mathbb{R}$ . Then*

1.  $\mu(B, \sigma) = \alpha \otimes \mu(A, \sigma)$  for every cycle  $\sigma$ ,
2.  $\lambda(B) = \alpha \otimes \lambda(A)$ ,
3.  $B$  is irreducible and  $E(A) = E(B)$ ,
4.  $\dim(A) = \dim(B)$ ,
5.  $E(A)$  and  $E(B)$  have the same equivalence classes,
6.  $V(A) = V(B)$ .

**Proof.** All statements are proved straightforwardly from the definitions.  $\square$

**Corollary 2.1.** Let  $A \in \overline{\mathbb{R}}^{n \times n}$  be irreducible and  $\lambda(A) > \varepsilon$ . Then  $\lambda(A)^{-1} \otimes A$  is definite.

**Lemma 2.2.** Let  $A, B \in \overline{\mathbb{R}}^{n \times n}$  and  $B = P^{-1} \otimes A \otimes P$ , where  $P$  is a permutation matrix. Then

1.  $A$  is irreducible if and only if  $B$  is irreducible,
2. the sets of cycle lengths in  $D_A$  and  $D_B$  are equal,
3.  $A$  and  $B$  have the same eigenvalues and
4. there is a bijection between  $V(A)$  and  $V(B)$  described by:

$$V(B) = \{P^{-1} \otimes x; x \in V(A)\}.$$

**Proof.** To prove (1) and (2) note that  $B$  is obtained from  $A$  by a simultaneous permutation of the rows and columns. Hence  $D_B$  differs from  $D_A$  by the numbering of the nodes only and the statements follow. For (3) and (4) we observe that  $B \otimes z = \lambda \otimes z$  if and only if  $A \otimes P \otimes z = \lambda \otimes P \otimes z$ , that is if and only if  $z = P^{-1} \otimes x$  for some  $x \in V(A)$ .  $\square$

Note that if  $D = \text{diag}(d_1, \dots, d_n), d_1, \dots, d_n \in \mathbb{R}$  then  $D^{-1} = \text{diag}(d_1^{-1}, \dots, d_n^{-1})$ .

**Lemma 2.3.** Let  $A, B \in \overline{\mathbb{R}}^{n \times n}$ ,  $A$  irreducible and  $B = D^{-1} \otimes A \otimes D$ , where  $D = \text{diag}(d_1, \dots, d_n), d_1, \dots, d_n \in \mathbb{R}$ . Then

1.  $w(A, \sigma) = w(B, \sigma)$  for every cycle  $\sigma$ ,
2.  $\lambda(A) = \lambda(B)$ ,
3.  $B$  is irreducible and  $E(A) = E(B)$ ,
4.  $\dim(A) = \dim(B)$ ,
5.  $E(A)$  and  $E(B)$  have the same equivalence classes,
6.  $\Gamma(B) = D^{-1} \otimes \Gamma(A) \otimes D$ .

**Proof.** The first property is well known and easily proved. The remaining statements follow immediately from the first one. For instance in the case of the last one we have:

$$\begin{aligned} \Gamma(B) &= \\ \sum_{j=1, \dots, n}^{\oplus} (D^{-1} \otimes A \otimes D)^{(j)} &= \\ D^{-1} \otimes \left( \sum_{j=1, \dots, n}^{\oplus} A^{(j)} \right) \otimes D &= \\ D^{-1} \otimes \Gamma(A) \otimes D. &\quad \square \end{aligned}$$

**Theorem 2.1.** Let  $A \in \overline{\mathbb{R}}^{n \times n}$  be irreducible and definite. If  $x = (x_1, \dots, x_n)^T \in V(A)$ ,  $D = \text{diag}(x_1, \dots, x_n)$  and  $B = D^{-1} \otimes A \otimes D$  then  $B$  is non-positive, irreducible and definite. Hence  $\sigma$  is a critical cycle for  $A$  if and only if  $\sigma$  is a zero cycle for  $B$ .

