On integer images of max-plus linear mappings

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Abstract

Let us extend the pair of operations $(\oplus, \otimes) = (\max, +)$ over real numbers to matrices in the same way as in conventional linear algebra.

We study integer images of mappings $x \to A \otimes x$, where $A \in \mathbb{R}^{m \times n}$ and $x \in \mathbb{R}^n$. The question whether $A \otimes x$ is an integer vector for at least one $x \in \mathbb{R}^n$ has been studied for some time but polynomial solution methods seem to exist only in special cases. In the terminology of combinatorial matrix theory this question reads: is it possible to add constants to the columns of a given matrix so that all row maxima are integer? This problem has been motivated by attempts to solve a class of job-scheduling problems.

We present two polynomially solvable special cases aiming to move closer to a polynomial solution method in the general case.

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Dedicated to Professor Karel Zimmermann.

1 Introduction

Since the 1960s max-algebra provides modelling and solution tools for a class of problems in discrete mathematics and matrix algebra. The key feature is the development of an analogue of linear algebra for the pair of operations $(\oplus, \otimes)$ where

$$a \oplus b = \max(a, b)$$

and

$$a \otimes b = a + b$$

for $a, b \in \overline{\mathbb{R}} \overset{def}{=} \mathbb{R} \cup \{-\infty\}$. This pair is extended to matrices and vectors as in conventional linear algebra. That is if $A = (a_{ij}),\ B = (b_{ij})$ and $C = (c_{ij})$ are matrices of compatible sizes with entries from $\overline{\mathbb{R}}$, we write $C = A \oplus B$ if $c_{ij} = a_{ij} \oplus b_{ij}$ for all i, j and $C = A \otimes B$ if

$$c_{ij} = \bigoplus_k a_{ik} \otimes b_{kj} = \max_k (a_{ik} + b_{kj})$$

for all i, j. If $\alpha \in \overline{\mathbb{R}}$ then $\alpha \otimes A = (\alpha \otimes a_{ij})$. For simplicity we will use the convention of not writing the symbol $\otimes$. Thus in what follows the symbol $\otimes$ will not be used (except when necessary for clarity), and unless explicitly stated otherwise, all multiplications indicated are in max-algebra.

The interest in max-algebra (today also called tropical linear algebra) was originally motivated by the possibility of dealing with a class of non-linear problems in pure and applied mathematics, operational research, science and engineering as if they were linear due to the fact that $\left(\overline{\mathbb{R}}, \oplus, \otimes\right)$ is a commutative and idempotent semifield. Besides the main advantage of using linear rather than non-linear techniques,

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max-algebra enables us to efficiently describe and deal with complex sets [6], reveal combinatorial aspects of problems [5] and view a class of problems in a new, unconventional way. The first pioneering papers appeared in the 1960s [17], [18] and [36], followed by substantial contributions in the 1970s and 1980s such as [19], [23], [24], [37] and [16]. Since 1995 we have seen a remarkable expansion of this research field following a number of findings and applications in areas as diverse as algebraic geometry [31] and [35], geometry [27], control theory and optimization [1], phylogenetic [34], modelling of the cellular protein production [3] and railway scheduling [25]. A number of research monographs have been published [1], [7], [25] and [30]. A chapter on max-algebra appears in a handbook of linear algebra [26] and a chapter on idempotent semirings can be found in a monograph on semirings [22].

Max-algebra covers a range of linear-algebraic problems in the max-linear setting, such as systems of linear equations and inequalities, linear independence and rank, bases and dimension, polynomials, characteristic polynomials, matrix scaling, matrix equations, matrix orbits and periodicity of matrix powers [1], [7], [19], [14] and [25]. Among the most intensively studied questions was the *eigenproblem*, that is the question, for a given square matrix A to find all values of λ and non-trivial vectors x such that $Ax = \lambda x$. This and related questions such as z-matrix equations $Ax \oplus b = \lambda x$ [15] have been answered [10], [19], [24], [20], [2] and [7] with numerically stable low-order polynomial algorithms. The same applies to the *subeigenproblem* that is the problem of finding solutions to $Ax \leq \lambda x$ [33] and the *supereigenproblem* that is solution to $Ax \geq \lambda x$, [8] and [32]. Max-linear and integer max-linear programs have also been studied [37], [7], [9], [21] and [13].

A specific area of interest is in solving the above mentioned problems with integrality requirements. It seems in general there is no polynomial solution method to find an integer eigenvector of a real matrix in max-algebra or to decide that there is none. A closely related [13] is the question whether the mapping $x \to Ax$ has an integer image, that is whether Ax is an integer vector for at least one $x \in \mathbb{R}^n$. The motivation for the latter comes from operational problems such as the following job-scheduling task [19] and [7]: Products $P_1, ..., P_m$ are prepared using n machines (processors), every machine contributing to the completion of each product by producing a component. It is assumed that each machine can work for all products simultaneously and that all these actions on a machine start as soon as the machine starts to work. Let a_{ij} be the duration of the work of the j^{th} machine needed to complete the component for P_i ($i = 1, ..., m; j = 1, ..., n$). If this interaction is not required for some i and j then a_{ij} is set to $-\infty$. The matrix $A = (a_{ij})$ is called the *production matrix*. Let us denote by x_j the starting time of the j^{th} machine ($j = 1, ..., n$). Then all components for P_i ($i = 1, ..., m$) will be ready at time

$$\max(x_1 + a_{i1}, ..., x_n + a_{in}).$$

Hence if $b_1, ..., b_m$ are given completion times then the starting times have to satisfy the system of equations:

$$\max(x_1 + a_{i1}, ..., x_n + a_{in}) = b_i \quad \text{for all } i = 1, ..., m.$$

Using max-algebra this system can be written in a compact form as a system of linear equations:

$$Ax = b. \tag{1}$$

A system of the form (1) is called a *one-sided system of max-linear equations* (or briefly a *one-sided max-linear system* or just a *max-linear system*). Such systems are easily solvable [17], [37] and [7], see also Section 2. However, sometimes the vector b of completion times is not given explicitly, instead it is only required that completions of individual products occur at discrete time intervals, for instance at integer times. This motivates the study of integer images of max-linear mappings to which this paper aims to contribute. More precisely, we deal with the question: Given a real matrix A, find a real vector x such that Ax is integer or decide that none exists. In the terminology of combinatorial matrix theory this question reads: is it possible to add constants to the columns of a given matrix so that all row maxima are integer? We will call this problem the *Integer Image Problem* (IIP). This problem has been studied for some time [12], [13] and [29], yet it seems to be still open whether it can be answered in polynomial time. In this paper we present two polynomially solvable special cases aiming to suggest a direction in which an efficient method could be found for general matrices in the future. We also provide a brief summary of a selection of already achieved results.

