Non-linear programs with max-linear constraints:  
A heuristic approach

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Abstract

Let $a \oplus b = \max(a, b)$ and $a \otimes b = a + b$ for $a, b \in \mathbb{R}$ and extend the pair of operations to matrices and vectors in the same way as in linear algebra. We present neighborhood local search methods for minimizing non-linear functions $f(x)$ subject to two-sided max-linear constraints $A \otimes x \oplus c = B \otimes x \oplus d$. The methods are tested on a number of instances and if the objective function is max-linear, performance of the methods is compared with the exact method for solving max-linear programs with two-sided constraints.

Keywords: Max-linear Systems; Max-linear Programming; Neighborhood Local Search

1 Introduction

Let $a \oplus b = \max(a, b)$ and $a \otimes b = a + b$ for $a, b \in \mathbb{R}$ and extend the pair of operations to matrices and vectors in the same way as in linear algebra. That is if $A = (a_{ij}), B = (b_{ij})$ and $C = (c_{ij})$ are matrices of compatible sizes with entries from $\mathbb{R}$, we write $C = A \oplus B$ if $c_{ij} = a_{ij} \oplus b_{ij}$ for all $i, j$ and $C = A \otimes B$ if $c_{ij} = \sum_{k} a_{ik} \otimes b_{kj} = \max_k (a_{ik} + b_{kj})$ for all $i, j$. Also, if $\alpha \in \mathbb{R}$ then $\alpha \otimes A = (\alpha \otimes a_{ij})$. Max-algebra has been studied by many authors and the reader is referred to [8], [9], [12], [4], [5] or [2].

The aim of this paper is for finding an $x \in \mathbb{R}$ that minimizes the non-linear function $f(x)$ subject to

$$A \otimes x \oplus c = B \otimes x \oplus d,$$

where $A = (a_{ij}), B = (b_{ij}) \in \mathbb{R}^{m \times n}, c = (c_1, \ldots, c_m)^T, d = (d_1, \ldots, d_m)^T$ are given matrices and vectors. Optimization problems of this type will be called 'non-linear programs with two-sided max-linear constraints' or just 'non-linear programs with max-linear constraints (NLPMs)'.

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Systems of max-linear equations were investigated already in the first publications dealing with the algebraic structure called max-algebra (sometimes also extremal algebra, path algebra or tropical algebra). In these publications, systems of equations with all variables on one side were considered [8], [5], [14], [16]. Other systems with a special structure were studied in the context of solving eigenvalue problems in the corresponding algebraic structures or synchronization in discrete event systems [4]. Using the \((\oplus, \otimes)\)-notation, the studied systems had one of the following forms: \(A \otimes x = b, A \otimes x = x\) or \(A \otimes x = x \oplus b\), where \(A\) is a given matrix and \(b\) is a given vector. Infinite-dimensional generalisations can be found, e.g. in [3].

General two-sided max-linear systems have also been studied [7], [10], [12], [15]. A general solution method was presented in [15], however, no complexity bound was given. In [10] a pseudopolynomial algorithm, called the Alternating Method, has been developed. In [7] it was shown that the solution set is generated by a finite number of vectors and an elimination method was suggested. A general iterative approach suggested in [11] assumes that finite upper and lower bounds for all variables are given.

Solution methods for max-linear programs with two-sided constraints (MLP) for both minimisation and maximisation problems have been presented in [6] also see [1]. It was proved that the methods are pseudopolynomial if all entries are integers. The methods are based on the Alternating Method [10]. Note that MLP is also NLPM with the objective function max-linear. NLPM has a number of applications among them is the multiprocessor interactive process (MPIS) for more information, see [6].

2 The one-sided case

One-sided systems of max-linear equations have been studied for many years and they are very well understood, see [8], [16] and [5]. These problems have the following form

\[ A \otimes x = b, \]  

(2)

where \(A = (a_{ij}) \in \mathbb{R}^{n \times n}\) and \(b = (b_1, \ldots, b_m)^T \in \mathbb{R}^m\).

A method for minimizing max-linear function \(f(x) = f^T \otimes x = \max(f_1 + x_1, \ldots, f_n + x_n)\) subject to (2) is presented in [6]. Note that the problem of minimizing the non-linear function \(f(x) = 2^x_1 + 2^x_2 + \cdots + 2^x_n\) is \(NP\)-complete [6]. Since a one-sided system is a special case of a two-sided system [1] where \(a_{ij} > b_{ij}\) and \(c_{ij} < d_{ij}\) for every \(i\) and \(j\), NLPM is also \(NP\)-complete.