**Proof.** We have

$$\begin{aligned} & A \otimes x = x \\ \max_{j=1, \dots, n} & (a_{ij} + x_j) = x_i \quad (i = 1, \dots, n) \\ & -x_i + a_{ij} + x_j \leq 0 \quad (i, j = 1, \dots, n) \end{aligned}$$

and so  $B$  is non-positive. Also,  $\lambda(B) = \lambda(A) = 0$  and so critical cycles in  $B$  and in  $A$  are exactly the zero cycles in  $B$ .  $\square$

Theorem 2.2 below applies to definite, non-positive matrices. Note that the statements proved above in this section show how a general irreducible matrix can be transformed to a definite, non-positive matrix.

A matrix  $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}$  is called 0-irreducible if the digraph  $D_A^0 = (N, \{(i, j); a_{ij} = 0\})$  is strongly connected. Since a strongly connected digraph with 2 or more nodes contains at least one cycle, every 0-irreducible non-positive matrix of order 2 or more is definite.

Theorem 2.2 below is an application of [2, Theorem 3.4.5].

**Theorem 2.2** (Brualdi–Ryser). *Let  $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}$  be a 0-irreducible and non-positive matrix and  $g$  be the gcd of the lengths of zero cycles. Let  $k$  be a positive integer. Then there is a permutation matrix  $P$  such that  $P^{-1} \otimes A^{(k)} \otimes P$  has  $r$  0-irreducible diagonal blocks where  $r = \text{gcd}(k, g)$  and all elements outside these blocks are negative.*

**Corollary 2.2.** *Let  $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}$  be a matrix with  $\lambda(A) > \varepsilon$  whose critical digraph  $C(A)$  is strongly connected and  $g$  be the gcd of the lengths of all critical cycles. Let  $k$  be a positive integer. Then  $C(A^{(k)})$  has  $r$  connected components where  $r = \text{gcd}(k, g)$ .*

**Proof.** Follows from Theorems 2.1, 2.2 and Lemma 2.2.  $\square$

The following is a classical result corresponding to the case  $k = 1$  in [2, Theorem 3.4.5].

**Corollary 2.3.** *Let  $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}$  be irreducible. Then  $A$  is strongly irreducible if and only if the lengths of all cycles in  $D_A$  are co-prime.*

**Theorem 2.3.** *Let  $k, n$  be positive integers,  $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}$  be strongly irreducible. Then*

1.  $\lambda(A^{(k)}) = (\lambda(A))^{(k)}$  and  $V(A) \subseteq V(A^{(k)})$ .
2.  $\Gamma((\lambda(A^{(k)}))^{-1} \otimes A^{(k)}) \leq \Gamma((\lambda(A))^{-1} \otimes A)$ .
3.  $E(A) = E(A^{(k)})$  and the equivalence classes of  $E(A^{(k)})$  are either equal to the equivalence classes of  $E(A)$  or are their refinements.
4. If  $v_j, u_j$  ( $j \in E(A)$ ) are the fundamental eigenvectors (FEV) of  $A$  and  $A^{(k)}$  respectively, then  $v_j \geq u_j$  for all  $j \in E(A)$ .
5.  $\dim(A^{(k)}) = \sum_i \text{gcd}(r_i, k)$  where  $r_i$  is the gcd of the lengths of all critical cycles of  $A$  in the  $i$ th connected component of  $C(A)$ .

**Proof**

1. If  $A \otimes x = \lambda \otimes x$  then  $A^{(k)} \otimes x = \lambda^{(k)} \otimes x$ . Since  $A^{(k)}$  is irreducible,  $\lambda(A^{(k)})$  is its unique eigenvalue and the statements follow.

2. Denote  $\lambda(A)^{(-1)} \otimes A$  as  $B$ . Then LHS is  $\Gamma(B^{(k)}) = B^{(k)} \oplus B^{(2k)} \oplus \dots \oplus B^{(nk)} \leq \Gamma(B)$  because (as is well known)  $B^{(r)} \leq \Gamma(B)$  for every natural  $r > 0$ .
3.  $E(A) \supseteq E(A^{(k)})$  follows from Part 2 immediately since in a metric matrix all diagonal elements are non-positive and the  $j$ th diagonal entry is zero if and only if  $j$  is a critical node. Now let  $j \in E(A)$  and  $\sigma = (j = j_0, j_1, \dots, j_r = j)$  be any critical cycle in  $A$  (and  $B$ ) containing  $j$ , thus  $w(\sigma, B) = 0$ . Let us denote