2 Definitions, notation and previous results

Throughout the paper we denote $-\infty$ by ε (the neutral element with respect to $\oplus$) and for convenience we also denote by the same symbol any vector, whose all components are $-\infty$, or a matrix whose all entries are $-\infty$. A matrix or vector with all entries equal to 0 will also be denoted by 0. If $a \in \mathbb{R}$ then the symbol a^{-1} stands for $-a$. Matrices and vectors whose all entries are real numbers are called *finite*. We assume everywhere that $m, n \geq 1$ are integers and denote $M = \{1, ..., m\}$ and $N = \{1, ..., n\}$.

It is easily proved that if A, B, C and D are matrices of compatible sizes (including vectors considered as $m \times 1$ matrices) then the usual laws of associativity and distributivity hold and also isotonicity is satisfied:

$$A \geq B \Longrightarrow AC \geq BC \text{ and } DA \geq DB. \tag{2}$$

A square matrix is called *diagonal* if all its diagonal entries are real numbers and off-diagonal entries are ε. More precisely, if $x = (x_1, ..., x_n)^T \in \mathbb{R}^n$ then $diag\,(x_1, ..., x_n)$ is the $n \times n$ diagonal matrix

$$\begin{pmatrix} x_1 & \varepsilon & ... & \varepsilon \\ \varepsilon & x_2 & ... & \varepsilon \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon & \varepsilon & ... & x_n \end{pmatrix}.$$

The matrix $diag\,(0)$ is called the *unit matrix* and denoted I. Obviously, $AI = IA = A$ whenever A and I are of compatible sizes. A matrix obtained from a diagonal matrix by permuting the rows and/or columns is called a *generalized permutation matrix*. It is known that in max-algebra generalized permutation matrices are the only type of invertible matrices [19] and [7].

If A is a square matrix then the iterated product $AA...A$ in which the symbol A appears k-times will be denoted by A^k. By definition $A^0 = I$.

Given $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}$ the symbol D_A stands for the weighted digraph (N, E, w) (called *associated with* A) where $E = \{(i, j)\,; a_{ij} > \varepsilon\}$ and $w\,(i, j) = a_{ij}$ for all $(i, j) \in E$. The symbol $\lambda(A)$ denotes the *maximum cycle mean* of A, that is:

$$\lambda(A) = \max_{\sigma} \mu(\sigma, A), \tag{3}$$

where the maximization is taken over all elementary cycles in D_A, and

$$\mu(\sigma, A) = \frac{w(\sigma, A)}{l\,(\sigma)} \tag{4}$$

denotes the *mean* of a cycle σ. With the convention $\max \emptyset = \varepsilon$ the value $\lambda\,(A)$ always exists since the number of elementary cycles is finite. It can be computed in $O\left(n^3\right)$ time [28], see also [7]. We say that A is *definite* if $\lambda\,(A) = 0$ and *strongly definite* if it is definite and all diagonal entries of A are zero.

Given $A \in \overline{\mathbb{R}}^{n \times n}$ it is usual [19], [1], [25] and [7] in max-algebra to define the infinite series

$$A^* = I \oplus A \oplus A^2 \oplus A^3 \oplus ... . \tag{5}$$

The matrix A^* is called the *strong transitive closure* of A, or the *Kleene Star*.

It follows from the definitions that every entry of the matrix sequence

$$\left\{I \oplus A \oplus A^2 \oplus ... \oplus A^k\right\}_{k=0}^{\infty}$$

is a nondecreasing sequence in $\overline{\mathbb{R}}$ and therefore either it is convergent to a real number (if bounded) or its limit is $+\infty$. If $\lambda(A) \leq 0$ then

$$A^* = I \oplus A \oplus A^2 \oplus ... \oplus A^{k-1}$$

for every $k \geq n$ and can be found using the Floyd-Warshall algorithm in $O\left(n^3\right)$ time [7].

The matrix $\lambda^{-1}A$ for $\lambda \in \mathbb{R}$ will be denoted by A_λ and $(A_\lambda)^*$ will be shortly written as A^*_λ.

The *eigenvalue-eigenvector problem* (briefly *eigenproblem*) is the following:
Given $A \in \overline{\mathbb{R}}^{n\times n}$, *find all* $\lambda \in \overline{\mathbb{R}}$ *(eigenvalues) and* $x \in \overline{\mathbb{R}}^{n}, x \neq \varepsilon$ *(eigenvectors) such that*

$$Ax = \lambda x.$$

This problem has been studied since the work of R.A.Cuninghame-Green [18]. An $n \times n$ matrix has up to n eigenvalues with $\lambda(A)$ always being the largest eigenvalue (called *principal*). This finding was first presented by R.A.Cuninghame-Green [19] and M.Gondran and M.Minoux [23], see also N.N.Vorobyov [36]. The full spectrum was first described by S.Gaubert [20] and R.B.Bapat, D.Stanford and P. van den Driessche [2]. The spectrum and bases of all eigenspaces can be found in $O(n^3)$ time [10] and [7].

The aim of this paper is to study the existence of integer images of max-linear mappings and therefore we summarize here only the results on finite solutions and for finite A. For $A \in \mathbb{R}^{n\times n}$ and $\lambda \in \mathbb{R}$ we denote

$$V(A, \lambda) = \{x \in \mathbb{R}^n : Ax = \lambda x\}.$$

In this case there are no eigenvalues other than the principal and we can easily describe all eigenvectors:

Theorem 2.1 *[18], [19], [23] If* $A \in \mathbb{R}^{n\times n}$ *then* $\lambda(A)$ *is the unique eigenvalue of* A *and all eigenvectors of* A *are finite. If* $\lambda = \lambda(A)$ *then*

$$V(A, \lambda) = \{A_\lambda^* u : u \in \mathbb{R}^n\}.$$

In what follows $V(A)$ will stand for $V(A, \lambda(A))$.

As usual for any $a \in \mathbb{R}$ we denote the lower integer part, upper integer part and fractional part of a by $\lfloor a \rfloor$, $\lceil a \rceil$ and $\mathrm{fr}(a)$. Hence $\mathrm{fr}(a) = a - \lfloor a \rfloor$. For any matrix A the symbol $\lfloor A \rfloor$ stands for the matrix obtained by replacing every entry of A by its lower integer part, similarly $\lceil A \rceil$ and $\mathrm{fr}(A)$. The same conventions apply to vectors. The set of integer eigenvectors of $A \in \mathbb{R}^{n\times n}$ will be denoted by $\mathrm{IV}(A)$, that is

$$\mathrm{IV}(A) = V(A) \cap \mathbb{Z}^n.$$

Given an $A \in \mathbb{R}^{m\times n}$ we will use the following notation:

$$\mathrm{Im}(A) = \{Ax : x \in \mathbb{R}^n\}$$

and

$$\mathrm{IIm}(A) = \mathrm{Im}(A) \cap \mathbb{Z}^m.$$

We call $\mathrm{Im}(A)$ [$\mathrm{IIm}(A)$] the *image set of* A [*integer image set of* A].