3 Definitions and problem formulation

Since NLPM is \(NP\)-complete for a general non-linear function, an exact solution method is unlikely to be efficient. Therefore we will develop heuristic methods for solving this problem. That is we develop heuristic methods for minimizing non-linear function \(f(x)\) subject to two-sided max-linear system

\[ A \otimes x \oplus c = B \otimes x \oplus d, \]  

(3)

where \(A = (a_{ij}), B = (b_{ij}) \in \mathbb{R}^{m \times n}\) and \(c = (c_1, \ldots, c_m)^T, d = (d_1, \ldots, d_m)^T \in \mathbb{R}^m\) are given matrices and vectors. A practical motivation for this research
is that no polynomial method is known for solving max-linear programs with two-sided constraints (MLP). We denote the set of solutions for (3) by $S$. Note that we may consider any matrix $A \in \mathbb{R}^{m \times n}$ to be made up of $n$ column vectors with $m$ entries in each vector, that is

$$A = (A_1, A_2, \ldots, A_n), \quad A_j = (a_{1j}, a_{2j}, \ldots, a_{mj})^T.$$ 

A function $f(x) : \mathbb{R}^n \to \mathbb{R}$ is said to be isotone if for every $x, y \in \mathbb{R}^n$, $x \leq y$ we have $f(x) \leq f(y)$. In what follows we assume that the objective function $f(x)$ is non-linear, isotone and computable in polynomial time.

**Definition 3.1.**

Let $x = (x_1, \ldots, x_n) \in S$ and $r$ be a positive integer, $r \leq n$. The $r$-neighborhood $N_r(x)$ of $x$ is the set of feasible solutions obtainable by changing at most $r$ components of $x$ (and fixing the remaining variables).

**Definition 3.2.**

A solution $x \in S$ is called a local optimum, if there is no $z \in N_r(x)$ such that $f(z) < f(x)$.

Note that $r$ will be omitted if the value of $r$ is known and thus instead of $r$-neighborhood we will just say neighborhood.

The aim here is to develop local search methods, based on $r$-optimum neighborhood for $r = 1$ and $r = 2$, to minimize non-linear isotone function $f(x)$ subject to

$$A \odot x \oplus c = B \odot x \oplus d,$$

where $A = (a_{ij}), B = (b_{ij}) \in \mathbb{R}^{m \times n}$ and $c = (c_1, \ldots, c_m)^T, d = (d_1, \ldots, d_m)^T \in \mathbb{R}^m$ are given matrices and vectors.

## 4 Max-linear programs: Some special cases

Max-linear program with two-sided constraints (minimization) is the following

$$f(x) = f^T \odot x \longrightarrow \min$$

subject to

$$A \odot x \oplus c = B \odot x \oplus d.$$ 

where $f = (f_1, \ldots, f_n)^T \in \mathbb{R}^n$, $c = (c_1, \ldots, c_n)^T, d = (d_1, \ldots, d_n)^T \in \mathbb{R}^m$, $A = (a_{ij})$ and $B = (b_{ij}) \in \mathbb{R}^{m \times n}$. This problem is denoted by $\text{MLP}^\min$. Solution methods for $\text{MLP}^\min$ are developed in [1]. We denote the optimal solution set for $\text{MLP}^\min$ by $S^\min$.

### 4.1 Basic properties

Before we develop methods for solving max-linear programs for special cases we summarize some basic properties. To do so we will denote $\inf_{x \in S} f(x)$ by $f^\min$ and $M^\vartriangleright = \{i \in M; c_i > d_i\}$. Also if $\alpha \in \mathbb{R}$, then the symbol $\alpha^{-1}$ stands for $-\alpha$.

**Theorem 4.1.** $\text{MLP}^\min$

$f^\min = -\infty$ if and only if $c = d$. 

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Lemma 4.2. \[\text{\textbullet} \]
Let \(c \geq d\). If \(x \in S\) and \((A \otimes x)_i \geq c_i\) for all \(i \in M\), then \(x' = \alpha \otimes x \in S\) and \((A \otimes x)_i = c_i\) for some \(i \in M\), where

\[
\alpha = \max_{i \in M} (c_i \otimes (A \otimes x)_i^{-1})
\]

(5)

For \(j \in N\) we denote by

\[
h'_j = \min \left( \min_{r \in M} a_{rj}^{-1} \otimes c_r, \min_{r \in M} b_{rj}^{-1} \otimes d_r \right),
\]

(6)

and \(h' = (h'_1, \ldots, h'_n)^T\).