$$\pi = (j = j_0, j_{k(\text{mod } r)}, j_{2k(\text{mod } r)}, \dots, j_{rk(\text{mod } r)} = j)$$

and  $B^{(k)}$  by  $C = (c_{ij})$ . Then for all  $i = 0, 1, \dots, r - 1$  we have (all indices are mod  $r$  and, for convenience, we write here  $c(i, j)$  rather than  $c_{ij}$ , similarly  $b(i, j)$ ):

$$c(j_{ik}, j_{ik+k}) \geq b(j_{ik}, j_{ik+1}) + b(j_{ik+1}, j_{ik+2}) + \dots + b(j_{ik+k-1}, j_{ik+k})$$

since  $c(j_{ik}, j_{ik+k})$  is the weight of a heaviest path of length  $k$  from  $j_{ik}$  to  $j_{ik+k}$  w.r.t.  $B$  and the RHS is the weight of one such path. Therefore

$$w(\pi, (\lambda(A^{(k)}))^{-1} \otimes A^{(k)}) = w(\pi, B^{(k)}) \geq (w(\sigma, B))^{(k)} = 0.$$

Hence, equality holds, as there are no positive cycles in  $(\lambda(A^{(k)}))^{-1} \otimes A^{(k)}$ . This implies that  $\pi$  is a critical cycle w.r.t.  $A^{(k)}$  and so  $j \in E(A^{(k)})$ .

If  $w$  is the weight of an arc  $(u, v)$  on a critical cycle for  $A^{(k)}$  then there is a path from  $u$  to  $v$  having the total weight  $w$  w.r.t.  $A$ . Therefore all nodes on a critical cycle for  $A^{(k)}$  belong to one critical cycle for  $A$ . Hence the refinement statement.

4. Follows from Part 2 immediately.
5. It now follows from Theorems 2.1 and 2.2.  $\square$

Note that the first two statements of Theorem 2.3 also hold without the assumption of strong irreducibility. However, for the key part 5. we need to assume that  $A$  is strongly irreducible. Actually the proof of 2. presented above does not require  $A$  to be strongly irreducible and the proof of the inclusion in 1. is straightforward from the definition for any matrix. The proof of the identity in 1. for a general matrix (although not very difficult) is beyond the scope of the present paper. It easily follows from the results of general spectral theory, see e.g. [3].

### 3. Robust matrices

Let  $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}$  and  $x \in \overline{\mathbb{R}}^n$ . The orbit of  $A$  with starting vector  $x$  is the sequence  $O(A, x) = \{A^{(r)} \otimes x; r = 0, 1, \dots\}$ .

Let

$$T(A) = \{x \in \overline{\mathbb{R}}^n; O(A, x) \cap V(A) \neq \emptyset\}.$$

As mentioned at the beginning of this paper the set  $T(A)$  is of interest in questions of system stability. Obviously,

$$V(A) \subseteq T(A) \subseteq \overline{\mathbb{R}}^n - \{\varepsilon\}$$

holds for every matrix  $A \in \overline{\mathbb{R}}^{n \times n}$ .

It may happen that  $T(A) = V(A)$ , for instance when  $A$  is the irreducible matrix  $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ : Here  $\lambda(A) = 0$  and

$$V(A) = \{\alpha \otimes (0, 0)^T; \alpha \in \mathbb{R}\}.$$

Since

$$A \otimes \begin{pmatrix} a \\ b \end{pmatrix} = (\max(a - 1, b), \max(a, b - 1))^T,$$

we have that  $A \otimes \begin{pmatrix} a \\ b \end{pmatrix} \in V(A)$  if and only if  $a = b$ , that is  $A \otimes x \in V(A)$  if and only if  $x \in V(A)$ . Hence  $T(A) = V(A)$ . We generalize:

**Theorem 3.1.** *Let  $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}$  be irreducible. Then  $T(A) = V(A)$  if and only if for every  $x \in \overline{\mathbb{R}}^n - \{\varepsilon\}$ :*

$$A \otimes x \in V(A) \iff x \in V(A).$$

**Proof.** Recall that  $x \in V(A) \implies A \otimes x \in V(A)$  holds for every irreducible matrix  $A$  by Lemma 1.1.