If we randomly generate two real numbers then their fractional parts are different with probability 1. Being motivated by this we say that a real vector v is *typical* if no two components of v have the same fractional part. On the other hand if every component of a vector v has the same fractional part then we say that v is *uniform*. If every column of a real matrix A is typical [uniform] then we say that A is *column typical [column uniform]*.

Remark 2.2 *Observe that* $\mathrm{IIm}(A) \neq \emptyset$ *if* A *has at least one uniform column.*

As the next theorem shows strongly definite matrices are an important class for which there is an easy solution to the integer eigenvalue problem.

Theorem 2.3 *[12] Let* $A \in \mathbb{R}^{n\times n}$ *be strongly definite. Then*

1. $\mathrm{IV}(A) \neq \emptyset$ *if and only if* $\lambda(\lceil A \rceil) = 0$.
2. *If* $\mathrm{IV}(A) \neq \emptyset$ *then* $\mathrm{IV}(A) = \left\{\lceil A \rceil^* z : z \in \mathbb{Z}^n\right\}$.

The *max-algebraic permanent* of a matrix $A \in \mathbb{R}^{n \times n}$ is an analogue of the conventional permanent:

$$per(A) = \sum\nolimits_{\pi \in P_n}^{\oplus} \prod\nolimits_{i \in N}^{\otimes} a_{i,\pi(i)}$$

where P_n is the set of all permutations of N. In conventional notation this reads:

$$per(A) = \max_{\pi \in P_n} \sum\nolimits_{i \in N} a_{i,\pi(i)}$$

which is the optimal value for the linear assignment problem for the matrix A [5], [4] and [7]. Using the notation

$$w(\pi, A) = \sum\nolimits_{i \in N} a_{i,\pi(i)}$$

for $\pi \in P_n$ we can then define the set of optimal permutations:

$$ap(A) = \{\pi \in P_n : w(\pi, A) = per(A)\}.$$

Uniqueness of optimal permutations plays a significant role in max-algebra, see for instance the question of regularity of matrices [11] and [5]. It is also important for integer images as shown in Theorem 2.4 below. Note that it follows from the definitions that $\mathrm{IIm}(A) = \mathrm{IIm}(AQ)$ for any generalized permutation matrix Q. It is known [7] that for every $A \in \mathbb{R}^{n \times n}$ with $|ap(A)| = 1$ there exists a unique generalized permutation matrix Q such that AQ is strongly definite.

Theorem 2.4 *[12] Let $A \in \mathbb{R}^{n \times n}$ be column typical.*

1. *If $|ap(A)| > 1$ then $\mathrm{IIm}(A) = \emptyset$.*
2. *If $|ap(A)| = 1$ and Q is the (unique) generalized permutation matrix such that AQ is strongly definite then*
$$\mathrm{IIm}(A) = \mathrm{IIm}(AQ) = \mathrm{IV}(AQ).$$

Theorem 2.4 effectively solves the IIP for column typical square matrices. It is not difficult to see that in general $m \leq n$ is a necessary condition for $\mathrm{IIm}(A) \neq \emptyset$ if $A \in \mathbb{R}^{m \times n}$ is column typical. If $m \leq n$ then a necessary and sufficient condition for $\mathrm{IIm}(A) \neq \emptyset$ is existence of a submatrix $A' \in \mathbb{R}^{m \times m}$ for which $\mathrm{IIm}(A') \neq \emptyset$. Hence there is the possibility of solving the IIP for $m \times n$ matrices by checking all $m \times m$ submatrices. The number of such submatrices is $\binom{n}{m}$ which is polynomial when m is fixed. In particular, this immediately yields an $O\left(n^3\right)$ method for answering the problem for $3 \times n$ column typical matrices. One of the aims of this paper is to present an $O\left(n^2\right)$ method for this particular special case.

It will be useful to also define min-algebra over $\mathbb{R}$ [19] and [7]:

$$a \oplus' b = \min(a, b)$$

and

$$a \otimes' b = a \otimes b$$

for all a and b. We extend the pair of operations $(\oplus', \otimes')$ to matrices and vectors in the same way as in max-algebra. We also define the *conjugate* $A^{\#} = -A^T$. Note that isotonicity holds for $(\oplus', \otimes')$ similarly as for $(\oplus, \otimes)$, see (2).

We will usually not write the operator $\otimes'$and for matrices the convention applies that if no multiplication operator appears then the product is in min-algebra whenever it follows the symbol #, otherwise it is in max-algebra. In this way a residuated pair of operations (a special case of Galois connection) has been defined, namely

$$Ax \leq y \Longleftrightarrow x \leq A^{\#}y \tag{6}$$

for all $x, y \in \mathbb{R}^n$. Hence $Ax \leq y$ implies $A(A^{\#}y) \leq y$. It follows immediately that a one-sided system $Ax = b$ has a solution if and only if $A\left(A^{\#}b\right) = b$ (see Corollary 3.3 below) and using isotonicity then the system $Ax \leq b$ always has an infinite number of solutions with $A^{\#}b$ being the greatest solution.

3 Finding an integer image for a $3 \times n$ matrix

We start with historically the first result in max-algebra. In what follows if $A \in \mathbb{R}^{m \times n}$ and $j \in N$ then A_j will denote the j^{th} column of A.

Problem P1: Given $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, find all $x = (x_1, ..., x_n)^T \in \mathbb{R}^n$ such that $Ax = b$ or decide that none exist.

For $A = (a_{ij}) \in \mathbb{R}^{m \times n}$ and $b = (b_1, ..., b_m)^T \in \mathbb{R}^m$ define

$$\overline{x} = A^{\#} b, \tag{7}$$

that is

$$\overline{x}_j = \min_{i \in M} (b_i - a_{ij}) = - \max_{i \in M} (a_{ij} - b_i)$$

for all $j \in N$ and

$$M_j (A, b) = \{ i \in M : \overline{x}_j = b_i - a_{ij} \}, j \in N.$$

The notation $M_j (A, b)$ will be shortened to M_j if no confusion can arise. The answer to P1 is summarized in the following statement.

Proposition 3.1 *[17], [19], [7] Let $\overline{x}$ be as defined in (7) and $x \in \mathbb{R}^n$. Then*

(a) $A\overline{x} \leq b$.

(b) $Ax \leq b$ *if and only if* $x \leq \overline{x}$.

(c) $Ax = b$ *if and only if* $x \leq \overline{x}$ *and*

$$\bigcup_{x_j = \overline{x}_j} M_j = M.$$

Corollary 3.2 $A\overline{x} = b$ *if and only if* $\bigcup_{j \in N} M_j = M$.

Corollary 3.3 $Ax = b$ *has a solution if and only if* $A\overline{x} = b$.

Problem P2: Given $A = (a_{ij}) \in \mathbb{R}^{m \times n}$, $b = (b_1, ..., b_m)^T \in \mathbb{R}^m$ and $d = (d_1, ..., d_n)^T \in \mathbb{R}^n$, find an $x = (x_1, ..., x_n)^T \in \mathbb{R}^n$ such that $Ax = b, x \leq d$ or decide that none exists.