Remark 4.1.
Each component \(h'_j\) for \(j \in N\) can be determined in \(O(m)\) time. Hence the vector \(h'\) is \(O(mn)\).

Proposition 4.1.
Let \(c \geq d\) and \(c \neq d\). If \(S \neq \emptyset\) then for every \(x \in S_{\text{min}}^\alpha\) there is an \(i \in M^>\) and \(j \in N\) such that \((B \otimes x)_i = c_i \geq (A \otimes x)_i\), and \(x_j = c_j \otimes b_{ij}^{-1}\).

Proof.
Suppose \(c \geq d\) and \(c \neq d\). Let \(x \in S_{\text{min}}^\alpha\) such that \(\exists i \in M^>, j \in N, x_j < c_i \otimes b_{ij}^{-1}\). Then we have

\[(B \otimes x)_i < c_i \leq (A \otimes x)_i \oplus c_i,
\]

which contradicts the assumption that \(x \in S\). Hence \(x_j \geq c_i \otimes b_{ij}^{-1}, \forall i \in M^>\).

If \(x_j = c_i \otimes b_{ij}^{-1}\) for some \(i \in M^>\) then the statement follows. Suppose \(\forall i \in M^>, j \in N, x_j > c_i \otimes b_{ij}^{-1}\). Since \(x \in S\) we have

\[(A \otimes x)_i \oplus c_i = (B \otimes x)_i, i \in M^>,
\]

(7)

\[(A \otimes x)_r \oplus c_r = (B \otimes x)_r \oplus c_r, r \notin M^>.
\]

(8)

Now \(x_j > c_i \otimes b_{ij}^{-1}, \forall i \in M^>, j \in N\) and \(x \in S\), therefore from (7) we have

\[(A \otimes x)_i = (B \otimes x)_i > c_i, \forall i \in M^>.
\]

(9)

and it follows from Lemma 4.2 that \(\alpha \otimes x = x' \in S\), where in this case \(\alpha\) as defined in (6) is

\[
\alpha = \max_{i \in M^>} (c_i \otimes (A \otimes x)_i^{-1}) < 0.
\]

Clearly \(f(x') < f(x)\), a contradiction with optimality of \(x\).

4.2 The \(m\)-by-one case

Given \(f \in \mathbb{R}\) and \(a, b, c, d \in \mathbb{R}^{m \times 1}\), the problem of minimizing the function \(f(x) = f \otimes x\) (of one variable) subject to

\[
a_i \otimes x \oplus c_i = b_i \otimes x, \ i \in M^>,
\]

\[
a_i \otimes x \oplus c_i = b_i \otimes x \oplus d_i, \ i \notin M^>.
\]

(10)

is denoted \(\text{MLP}_{m_1}^{\text{min}}\). We define by \(M^m = \{i \in M; c_i = d_i\}\) and \(\overline{M}^m = \{i \in M^m; a_i \neq b_i\}\). We denote by \(S_{m1}\) the solution set of (10) and \(S_{m1}^{\text{min}}\) the set of minimisers of (10).
Proposition 4.2.
Let $c \geq d$ and $c \neq d$. Then
i) If $\exists i \in M^>$ such that $a_i > b_i$ then $S_{m1} = \emptyset$.
ii) If $\exists i \in M^>$ such that $a_i < b_i$ then $S_{m1} \neq \emptyset$ implies that $S_{m1} = \{ c_i \otimes b_i^{-1} \}$.

Proof.
i) Suppose there is an equation $i \in M^>$ such that $a_i > b_i$. Then it follows from (10) that
\[ b_i \otimes x < a_i \otimes x \leq a_i \otimes x \oplus c_i. \]
Thus $S_{m1} = \emptyset$.
ii) Suppose there is an equation $i \in M^>$ such that $a_i < b_i$. Then from (10) we have
\[ a_i \otimes x \oplus c_i = b_i \otimes x \]
\[ \implies c_i = b_i \otimes x \]
\[ \implies x = c_i \otimes b_i^{-1}. \]
Therefore it follows that if $S_{m1} \neq \emptyset$ then $x = c_i \otimes b_i^{-1} \in S_{m1}$ and it is unique. \(\square\)

Due to Proposition 4.2, we recognise that there is at most one feasible solution to $\text{MLP}_{m1}^{\text{lin}}$ if $\exists i \in M^>$ such that $a_i > b_i$ or $a_i < b_i$. For that reason, we may assume without loss of generality that in $\text{MLP}_{m1}^{\text{lin}}$ we have $a_i = b_i$ for all $i \in M^>$.