Suppose first that  $T(A) = V(A)$  and  $A \otimes x \in V(A)$ ,  $x \in \overline{\mathbb{R}}^n - \{\varepsilon\}$ . Then  $x \in T(A)$  and hence also  $x \in V(A)$ .

Suppose now that

$$A \otimes x \in V(A) \implies x \in V(A)$$

holds for every  $x \in \overline{\mathbb{R}}^n - \{\varepsilon\}$  and let  $x \in T(A)$ . Then  $A^{(k)} \otimes x \in \overline{\mathbb{R}}^n - \{\varepsilon\}$  by Lemma 1.1 for all  $k$  and  $A^{(k)} \otimes x \in V(A)$  for some  $k$ , thus  $A^{(k-1)} \otimes x \in V(A)$ ,  $A^{(k-2)} \otimes x \in V(A)$ ,  $\dots$ ,  $x \in V(A)$ .  $\square$

Note that  $T(A)$  may be different from both  $V(A)$  and  $\overline{\mathbb{R}}^n - \{\varepsilon\}$ : Consider the irreducible matrix

$$A = \begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ -1 & -1 & 0 \end{pmatrix}.$$

Here  $\lambda(A) = 0$  and  $x = (-2, -2, 0)^T \notin V(A)$ , but  $A \otimes x = (-1, -1, 0)^T \in V(A)$ , showing that  $T(A) \neq V(A)$ . At the same time if  $y = (0, -1, 0)^T$  then  $A^{(k)} \otimes y$  is  $y$  for  $k$  even and  $(-1, 0, 0)^T$  for  $k$  odd, showing that  $y \notin T(A)$ .

**Definition 3.1.** If  $T(A) = \overline{\mathbb{R}}^n - \{\varepsilon\}$  then  $A$  is called robust.

Now we present a characterisation of irreducible robust matrices which enables us to check this property easily. Then we show special criteria for strongly irreducible matrices. We say that  $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}$  is *ultimately periodic* if there is a natural number  $p$  such that the following holds for some  $\lambda \in \mathbb{R}$  and  $k_0$  natural:

$$A^{(k+p)} = \lambda \otimes A^{(k)} \quad \text{for all } k \geq k_0.$$

If  $p$  is the smallest natural number with this property then we call  $p$  the *period* of  $A$  and denote it  $\text{per}(A)$ . If  $A$  is not ultimately periodic then we set  $\text{per}(A) = +\infty$ . It is easily seen that  $\lambda = \lambda(A)^{(p)}$  if  $A$  is an irreducible matrix and  $p = \text{per}(A) < +\infty$ .

**Theorem 3.2.** *Let  $A \in \overline{\mathbb{R}}^{n \times n}$  be irreducible. Then  $A$  is robust if and only if  $\text{per}(A) = 1$ .*

**Proof.** Let  $\text{per}(A) = 1$ ,  $x \in \overline{\mathbb{R}}^n - \{\varepsilon\}$  and  $k \geq k_0$ . Then  $A^{(k)} \otimes x \in \overline{\mathbb{R}}^n - \{\varepsilon\}$  by Lemma 1.1,  $A^{(k+1)} \otimes x = \lambda(A) \otimes A^{(k)} \otimes x$  and so  $A^{(k)} \otimes x \in V(A)$ . Hence  $A$  is robust.



Now let  $A$  be robust and  $x = a_j$  ( $j$ th column of  $A$ ). Then  $x \in \overline{\mathbb{R}}^n - \{\varepsilon\}$  and there is an integer  $k_j$  such that  $A^{(k+1)} \otimes a_j = \lambda(A) \otimes A^{(k)} \otimes a_j$  for all  $k \geq k_j$ . So, if  $k_0 = \max(k_1, \dots, k_n)$  then  $A^{(k+2)} = \lambda(A) \otimes A^{(k+1)}$  for all  $k \geq k_0$ , and thus  $\text{per}(A) = 1$ .  $\square$

A by-product of this investigations is

**Corollary 3.1.** *Let  $A \in \overline{\mathbb{R}}^{n \times n}$  be irreducible. If  $\text{per}(A) = 1$ ,  $x \in \overline{\mathbb{R}}^n - \{\varepsilon\}$  then there is a positive integer  $k_0$  such that  $A^{(k)} \otimes x$  is finite for all  $k \geq k_0$ .*

**Proof.**  $A^{(k)} \otimes x \in V(A)$  for some  $k$  but  $V(A)$  only contains finite eigenvectors since  $A$  is irreducible. If  $x$  is finite then  $A \otimes x$  is finite too.  $\square$

Matrix period has been studied in several papers. We recall the following.