Proposition 3.4 $(\exists x \in \mathbb{R}^n)\, Ax = b, x \leq d \iff Az = b$, *where* $z = d \oplus' \overline{x}$.

Proof. "If" is obvious since $z \leq d$.

Suppose now that $Ax = b, x \leq d$ for some $x \in \mathbb{R}^n$. By Proposition 3.1 (c) then $x \leq \overline{x}$ and $\bigcup_{x_j = \overline{x}_j} M_j = M$. It follows from the definition of z that

$$z_j = \overline{x}_j \iff \overline{x}_j \leq d_j$$

and $z \leq \overline{x}$. Hence if $x_j = \overline{x}_j$ then $\overline{x}_j \leq d_j$. Therefore $z_j = \overline{x}_j$ and thus

$$M = \bigcup_{x_j = \overline{x}_j} M_j \subseteq \bigcup_{z_j = \overline{x}_j} M_j \subseteq M,$$

from which the statement follows. ∎

Problem P3: Given $A \in \mathbb{R}^{m \times n}$, find a point in $\mathrm{IIm}\,(A)$ or decide that there is none.

Solution to P3 is easy for $m = 2$ as can be seen from the next few lines where we give a full description of $\mathrm{Im}\,(A)$ and $\mathrm{IIm}\,(A)$. The primary objective of this paper is to present a solution for $m = 3$ provided that A is column typical, which is done later on.

Proposition 3.5 *[12] Let $A \in \mathbb{R}^{2\times n}$. Then*

$$\operatorname{Im}(A) = \left\{ (y_1, y_2)^T \in \mathbb{R}^2 : y_2 - y_1 \in \left[\underline{\alpha}(A), \overline{\alpha}(A) \right] \right\},$$

where

$$\underline{\alpha}(A) = \min_{j \in N} (a_{2j} - a_{1j})$$

and

$$\overline{\alpha}(A) = \max_{j \in N} (a_{2j} - a_{1j}).$$

Proof. Let $y \in \mathbb{R}^2$. Then by Corollaries 3.2 and 3.3 $y \in \operatorname{Im}(A)$ if and only if

$$a_{1j} - y_1 \geq a_{2j} - y_2$$

and

$$a_{2l} - y_2 \geq a_{1l} - y_1$$

for some $j, l \in N$.

Equivalently,

$$y_2 - y_1 \geq a_{2j} - a_{1j}$$

and

$$y_2 - y_1 \leq a_{2l} - a_{1l}$$

for some $j, l \in N$, from which the statement follows. ■

Note that we will write shortly $\underline{\alpha}, \overline{\alpha}$ instead of $\underline{\alpha}(A), \overline{\alpha}(A)$ if no confusion can arise.

Corollary 3.6 *Let $A \in \mathbb{R}^{2\times n}$. Then*

$$\operatorname{IIm}(A) = \left\{ (y_1, y_2)^T \in \mathbb{Z}^2 : y_2 - y_1 \in [\underline{\alpha}, \overline{\alpha}] \right\}.$$

Corollary 3.7 *Let $A \in \mathbb{R}^{2\times n}$. Then $\operatorname{IIm}(A) \neq \emptyset$ if and only if $[\underline{\alpha}, \overline{\alpha}] \cap \mathbb{Z} \neq \emptyset$.*

Problem P4: Given $A \in \mathbb{R}^{2\times n}$ and $L = (l_1, l_2)^T, U = (u_1, u_2)^T \in \mathbb{R}^2$, describe the set

$$S = \{ y \in \operatorname{IIm}(A) : L \leq y \leq U \}.$$

By Corollary 3.6 the set S (if non-empty) consists of integer points on adjacent parallel line segments. We may assume $L \in \mathbb{Z}^2$ (or take $\lceil L \rceil$ if necessary). An answer to P4 is in the next proposition which follows from Corollary 3.6 immediately.

Proposition 3.8 *The set $S = \{ y \in \operatorname{IIm}(A) : L \leq y \leq U \}$ consists of integer points on line segments described by the following conditions:*

$$\left. \begin{array}{c} y_2 = y_1 + \alpha, \alpha \in [\underline{\alpha}, \overline{\alpha}] \cap \mathbb{Z}, \\ l_1 \leq y_1 \leq u_1, \\ l_2 \leq y_2 \leq u_2. \end{array} \right\} \tag{8}$$

Problem P5: Given $A \in \mathbb{R}^{2\times n}$ and $L \in \mathbb{R}^2$, find an $x \in \mathbb{R}^n$ satisfying the following conditions:

$$\left. \begin{array}{c} Ax \in \mathbb{Z}^2 \\ x \leq 0 \\ Ax \geq L \end{array} \right\} \tag{9}$$

or decide that there is none.

In order to solve P5 we first prove a few auxiliary statements.

A set $S \subseteq \mathbb{R}^n$ is called *max-convex* if $\lambda x \oplus \mu y \in S$ for any $x, y \in S$ and $\lambda, \mu \in \mathbb{R}$ satisfying $\lambda \oplus \mu = 0$.

Proposition 3.9 *Let $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$ and $d \in \mathbb{R}^n$. Then the sets*

$$S = \{x \in \mathbb{R}^n : Ax = b\}$$

and

$$S' = \{x \in \mathbb{R}^n : Ax = b, x \leq d\}$$

are max-convex.

Proof. Since $\left(\overline{\mathbb{R}}, \oplus, \otimes\right)$ is a semifield the proof follows the lines of the proofs of the corresponding conventional statements:

$$\begin{aligned} A\left(\lambda x \oplus \mu y\right) &= \lambda Ax \oplus \mu Ay \\ &= \lambda b \oplus \mu b \\ &= \left(\lambda \oplus \mu\right) b \\ &= 0b = b \end{aligned}$$

and

$$\begin{aligned} \lambda x \oplus \mu y &\leq \lambda d \oplus \mu d \\ &= \left(\lambda \oplus \mu\right) d \\ &= 0d = d. \end{aligned}$$

■

Proposition 3.10 *Let $S \subseteq \mathbb{R}^n$ be a max-convex set and $f(x) = c^T x$, where $c \in \mathbb{R}^n$. If $f(x) \leq f(y)$ for some $x, y \in S$ then for every $f \in [f(x), f(y)]$ there exists a $z \in S$ such that $f(z) = f$.*

Proof. Denote $f' = f(x)$, $f'' = f(y)$ and suppose $f \in [f', f'']$. Let $\lambda = 0, \mu = f - f'' \leq 0$. Then $\lambda \oplus \mu = 0$ and thus by Proposition 3.9 $z \in S$ where $z = \lambda x \oplus \mu y$. Also,

$$\begin{aligned} f(z) &= c^T\left(\lambda x \oplus \mu y\right) \\ &= \lambda c^T x \oplus \mu c^T y \\ &= \lambda f' \oplus \mu f'' \\ &= f' \oplus f = f. \end{aligned}$$

■

Proposition 3.11 *Let $a, c \in \mathbb{R}^n, b \in \mathbb{R}, f(x) = c^T x$ and*

$$S = \left\{x \in \mathbb{R}^n : a^T x = b\right\}.$$

Then the set $F = \{f(x) : x \in S\}$ is a non-empty closed interval.