Let us define by
\[ \tilde{x} = \max_{i \in M^>} (c_i \otimes b_i^{-1}). \quad (11) \]

Proposition 4.3.
Let $a_i = b_i$ for all $i \in M^>$. If $S_{m1} \neq \emptyset$ then $\tilde{x} \in S_{m1}$.

Proof.
Let $a_i = b_i$ for all $i \in M^>$. Suppose $\exists x \in S_{m1}$. It follows from (10) that
\[ b_i \otimes x = a_i \otimes x \oplus c_i, \quad i \in M^> \]
(12)
and
\[ b_i \otimes x \oplus d_i = a_i \otimes x \oplus c_i, \quad i \in M^= \]
(13)
From (12) we have
\[ b_i \otimes x \geq c_i, \forall i \in M^> \]
\[ \implies x \geq b_i \otimes c_i^{-1}, \forall i \in M^> \]
\[ \implies x \geq \max_{i \in M^>} (c_i \otimes b_i^{-1}) = \tilde{x}. \]
Now, if $x > \tilde{x}$ then $b_i \otimes x = a_i \otimes x, \quad i \in M^>$. It follows from Lemma 4.2 that we can scale down $x$ to the level where $b_i \otimes x = c_i$ for some $i \in M^>$. Let $\alpha = \tilde{x} \otimes x^{-1}$. Then we have $\alpha \otimes x = \tilde{x} \in S_{m1}$. Hence the proof. \(\square\)

Algorithm 4.1. ONEVARIABLE (Non-homogeneous max-linear systems with one variable)
Input: $x = (c_i), d = (d_i), A = (a_i), B = (b_i) \in \mathbb{R}^{m \times 1}$.
Output: $x \in S_{m1}$ or an indication that $S_{m1} = \emptyset$. 5
1. If $c = d$ then stop ($x = h'$ where $h'$ is defined in (6)) and $x \in S_{m1}$

2. for all $i \in M^>$ do
   begin
   if $a_i > b_i$ then stop ("$S_{m1} = \emptyset"$
   if $a_i < b_i$ then
   begin
   $x := c_i \otimes b_i^{-1}$
   if $x \in S_{m1}$ then stop ($x \in S_{m1}$)
   else stop ("$S_{m1} = \emptyset"$
   end
   end

3. $x := \max\{c_i - b_i; i \in M^>\}$
   if $x \in S_{m1}$ then stop ($x \in S_{m1}$)
   else stop ("$S_{m1} = \emptyset"$

Theorem 4.3.
Algorithm ONEVARIABLE is correct and its computational complexity is $O(m)$.

It follows from Proposition 4.3 that the existence of an optimal solution for $MLP_{m1}^{\text{min}}$ is reduced to the checking whether $\hat{x} \in S_{m1}$.

Proposition 4.4.
If $a_i = b_i$ for all $i \in M^>$ and $S_{m1} \neq \emptyset$ then $\hat{x} \in S_{m1}^{\text{min}}$.

Proof.
Suppose $a_i = b_i$ for all $i \in M^>$ and $x \in S_{m1}$ such that $x < \hat{x} = \max_{i \in M^>}(c_i \otimes b_i^{-1})$. Therefore we have

$$x < c_k \otimes b_k^{-1} \text{ for some } k \in M^>$$

$$\implies b_k \otimes x < c_k,$$

which contradicts the assumption that $x \in S_{m1}$. Therefore $\hat{x} \in S_{m1}^{\text{min}}$.

Based on Propositions 4.2, 4.3 and 4.4 we will formulate the following algorithm for finding a solution to $MLP_{m1}^{\text{min}}$ or to indicate that $S_{m1} = \emptyset$ or the objective function is unbounded.

Algorithm 4.2. MAXLINMINm1 (Max-linear minimisation for $m \times 1$ matrices with $c \geq d$)

Input: $f \in \mathbb{R}$, $c = (c_i)$, $d = (d_i)$, $A = (a_i)$, $B = (b_i)$ \in $\mathbb{R}^{m \times 1}$.

Output: $x \in S_{m1}^{\text{min}}$, an indication that $S_{m1} = \emptyset$ or $f^{\text{min}} = -\infty$.