**Theorem 3.3** [9]. *Let  $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}$  be an irreducible matrix and  $g_s$  be the gcd of the lengths of critical cycles in the  $s$ th strongly connected component of  $C(A)$ . Then*

$$\text{per}(A) = \text{lcm}(g_1, g_2, \dots).$$

Note that an  $O(n^3)$  algorithm for finding  $\text{per}(A)$  is presented in [9].

As a corollary of Theorems 3.2 and 3.3 we have:

**Theorem 3.4.** *An irreducible matrix  $A \in \overline{\mathbb{R}}^{n \times n}$  is robust if and only if in every strongly connected component of  $C(A)$  the lengths of all critical cycles are co-prime.*

We can use this result to derive a simple method for checking that a given strongly irreducible matrix  $A \in \overline{\mathbb{R}}^{n \times n}$  is robust. We can assume without loss of generality that  $C(A)$  is strongly connected (as we can investigate each component separately) and so the check reduces to finding out whether the lengths of all zero cycles are co-prime. Since no cycle of length greater than  $n$  needs to be considered, Theorem 2.2 offers a quick tool for this check. Let us calculate the powers  $A, A^{(2)}, A^{(3)}, \dots, A^{(n)}$  and then find the dimensions  $d_1, d_2, \dots, d_n$  of the eigenspaces of these matrices. Since  $d_k = \text{gcd}(k, g)$ ,  $k = 1, \dots, n$ , where  $g$  is the gcd of the lengths of all critical cycles, it is now clear that  $g = 1$  if and only if  $d_1 = d_2 = \dots = d_n = 1$ . We can summarize:

**Theorem 3.5.** *A strongly irreducible matrix  $A \in \overline{\mathbb{R}}^{n \times n}$  is robust if and only if the eigenspaces of  $A, A^{(2)}, A^{(3)}, \dots, A^{(n)}$  have the same dimension.*

Since all fundamental eigenvectors of  $A$  are also eigenvectors of any power of  $A$  (see Theorem 2.3), any maximal set of non-equivalent fundamental eigenvectors also generates the eigenspace of any power of  $A$  (see (3)). Hence we arrive at an even stronger version of Theorem 3.5.

**Theorem 3.6.** *A strongly irreducible matrix  $A \in \overline{\mathbb{R}}^{n \times n}$  is robust if and only if the eigenspaces of  $A, A^{(2)}, A^{(3)}, \dots, A^{(n)}$  coincide.*

Obviously, if  $g = 1$  then  $d_k = 1$  for all positive integers  $k$ . Hence:

**Corollary 3.2.** *A strongly irreducible matrix  $A \in \overline{\mathbb{R}}^{n \times n}$  is robust if and only if the eigenspaces of all powers of  $A$  coincide.*

The following, slightly extended, sufficient condition for  $A$  to be robust has already been known for some time [6] and now follows as a direct corollary of Theorems 3.2 and 3.3.

**Corollary 3.3.** *Let  $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}$  be irreducible. Then  $A$  is robust if  $a_{ii} = \lambda(A)$  for every  $i \in E(A)$ .*

**Proof.**  $a_{ii} = \lambda(A)$  for every  $i \in E(A)$  implies that in every component of the critical digraph there is a critical cycle of length 1, hence  $g_i = 1$  for all  $i$  and  $\text{per}(A) = 1$ .  $\square$

#### 4. Robust matrices and discrete-event dynamic systems

The reader is referred to [1] or [8] for basic information on the theory of max-algebraic discrete-event dynamic system. If  $A \in \overline{\mathbb{R}}^{m \times n}$ ,  $b \in \overline{\mathbb{R}}^n$ ,  $c \in \overline{\mathbb{R}}^m$  then a real sequence  $\{g_k\}_{k=0}^\infty$ , where

$$g_k = c^T \otimes A^{(k)} \otimes b$$

for all  $k$ , is called the sequence of *Markov parameters* of the *discrete-event dynamic system*  $(A, b, c)$ . The triple  $(A, b, c)$  is called a *realisation* of the sequence of Markov parameters. We also say that  $(A, b, c)$  *realises*  $\{g_k\}_{k=0}^\infty$ .