Proof. Let us denote $a = (a_1, ..., a_n)^T$ and $c = (c_1, ..., c_n)^T$. Let A be the $1 \times n$ matrix $A = (a_1, ..., a_n)$ and as before $\overline{x} = A^{\#} b = (b - a_1, ..., b - a_n)^T$. Then clearly $\overline{x} \in S$ and $x \leq \overline{x}$ for any $x \in S$ by Proposition 3.1. It follows by isotonicity that $f(x) \leq f(\overline{x})$ for every $x \in S$ and so $f(\overline{x})$ is an upper bound of F attained on S.

On the other hand, define for $k = 1, ..., n$:

$$x^{(k)} = \left(x_1^{(k)}, ..., x_{k-1}^{(k)}, x_k^{(k)}, x_{k+1}^{(k)}, ..., x_n^{(k)}\right)^T,$$

where $x_k^{(k)} = \overline{x}_k = b - a_k$ and $x_j^{(k)}$ is any value not exceeding $c_k + \overline{x}_k - c_j$ for $j \neq k$. Hence $f\left(x^{(k)}\right) = c_k + \overline{x}_k$. For every $x \in S$ there exists a $k \in N$ such that $x_k = \overline{x}_k$ and for this k we have:

$$f(x) \geq c_k + \overline{x}_k = f\left(x^{(k)}\right) \geq \min_{j \in N} f\left(x^{(j)}\right).$$

Since $\min_{j \in N} f\left(x^{(j)}\right) = f\left(x^{(j_0)}\right)$ for some $j_0 \in N$ and $x^{(j_0)} \in S$ we have that F has a lower bound and this bound is attained on S. The statement now follows from Propositions 3.9 and 3.10. ■

Corollary 3.12 *Let* $a, c \in \mathbb{R}^n, b \in \mathbb{R}, f(x) = c^T x, d \in \mathbb{R}^n$ *and*

$$S' = \left\{ x \in \mathbb{R}^n : a^T x = b, x \leq d \right\}.$$

If $S' \neq \emptyset$ *then the set* $F = \{f(x) : x \in S'\}$ *is a (non-empty) closed interval.*

Proof. If $S' \neq \emptyset$ then $\overline{x}_j \leq d_j$ for at least one $j \in N$. The rest of the proof follows the lines of the proof of Proposition 3.11 where $\overline{x}$ is replaced by $\overline{x} \oplus' d$. See Proposition 3.4. ∎

Let us return to P5. We denote by X the set of vectors x satisfying (9). Note that we may assume without loss of generality that $L = (l_1, l_2)^T \in \mathbb{Z}^2$ (otherwise we replace L by $\lceil L \rceil$). By isotonicity we have $Ax \leq A0$ for every $x \in X$ and we denote $A0$ by $U = (u_1, u_2)^T$. If $L \leq U$ is not satisfied then $X = \emptyset$ hence we will assume in what follows that $L \leq U$. So the task is to find integer points $y = (y_1, y_2)^T$ in the rectangle $L \leq y \leq U$ (see Figure 1) of the form $y = Ax, x \leq 0$ or to decide that there are none. For ease of reference we will denote

$$T = \left\{ y \in \mathbb{R}^2 : L \leq y \leq U \right\}.$$

Recall that the integer points of the form $y = Ax$ in this rectangle are described by (8). In Figure 1 the lines containing integer images of A are dashed. A little bit more challenging is the task to identify those of them (if any) that are of the form $y = Ax$ where $x \leq 0$.

First we observe in the following statement that every integer point in T (if any) can be "diagonally projected" on the left-hand side or bottom side of T.

Proposition 3.13 *If* $x \in X$ *then there exists* $\lambda \leq 0$ *such that the vector* $x' = \lambda x$ *is in* X *and satisfies either* $(Ax')_1 = l_1$ *or* $(Ax')_2 = l_2$.

Proof. Let us denote Ax by $b = (b_1, b_2)^T$.

Suppose first that $b_2 - b_1 \geq l_2 - l_1$. Then take $\lambda = l_1 - b_1 \leq 0$ and observe

$$A(\lambda x) = \lambda (Ax) = \lambda b = (l_1, b_2 + l_1 - b_1) \geq L.$$

Clearly, $Ax' = \lambda b \in \mathbb{Z}^2$ since both $\lambda \in \mathbb{Z}$ and $b \in \mathbb{Z}^2$. Also $x' = \lambda x \leq 0$, thus $x' \in X$ and $(Ax')_1 = l_1$.

The case $b_2 - b_1 \leq l_2 - l_1$ is proved similarly by taking $\lambda = l_2 - b_2 \leq 0$. ∎

We will also use the diagonal projection of the point U on the left-hand side or bottom side of T. For this we will need to distinguish two possibilities to which we will refer as Case 1 and Case 2:

Case 1: $u_2 - u_1 \geq l_2 - l_1$.

Case 2: $u_2 - u_1 \leq l_2 - l_1$.

Under the assumption of Case 1 the diagonal projection of U is $P = (p_1, p_2)^T = (l_1, l_1 + u_2 - u_1)^T$. In Case 2 it is $P' = (l_2 + u_1 - u_2, l_2)^T$.

Due to Proposition 3.13 it is sufficient to search integer points Ax satisfying $(Ax)_1 = l_1$ or $(Ax)_2 = l_2$ and find those (if any) for which the condition $x \leq 0$ is satisfied. As there is possibly a non-polynomial number of such points we will narrow the set of candidates. All candidates have the form $(l_1, l_1 + \alpha)$ or $(l_2 - \alpha, l_2)$, where $\alpha \in [\underline{\alpha}(A), \overline{\alpha}(A)] \cap \mathbb{Z}$ by Corollary 3.6.