1. If $c = d$ then stop ("$f^{\text{min}} = -\infty$"

2. Use the Algorithm ONEVARIABLE to find an $x \in S_{m1}$
   if it exists then $x \in S_{m1}^{\text{min}}$
   else $S_{m1} = \emptyset$

Theorem 4.4.
Algorithm MAXLINMINm1 is correct and its computational complexity is $O(m)$. 

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4.3 The $m$-by-two case

In this section we will study and develop polynomial solution methods for max-linear programming problem (MLP) with $m \times 2$ entry matrices for minimisation problems. That is:

\[
\begin{align*}
    f^T \otimes x \quad \rightarrow \min \\
    \text{subject to} \\
    A \otimes x \oplus c = B \otimes x \oplus d,
\end{align*}
\]

where $A = (a_{ij}), B = (b_{ij}) \in \mathbb{R}^{m \times 2}$, $f = (f_1, f_2)^T \in \mathbb{R}^2$, $c = (c_1, \ldots, c_m)^T$, $d = (d_1, \ldots, d_m)^T \in \mathbb{R}^m$.

An optimization problem of this type will be denoted by MLP$_{m2}^\min$. The set of feasible solutions for MLP$_{m2}^\min$ will be denoted by $S_{m2}$, the set of optimal solutions by $S_{m2}^\min$.

**Proposition 4.5.**  
Solving MLP$_{m2}^\min$ can be converted to a repeated solving of problem MLP$_{m1}^\min$.

**Proof.**
It follows from Proposition 4.1 that at least one of the decision variables of MLP$_{m2}^\min$ can be fixed to $x_j = c_i \otimes b_{ij}^{-1}$ for some $i \in M^>$ and $j \in \{1, 2\}$. If this value is substituted on to MLP$_{m2}^\min$ then this problem will have one variable to be determined and the statement follows.

Again, it follows from Proposition 4.1 that finding a feasible solution to MLP$_{m2}^\min$ is equivalent to checking whether a feasible solution can be found by fixing one of the two decision variables to $x_j = c_i \otimes b_{ij}^{-1}$, for some $i \in M^>$ and $j \in \{1, 2\}$. Due to Corollary 4.5 we can determine the optimal value of the free variable by solving the corresponding MLP$_{m1}^\min$. Therefore we may assume that a feasible solution exists. The use of Algorithm MAXLINMINm1 is applied to the corresponding MLP$_{m1}^\min$ for finding the free variable in order to minimize $f(x)$. The algorithm will first fix $x_1$ at a number of rows and determine $x_2$ using Algorithm MAXLINMINm1, then fix $x_2$ and determine $x_1$ and stop (when all these are done) with an output of either $x \in S_{m2}^\min$, an indication that $S_{m2} = \emptyset$ or $f_{\min} = -\infty$, where $f_{\min} = \inf_{x \in S_{m2}} f(x)$. We assume without loss of generality that $c \geq d$ otherwise we swap the sides of equations appropriately.

**Algorithm 4.3.** MAXLINMINm2 (Max-linear minimisation for $m \times 2$ matrices with $c \geq d$)

**Input:** $f = (f_1, f_2)^T \in \mathbb{R}^2$, $c = (c_1, \ldots, c_m)^T$, $d = (d_1, \ldots, d_m)^T \in \mathbb{R}^m$, $A = (a_{ij}), B = (b_{ij}) \in \mathbb{R}^{m \times 2}$.

**Output:** $x \in S_{m2}^\min$, an indication that $S_{m2} = \emptyset$ or $f_{\min} = -\infty$.

1. If $c = d$ then stop ("$f_{\min} = -\infty$"")
2. Set $x^0 := (+\infty, +\infty)^T$, $x := x^0$
3. for $j = 1, 2$ do
   begin
   for all $i \in M^>$ do
   begin

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\[ y_j := c_i \otimes b_{ij}^{-1} \]
\[ k := 3 - j \]
for all \( r \in M \) do
begin
\[ A'_r := a_{rk} \]
\[ B'_r := b_{rk} \]
\[ c'_r := \max(c_r, a_{rj} + y_j) \]
\[ d'_r := \max(d_r, b_{rj} + y_j) \]
if \( c'_r < d'_r \) then swap the sides in the \( r^{th} \) equation
\[ M'_{\text{new}} := \{ r \in M ; c_r > d'_r \} \]
end
if \( M'_{\text{new}} = \emptyset \) then \( y_k := h'_k \) where \( h'_k \) is defined in (6)
e else find \( y_k \) by applying Algorithm \( \text{MAXLINMINm1} \) to \( A', B', c' \) and \( d' \)
if \( A \otimes y \oplus c = B \otimes y \oplus d \) and \( f(y) < f(x) \) then \( x := y \)
end end

Theorem 4.5.
Algorithm \( \text{MAXLINMINm2} \) is correct and its computational complexity is \( O(m) \).