A key question is the *realisation problem*: Given a sequence of Markov parameters, find a realisation  $(A, b, c)$ .

Let  $e_i$  stand for the vector from  $\overline{\mathbb{R}}^n$  whose  $i$ th component is 0 and all other are  $\varepsilon$ . We denote by  $a_{ij}^{[k]}$  the  $(i, j)$  entry of  $A^{(k)}$  (in contrast to  $a_{ij}^{(k)}$  which stands for the  $k$ th max-algebraic power of  $a_{ij}$ ).

A scalar sequence  $\{g_k \in \overline{\mathbb{R}}; k = 0, \dots, \infty\}$  is called *ultimately linear* if the following holds for some  $\lambda \in \mathbb{R}$  and  $k_0$  natural:

$$g_{k+1} = \lambda \otimes g_k \quad \text{for all } k \geq k_0.$$

**Theorem 4.1.** *Let  $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}$  be irreducible. Then  $A$  is robust if and only if for every  $b, c \in \overline{\mathbb{R}}^n$  the sequence of Markov parameters of the discrete-event dynamic system  $(A, b, c)$  is ultimately linear.*

**Proof.** Let  $A \in \overline{\mathbb{R}}^{n \times n}$  be irreducible and robust and  $b, c \in \overline{\mathbb{R}}^n$ . Then by Theorem 3.2

$$g_{k+1} = c^T \otimes A^{(k+1)} \otimes b = \lambda \otimes c^T \otimes A^{(k)} \otimes b = \lambda \otimes g_k \tag{4}$$

for  $k \geq k_0$  and  $\lambda = \lambda(A)$ .

Conversely, if (4) holds for every  $b, c \in \overline{\mathbb{R}}^n$  then in particular it holds for  $b = e_r, c = e_s$  ( $r, s = 1, \dots, n$ ) and all  $k$  starting from some value, say  $k(r, s)$ . However,  $e_s^T \otimes A^{(k+1)} \otimes e_r = a_{sr}^{[k+1]} = \lambda \otimes a_{sr}^{[k]}$ . Hence  $A^{(k+1)} = \lambda \otimes A^{(k)}$  for all  $k \geq \max_{r,s=1,\dots,n} k(r, s)$ . The result now follows by Theorem 3.2.  $\square$

Note that it follows from this proof that  $\lambda = \lambda(A)$  for any choice of  $b$  and  $c$  if  $A$  is irreducible.

**Theorem 4.2.** A real sequence  $\{g_k\}_{k=0}^\infty$  is ultimately linear if and only if there is an integer  $m$  and a discrete-event dynamic system  $(A, b, c)$  with  $A \in \mathbb{R}^{m \times m}$  robust, whose sequence of Markov parameters is  $\{g_k\}_{k=0}^\infty$ .

**Proof.** The “if” statement follows from Theorem 4.1.

Let  $\{g_k\}_{k=1}^\infty$  be a sequence satisfying

$$g_{k+1} = \lambda \otimes g_k \quad \text{for all } k \geq m$$

for some  $m$  natural. Set

$$c = (0, \varepsilon, \dots, \varepsilon)^T \in \overline{\mathbb{R}}^m,$$

$$A = \begin{pmatrix} \varepsilon & 0 & \varepsilon & \dots & \varepsilon \\ \varepsilon & \varepsilon & 0 & \dots & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \dots & \varepsilon \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \varepsilon & \varepsilon & \varepsilon & \dots & \lambda \end{pmatrix} \in \overline{\mathbb{R}}^{m \times m},$$

$$b = \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_m \end{pmatrix} \in \mathbb{R}^m.$$

It is easily verified that

$$c^T \otimes A^{(k)} \otimes b = g_k \tag{5}$$

for every  $k$  and that  $\varepsilon$  can be replaced by sufficiently small real values so that (5) is still true. Let this new (irreducible) matrix be denoted by  $B$ . Then it is easily seen that  $\lambda(B) = \lambda$  and  $C(B)$  contains only one cycle and its length is 1. Hence  $\text{per}(A) = 1$  and  $A$  is robust.  $\square$

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