Let us denote

$$S = \{Ax \in T : x \leq 0\},$$

$$S_1 = \{Ax \in T : x \leq 0, (Ax)_1 = l_1\}$$

and

$$S_2 = \{Ax \in T : x \leq 0, (Ax)_2 = l_2\}.$$

Clearly, $U \in S$ and $P \in S_1$ in Case 1 and $P' \in S_2$ in Case 2. We can also describe S_1 and S_2 as follows:

$$S_1 = \left\{ (l_1, (Ax)_2)^T : x \leq 0, (Ax)_1 = l_1 \right\} \cap \{(l_1, y_2) : y_2 \geq l_2\}$$

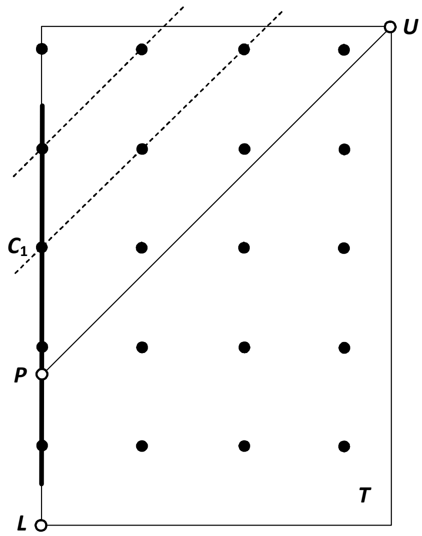

Figure 1: Rectangle T corresponding to Case 1a

and

$$S_2 = \left\{ \left((Ax)_1, l_2 \right)^T : x \leq 0, (Ax)_2 = l_2 \right\} \cap \left\{ (y_1, l_2) : y_1 \geq l_1 \right\}.$$

Note that in Figure 1 the set S_1 is shown as the bold line segment on the left-hand side of T. By Corollary 3.12 the set $\left\{ \left(l_1, (Ax)_2 \right)^T : x \leq 0, (Ax)_1 = l_1 \right\}$ is a closed interval or $\emptyset$ and so S_1 is a closed interval or $\emptyset$, similarly S_2. The point P is in S_1 in Case 1 and P' in S_2 in Case 2, so at least one of the sets S_1 and S_2 is non-empty in each case.

Consider now Case 1. Since the point $P = (l_1, l_1 + u_2 - u_1)^T$ is in $S_1 \cup S_2$ it is easy to check whether a point of the form $(l_1, l_1 + \alpha)$ or $(l_2 - \alpha, l_2)$ is in $S_1 \cup S_2$ where $\alpha \in [\underline{\alpha}(A), \overline{\alpha}(A)] \cap \mathbb{Z}$ - we only need to check those points of this form that are closest to P. More precisely, we distinguish 3 subcases:

- Subcase 1a: If $u_2 - u_1 < \underline{\alpha}(A)$ then check the point $C_1 = (l_1, l_1 + \underline{\alpha}(A))^T$. Note that $(l_1, l_1 + \underline{\alpha}(A))^T \geq L$.
- Subcase 1b: If $u_2 - u_1 > \overline{\alpha}(A)$ then check the point $C_2 = (l_1, l_1 + \overline{\alpha}(A))^T$. If $l_1 + \overline{\alpha}(A) < l_2$ then the checkpoint is at the bottom side of T, that is the point $(l_2 - \overline{\alpha}(A), l_2)^T \geq L$.
- Subcase 1c: If $\underline{\alpha}(A) \leq u_2 - u_1 \leq \overline{\alpha}(A)$ then check both

$$C_3 = (l_1, l_1 + \lfloor u_2 - u_1 \rfloor)^T$$

and

$$C_4 = (l_1, l_1 + \lceil u_2 - u_1 \rceil)^T.$$

Note that both these points are $\geq L$.

Case 2 is treated similarly.

"Checking" a point y means verifying that there is an $x \leq 0$ for which $Ax = y$. By Proposition 3.4 this can be done by checking that $A(\overline{x} \oplus' 0) = y$ which is $O(n)$. Finding $\underline{\alpha}(A), \overline{\alpha}(A)$ and U is obviously $O(n)$ as well so the whole method is $O(n)$.

Problem P6: Given $A \in \mathbb{R}^{3 \times n}$ find an $x \in \mathbb{R}^n$ such that $Ax \in \mathrm{IIm}(A)$ or decide there is none.

Remark 3.14 *Since $(AD)x = A(Dx)$ for any $D = diag(d_1, ..., d_n) \in \overline{\mathbb{R}}^{n \times n}$ we have $\mathrm{IIm}(AD) = \mathrm{IIm}(A)$. Therefore in P6 we may assume without loss of generality that $a_{3j} = 0$ for all $j \in N$ by taking $d_j = -a_{3j}$ for $j \in N$ if necessary.*

Remark 3.15 *Since* $(\alpha A)x = A(\alpha x)$ *for any* $\alpha \in \mathbb{R}$ *we have* $\mathrm{IIm}(\alpha A) = \mathrm{IIm}(A)$. *Therefore in P6 we may assume without loss of generality that for every* $i \in \{1,2,3\}$ *there is an* $x \in \mathbb{R}^n$ *such that* $(Ax)_i = 0$ *if* $\mathrm{IIm}(A) \neq \emptyset$ *by taking* $\alpha = -(Ax)_i$ *if necessary.*

Remark 3.16 *Since* $\alpha(Ax) = A(\alpha x)$ *for any* $\alpha \in \mathbb{R}$ *we have* $A(\alpha x) \in \mathbb{Z}^m$ *if* $Ax \in \mathbb{Z}^m$ *and* $\alpha \in \mathbb{Z}$. *Therefore in P6 we may assume without loss of generality that for every* $j \in N$ *there is an* $x \in \mathbb{R}^n$ *satisfying* $Ax \in \mathbb{Z}^m$ *and* $\lfloor x_j \rfloor = 0$ *whenever* $\mathrm{IIm}(A) \neq \emptyset$ *by taking* $\alpha = -\lfloor x_j \rfloor$ *if necessary.*

We are now ready to present the main result of this paper - a solution method for P6 for column typical matrices. So let $A \in \mathbb{R}^{3\times n}$ be a column typical matrix. We assume without loss of generality (see Remark 3.14) that $a_{3j} = 0$ for all $j \in N$. Suppose that $Ax \in \mathbb{Z}^3$ for some $x \in \mathbb{R}^n$ and again without loss of generality (see Remark 3.15) that $(Ax)_3 = 0$. Hence there is a $k \in N$ such that $x_k = 0 \geq x_j$ for every $j \in N$. Let $\overline{A}$ be the matrix obtained from A by removing row 3. Define

$$B^{(k)} = \left(\overline{A}_1, ..., \overline{A}_{k-1}, \overline{A}_{k+1}, ..., \overline{A}_n\right)$$

and

$$z = (x_1, ..., x_{k-1}, x_{k+1}, ..., x_n)^T.$$

Then

$$\left.\begin{array}{c} B^{(k)}z \in \mathbb{Z}^2, \\ B^{(k)}z \geq \overline{A}_k, \\ z \leq 0. \end{array}\right\} \tag{10}$$

The first inequality follows from the assumption that A is column typical since then a_{1k}, a_{2k} are non-integer as they cannot have the same fractional part as a_{3k} which is zero (note that this inequality is implicitly strict in each component).

Conversely, if a vector $z = (z_1, ..., z_{n-1})^T \in \mathbb{R}^{n-1}$ satisfies (10) then the vector

$$x = (z_1, ..., z_{k-1}, 0, z_k, ..., z_{n-1})^T$$

satisfies $Ax \in \mathbb{Z}^3$. So the method is to check for all $k = 1, ..., n$ that (10) has a solution and find one. If every check fails then A has no integer image. Each check (for a fixed k) is an instance of Problem 5 (with A replaced by $B^{(k)}$ and L is replaced by $\lceil \overline{A}_k \rceil$), which can be solved in $O(n)$ time, so in total P6 can be solved in $nO(n) = O(n^2)$ time.