Proof.
The correctness of \( \text{MAXLINMINm2} \) follows from Propositions 4.1 and 4.5. In Step 3 there are three loops. The first loop runs twice, the second in \( O(m) \) time and the third in \( O(m) \) time. The non-fixed variable can be determined in \( O(m) \) and checking whether \( y \in S_{m2} \) or not can be done in \( O(m) \) time. Thus the computational complexity of Algorithm \( \text{MAXLINMINm2} \) is \( O(m^2) \) + \( O(m) = O(m^2) \).

5 The local search methods

In this section we propose \( r \)-optimum local search methods for non-linear programs with two-sided max-linear constraints (NLPMs) for \( r = 1 \) and \( r = 2 \). The methods will repeatedly use Algorithm \( \text{MAXLINMINm1} \) if \( r = 1 \) and Algorithm \( \text{MAXLINMINm2} \) if \( r = 2 \). More on heuristic methods can be found in [13].

5.1 Steepest descent method with improvement

The steepest descent method with improvement (SDMI) is a method that will consider all candidates in the neighborhood of a current feasible solution \( x^c \), and one that gives the best objective function value is chosen to compete with the current feasible solution. This chosen solution is denoted as \( x \). If \( f(x^c) \) is worse than \( f(x) \) then \( x \) becomes the current feasible solution. Otherwise the method will produce \( x^c \) as the output.

5.1.1 1-optimum local search

The SDMI will start with an initial feasible solution \( x \). This solution can be found in pseudopolynomial time using (say) the Alternating Method [10] and
is called current feasible solution. Then fix all but one component \( x_r \), of the solution \( x \). These values for the fixed variables are then substituted in the problem which would yield a one variable problem with column matrices \( A' = A_r \) and \( B' = B_r \), \( r \in N \) and column vectors \( c'_i = \max_{j \neq r} (c_i, a_{ij} + x_j) \) and \( d'_i = \max_{j \neq r} (d_i, b_{ij} + x_j) \), \( i \in M \). Algorithm MAXLINMINm1 is then applied on to \( A', B', c' \) and \( d' \) to find the best value for \( x_r \) in \( O(m) \) time (Theorem 4.4).

After finding the best value for \( x_r \), the feasible solution \( x \) is denoted by \( y \) and \( y \in N_1(x) \). This process will be repeated for every \( r \). Choose among the feasible solutions in the neighborhood \( N_1(x) \) one with smallest objective function value and compare it with the current feasible solution. If the current feasible solution returns worse objective function value then this chosen solution becomes current feasible solution and the process will be repeated again. Otherwise the procedure stops and produces the current feasible solution as the output. The following algorithm sums up this method.

**Algorithm 5.1.** ONEOPT-SDMI (One-optimum steepest descent method with improvement for the NLPM)

**Input:** \( A = (a_{ij}) \), \( B = (b_{ij}) \in \mathbb{R}^{m \times n} \), \( c = (c_1, \ldots, c_m)^T \), \( d = (d_1, \ldots, d_m)^T \in \mathbb{R}^m \), a non-linear isotone function \( f \) and a maximum number of iterations \( U \).

**Output:** An \( x \in S \).

1. Find \( x^0 \in S \) (for example, using the Alternating Method)

2. Set \( x := x^0 \), \( T = (+\infty, \ldots, +\infty)^T \), \( flag := true \) and \( iteration := 0 \)

3. Repeat until \( flag = false \) or \( iteration = U \)

   \[ \begin{align*}
   &\text{begin} \\
   &\quad \text{for } r = 1 \text{ to } n \text{ do} \\
   &\quad \quad \text{begin} \\
   &\quad \quad \quad \text{for } j = 1 \text{ to } n \text{ do} \\
   &\quad \quad \quad \quad \text{begin} \\
   &\quad \quad \quad \quad \quad y_j := x_j \text{ if } j \neq r \\
   &\quad \quad \quad \quad \end{align*} \]