We conjecture that the above mentioned method for solving P6 in the case $m = 3$ and for column typical matrices can be extended to general matrices and any m. To do this one could try to develop a methodology to decide whether a matrix B obtained from A with $\mathrm{IIm}(A) \neq \emptyset$ by adding a row also has $\mathrm{IIm}(B) \neq \emptyset$. In the next section we will show that this can be done if $A \in \mathbb{R}^{m\times n}$ is column uniform.

We finish this section by an example illustrating the method for solving P6 on the 3×4 matrix

$$A = \begin{pmatrix} 6.9 & 10.0 & 4.6 & 3.7 \\ 10.6 & 6.7 & 2.7 & 1.8 \\ 3.7 & 4.4 & -2.8 & 1.3 \end{pmatrix}.$$

We start by normalization with respect to row 3, that is computing AD where $D = diag(-3.7, -4.4, 2.8, -1.3)$. We get

$$AD = \begin{pmatrix} 3.2 & 5.6 & 7.4 & 2.4 \\ 6.9 & 2.3 & 5.5 & 0.5 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

For simplicity of notation we will denote AD again by A. Hence

$$\overline{A} = \begin{pmatrix} 3.2 & 5.6 & 7.4 & 2.4 \\ 6.9 & 2.3 & 5.5 & 0.5 \end{pmatrix}.$$

We need to solve (10), or decide that there is no solution, for $k = 1, 2, 3, 4$ (until we find an integer image of A).

1. $k = 1:$
$$B^{(1)} = \begin{pmatrix} 5.6 & 7.4 & 2.4 \\ 2.3 & 5.5 & 0.5 \end{pmatrix}$$
$L = \lceil \overline{A}_1 \rceil = (4,7)^T, U = B^{(1)}0 = (7.4, 5.5)^T$. Since $L \nleq U$ there is no solution to (10).

2. $k = 2:$
$$B^{(2)} = \begin{pmatrix} 3.2 & 7.4 & 2.4 \\ 6.9 & 5.5 & 0.5 \end{pmatrix}$$
$L = \lceil \overline{A}_2 \rceil = (6,3)^T, U = B^{(2)}0 = (7.4, 6.9)^T$.

   $u_2 - u_1 = -0.5 \geq -3 = l_2 - l_1,\ P = (6, 5.5)^T$.

   $\overline{\alpha}\left(B^{(2)}\right) = \max(3.7, -1.9, -1.9) = 3.7, \underline{\alpha}\left(B^{(2)}\right) = \min(3.7, -1.9, -1.9) = -1.9$.

   $u_2 - u_1 \in \left[\underline{\alpha}\left(B^{(2)}\right), \overline{\alpha}\left(B^{(2)}\right)\right]$ thus this is an instance of Case 1c.

   $C_3 = (6,5)^T, C_4 = (6,6)^T$.

   Trying to solve $B^{(2)}x = C_3, x \leq 0 : \overline{x} = \left(B^{(2)}\right)^{\#} \begin{pmatrix} 6 \\ 5 \end{pmatrix} = (-1.9, -1.4, 3.6)^T, \overline{x} \oplus' 0 = (-1.9, -1.4, 0)^T$.

   $B^{(2)}(\overline{x} \oplus' 0) = C_3$ so $x = (-1.9, 0, -1.4, 0)^T$ yields $\overline{A}x = C_3$, which confirms that C_3 is an integer image of $\overline{A}$ and $(6,5,0)^T \in \mathrm{IIm}(A)$. (For the original matrix A take $D^{-1}x$.) Note that C_4 is also an integer image and can be obtained by changing the component -1.9 in $\overline{x}$ to -0.9.

   This is where the algorithm could stop but we continue to get more insight.

3. $k = 3:$
$$B^{(3)} = \begin{pmatrix} 3.2 & 5.6 & 2.4 \\ 6.9 & 2.3 & 0.5 \end{pmatrix}$$
$L = \lceil \overline{A}_3 \rceil = (8,6)^T, U = B^{(3)}0 = (5.6, 6.9)^T$. Since $L \nleq U$ there is no solution to (10).

4. $k = 4:$
$$B^{(4)} = \begin{pmatrix} 3.2 & 5.6 & 7.4 \\ 6.9 & 2.3 & 5.5 \end{pmatrix}$$
$L = \lceil \overline{A}_4 \rceil = (3,1)^T, U = B^{(4)}0 = (7.4, 6.9)^T$.

   $u_2 - u_1 = -0.5 \geq -2 = l_2 - l_1,\ P = (3, 2.5)^T$.

   $\overline{\alpha}\left(B^{(4)}\right) = \max(3.7, -3.3, -1.9) = 3.7, \underline{\alpha}\left(B^{(4)}\right) = \min(3.7, -3.3, -1.9) = -3.3$.

   $u_2 - u_1 \in \left[\underline{\alpha}\left(B^{(4)}\right), \overline{\alpha}\left(B^{(4)}\right)\right]$ thus this is an instance of Case 1c.

   $C_3 = (3,2)^T, C_4 = (3,3)^T$.

   Trying to solve $B^{(4)}x = C_3, x \leq 0 : \overline{x} = \left(B^{(4)}\right)^{\#} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = (-4.9, -2.6, -4.4)^T, \overline{x} \oplus' 0 = \overline{x}$.

   $B^{(4)}(\overline{x} \oplus' 0) = C_3$ so $x = (-4.9, -2.6, -4.4, 0)^T$ yields $\overline{A}x = C_3$, which confirms that C_3 is another integer image of $\overline{A}$ and $(3,2,0)^T \in \mathrm{IIm}(A)$.

4 Finding an integer image of an almost column uniform matrix

A matrix $A \in \mathbb{R}^{m \times n}$ is called *almost column uniform* if the matrix obtained by removing one row (called *exceptional*) of A is column uniform.

Problem P7: Given an almost column uniform matrix $A \in \mathbb{R}^{m \times n}$, find an $x \in \mathbb{R}^n$ such that $Ax \in \mathbb{Z}^m$ or decide that there is none.

We will show in this section how to solve P7 in polynomial time.

We may assume without loss of generality that the following is satisfied:

1. The exceptional row is row m, that is A is of the form

$$\begin{pmatrix} \overline{A} \\ a_{m1} ... a_{mn} \end{pmatrix}$$

where $\overline{A} \in \mathbb{R}^{(m-1)\times n}$ is column uniform.

2. A does not have a uniform column (see Remark 2.2), that is the following hold for every $j \in N$ and for every $r, s = 1, ..., m-1$:
$$\mathrm{fr}\,(a_{rj}) = \mathrm{fr}\,(a_{sj}) \neq \mathrm{fr}\,(a_{mj})$$

3. $a_{mj} = 0$ for all $j \in N$ (see Remark 3.14). Observe that the transformation suggested in Remark 3.14 does not affect the assumption that the matrix is almost column uniform.

For every $j \in N$ the common value of $\mathrm{fr}\,(a_{ij})$ for all $i = 1, ..., m-1$ will be denoted f_j.

Suppose $Ax \in \mathbb{Z}^m$ for some $x \in \mathbb{R}^n$. We may assume without loss of generality (see Remark 3.15) that $(Ax)_m = 0$. Hence there is a $k \in N$ such that $x_k = 0 \geq x_j$ for every $j \in N$. Let us now define

$$B^{(k)} = \left(\overline{A}_1, ...\overline{A}_{k-1}, \overline{A}_{k+1}, ..., \overline{A}_n\right)$$

and

$$z = (x_1, ..., x_{k-1}, x_{k+1}, ..., x_n)^T .$$

Since all entries of $\overline{A}_k$ are non-integral the vector z satisfies

$$\left.\begin{array}{c} B^{(k)}z \in \mathbb{Z}^{m-1}, \\ B^{(k)}z \geq \overline{A}_k, \\ z \leq 0. \end{array}\right\} \tag{11}$$

Conversely, if a vector $z = (z_1, ..., z_{n-1})^T \in \mathbb{R}^{n-1}$ satisfies (11) then $Ax \in \mathbb{Z}^m$, where

$$x = (z_1, ..., z_{k-1}, 0, z_{k+1}, ..., z_{n-1})^T .$$

Let us denote $F^{(k)} = (f_1, ..., f_{k-1}, f_{k+1}, ..., f_n)^T \in \mathbb{R}^{n-1}$.

Proposition 4.1 *The system (11) has a solution if and only if $-F^{(k)}$ is a solution.*

We prove a few auxiliary statements before the proof of Proposition 4.1.

Lemma 4.2 *If $a, b \in \mathbb{R}$ then*

$$\mathrm{fr}\,(a+b) = \mathrm{fr}\,(a) + \mathrm{fr}\,(b) \;\; \textit{if} \;\; \mathrm{fr}\,(a) + \mathrm{fr}\,(b) < 1$$

and

$$\mathrm{fr}\,(a+b) = \mathrm{fr}\,(a) + \mathrm{fr}\,(b) - 1 \;\; \textit{if} \;\; \mathrm{fr}\,(a) + \mathrm{fr}\,(b) \geq 1.$$

Proof. The statement follows immediately from the definition of the fractional part. ■

Lemma 4.3 *If $a, b, y \in \mathbb{R}, \mathrm{fr}\,(a) = \mathrm{fr}\,(b)$ then*

$$\lfloor a+y \rfloor - a = \lfloor b+y \rfloor - b.$$

Proof. For any $a, b, y \in \mathbb{R}$ we have

$$\lfloor a+y \rfloor - a = a + y - \mathrm{fr}\,(a+y) - a = y - \mathrm{fr}\,(a+y)$$

and similarly

$$\lfloor b+y \rfloor - b = y - \mathrm{fr}\,(b+y)\,.$$

If $\mathrm{fr}(a) + \mathrm{fr}(y) < 1$ or, equivalently $\mathrm{fr}(b) + \mathrm{fr}(y) < 1$, then by Lemma 4.2 this implies

$$\lfloor a + y \rfloor - a = y - \mathrm{fr}(a) - \mathrm{fr}(y) = \lfloor y \rfloor - \mathrm{fr}(a)$$

and similarly

$$\lfloor b + y \rfloor - b = \lfloor y \rfloor - \mathrm{fr}(b),$$

thus the statement follows.

The case $\mathrm{fr}(a) + \mathrm{fr}(y) \geq 1$ is proved in the same way. ∎

Lemma 4.4 *Let $H = (h_{ij}) \in \mathbb{R}^{m \times n}$ be column uniform, $w = (w_1, ..., w_n)^T \in \mathbb{R}^n$ and $\widetilde{w} = (\widetilde{w}_1, ..., \widetilde{w}_n)^T$ be defined as follows:*

$$\widetilde{w}_j = \lfloor h_{ij} + w_j \rfloor - h_{ij}$$

for any and therefore all $i \in M$ and for all $j \in N$ (see Lemma 4.3). Then $\widetilde{w} \leq w$ and $H\widetilde{w} = \lfloor Hw \rfloor$.

Proof. For any $j \in N$ we have

$$\widetilde{w}_j \leq h_{ij} + w_j - h_{ij} = w_j.$$

Let $i \in M$. Then

$$\left\lfloor \max_j (h_{ij} + w_j) \right\rfloor = \max_j \lfloor h_{ij} + w_j \rfloor = \max_j (h_{ij} + \widetilde{w}_j).$$

∎

Proof of Proposition 4.1. Suppose that $z \in \mathbb{R}^{n-1}$ satisfies (11). Then

$$B^{(k)}0 \geq B^{(k)}z \geq \overline{A}_k$$

and $B^{(k)}z \in \mathbb{Z}^{m-1}$. By taking $H = B^{(k)}$ and $w = 0$ in Lemma 4.4 we get using the notation of Lemma 4.4:

$$\begin{aligned} B^{(k)}\widetilde{w} &= B^{(k)}\widetilde{0} = \left\lfloor B^{(k)}0 \right\rfloor \geq \left\lfloor B^{(k)}z \right\rfloor = B^{(k)}z \geq \overline{A}_k \\ B^{(k)}\widetilde{w} &\in \mathbb{Z}^{m-1} \\ \widetilde{w} &\leq 0. \end{aligned}$$

Clearly, $\widetilde{0} = -F^{(k)}$, which completes the proof. ∎

Proposition 4.1 provides a simple way of solving Problem 7. Since for every $k \in N$ we have $-F^{(k)} \leq 0$ and $B^{(k)}\left(-F^{(k)}\right) \in \mathbb{Z}^{m-1}$ we only need to check whether for at least one $k \in N$ the vector $F^{(k)}$ satisfies

$$B^{(k)}\left(-F^{(k)}\right) \geq \overline{A}_k. \tag{12}$$

If it does then $B^{(k)}\left(-F^{(k)}\right)$ extended by zero in the m^{th} component is an integer image of A and $Ax \in \mathbb{Z}^m$ where

$$x = -\left(f_1, ..., f_{k-1}, 0, f_{k+1}, ..., f_n\right)^T.$$

If not then $\mathrm{IIm}(A) = \emptyset$.

Checking the condition (12) is $O(mn)$, normalisation of A with respect to the last row is also $O(mn)$. In general this check needs to be done for all $k = 1, ..., n$ and so the method is $O\left(mn^2\right)$.

5 Conclusions

We have shown in two special cases how to find an integer image of a matrix or decide that none exists, where the matrix is obtained by adding one row to a matrix for which the answer is known. It is self-suggesting an iterative procedure for answering this question in a general case, however the complexity issues remain to be solved.

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