   \[ \begin{align*}
   &\quad \quad \text{end} \\
   &\quad \quad A' := A_r \\
   &\quad \quad B' := B_r \\
   &\quad \text{for all } i \in M \text{ do} \\
   &\quad \quad \text{begin} \\
   &\quad \quad \quad c'_i := \max_{j \neq r} (c_i, a_{ij} + y_j) \\
   &\quad \quad \quad d'_i := \max_{j \neq r} (d_i, b_{ij} + y_j) \\
   &\quad \quad \quad \text{if } c'_i < d'_i \text{ then swap the sides of the } i^{th} \text{ equation} \\
   &\quad \quad \quad M'_{\text{new}} := \{ i \in M; c'_i > d'_i \} \\
   &\quad \quad \text{end} \\
   &\quad \text{Apply Algorithm MAXLINMINm1 to } A', B', c' \text{ and } d' \text{ to find } y_r \\
   &\quad \text{if } f(y) < f(T) \text{ then } T := y \\
   &\quad \text{end} \\
   &\quad \text{if } f(T) < f(x) \text{ then } x := T \\
   &\quad \text{if } x = x^0 \text{ then } flag := false \text{ (x is the best feasible solution found)} \\
   &\quad \text{else } x^0 := x \\
   &\quad \text{iteration := iteration + 1} \\
   &\quad \text{end} \\
   &\quad \text{end} \\
   &\text{end} \\
   &\text{end} \\
   &\text{end} \\
   &\text{end} \\
   &\text{end} \\
   &\text{end} \\
   &\text{end} \\
\]
5.1.2 2-optimum local search

The 2-optimum steepest descent method with improvement (SDMI) starts with an initial feasible solution $x$ which can again be found in pseudopolynomial time using say Alternating Method [10]. This solution is called a current feasible solution. Repeat the following process for all $r, s = 1, \ldots, n$, $r \neq s$. Fix all but two components $x_r$ and $x_s$ of the current feasible solution $x$. The values for the fixed variables are then substituted in the problem. This would produce a two variables problem whose matrices and vectors are $A' = (A_r, A_s) \in \mathbb{R}^{m \times 2}$ and $B' = (B_r, B_s) \in \mathbb{R}^{m \times 2}$ and

$$c'_i = \max_{j \neq r, j \neq s} (c_i, a_{ij} + x_j), i \in M$$

and

$$d'_i = \max_{j \neq r, j \neq s} (d_i, b_{ij} + x_j), i \in M,$$

where $i \in M$ and $j \in N$. Algorithm MAXLINMINm2 is then applied on to $A', B', c'$ and $d'$ to find the best value of the free variables, $x_r$ and $x_s$ in $O(m^2)$ time (Theorems 4.5). After finding the best value for $x_r$ and $x_s$, the feasible solution $x$ is denoted by $y$ and $y \in N_2(x)$. Select in the neighborhood $N_2(x)$ a solution with the smallest objective function value and compare it with the current feasible solution. If the current feasible solution returns worse objective function value, then the chosen solution becomes the current feasible solution and the process will be repeated again. Otherwise stop and declare current feasible solution the best feasible solution found. This procedure can be formulated in the following algorithm.

**Algorithm 5.2.** TWOOPT-SDMI (Two-optimum steepest descent method with improvement for the NLPM)

**Input:** $A = (a_{ij})$, $B = (b_{ij}) \in \mathbb{R}^{m \times n}$, $c = (c_1, \ldots, c_m)^T$, $d = (d_1, \ldots, d_m)^T \in \mathbb{R}^m$, a non-linear isotone function $f$ and a maximum number of iterations $U$.

**Output:** An $x \in S$.

1. Find $x^0 \in S$ (for example using the Alternating Method)
2. Set $x := x^0$, $T = (+\infty, \ldots, +\infty)^T$, $flag := true$ and $iteration := 0$
3. Repeat until $flag = false$ or $iteration = U$
   begin
   for $r = 1$ to $n$ do
     begin
     for $s = r + 1$ to $n$ do
       begin
       for $j = 1$ to $n$ do
         begin
         $y_j := x_j$ if $j \neq r$ and $j \neq s$
         end
       $A' := (A_r, A_s)$
       $B' := (B_r, B_s)$
       for all $i \in M$ do
     end
   end

\[ c_i' := \max(c_i, a_{ij} + y_j), \quad j \neq r, j \neq s \]
\[ d_i' := \max(d_i, b_{ij} + y_j), \quad j \neq r, j \neq s \]
if \( c_i' < d_i' \) then swap the sides of the \( i \)th equation
\[ M^\text{new} := \{ r \in M; c_i' > d_i' \} \]
end

Apply Algorithm MAXLINMINm2 to \( A', B', c' \) and \( d' \) to find \( y_r \) and \( y_s \)
if \( f(y) < f(T) \) then \( T := y \)
end

\[ 5.2 \text{ Steepest descent method without improvement} \]

In the SDMI we have seen that a best solution in the neighborhood of a current feasible solution \( x \) is chosen to compete with the current feasible solution. If this solution has a better objective function value than the current feasible solution, then it will become current, and the process will start again. Otherwise the current feasible solution is the best feasible solution found and the process stops. However, this method may not provide us with the best solution. As a consequence, we may get stuck in a local optimum. To overcome this we made some modifications to the SDMI.

\[ 5.2.1 \text{ 1-optimum local search} \]

This second version of the steepest descent method, will consider all candidates in the neighborhood \( N_1(x) \) of a current feasible solution \( x \) and chooses the one with the best objective function value among these solutions. The selected solution becomes current feasible solution regardless of the fact that its objective function value may be worse than that of the current feasible solution \( x \). This method is called \textit{steepest descent method without improvement}, for brevity SDMW. The method is formulated in the following algorithm:

\begin{algorithm}
\caption{ONEOPT-SDMW (One-optimum steepest descent method without improvement for the NLPM)}
\begin{algorithmic}
\STATE \textbf{Input:} \( A = (a_{ij}), B = (b_{ij}) \in \mathbb{R}^{m \times n}, c = (c_1, \ldots, c_m)^T, d = (d_1, \ldots, d_m)^T \in \mathbb{R}^m, \) a non-linear isotone function \( f \) and a maximum number of iterations \( U \).
\STATE \textbf{Output:} An \( x \in S \).
\STATE 1. Find \( x^0 \in S \) (for example using the Alternating Method)
\STATE 2. Set \( x := x^0, T = (+\infty, \ldots, +\infty)^T \) and \textit{Iteration} := 0
\STATE 3. Repeat until \textit{Iteration} = \( U \)
\STATE \hspace{1em} begin
\STATE \hspace{2em} for \( r = 1 \) to \( n \) do
\STATE \hspace{3em} end
\STATE \hspace{1em} end
\end{algorithmic}
\end{algorithm}
\begin{align*}
&\text{for } j = 1 \text{ to } n \text{ do} \\
&\quad \text{begin} \\
&\quad \quad y_j := x_j \text{ if } j \neq r \\
&\quad \text{end} \\
&\quad A' := A_r \\
&\quad B' := B_r \\
&\text{for all } i \in M \text{ do} \\
&\quad \text{begin} \\
&\quad \quad c_i' := \max_{j \neq r}(c_i, a_{ij} + y_j) \\
&\quad \quad d_i' := \max_{j \neq r}(d_i, b_{ij} + y_j) \\
&\quad \quad \text{if } c_i' < d_i' \text{ then swap the sides of the } i^{th} \text{ equation} \\
&\quad \quad M_{\text{new}} := \{i \in M; c_i > d_i'\} \\
&\quad \text{end} \\
&\text{Apply Algorithm MAXLINMINm1 on to } A', B', c', d' \text{ to find } y_r \\
&\text{if } f(y) < f(T) \text{ then } T := y \\
&\text{end} \\
&x := T \\
&x^0 := x \\
&\text{Iteration} := \text{Iteration} + 1 \\
&\text{end}
\end{align*}

### 5.2.2 2-optimum local search

Similarly as for 1-optimum SDMW, the 2-optimum SDMW will begin by finding an initial feasible (current) solution $x$ and chooses in the neighborhood $N_2(x)$ a feasible solution with the smallest objective function value regardless of the fact that its objective function value may be worse than the current feasible solution $x$. The chosen solution is now current feasible solution and the process starts again. The method stops if the maximum number of iterations (or time) is reached.

**Algorithm 5.4.** TWOOPT-SDMW(Two-optimum steepest descent method without improvement for the NLPM)

**Input:** $A = (a_{ij})$, $B = (b_{ij}) \in \mathbb{R}^{m \times n}$, $c = (c_1, \ldots, c_m)^T$, $d = (d_1, \ldots, d_m)^T \in \mathbb{R}^n$, a non-linear isotone function $f$ and a maximum number of iteration $U$.

**Output:** An $x \in S$.

1. Find $x^0 \in S$ (for example using the Alternating Method)
2. Set $x := x^0$, $T := (+\infty, \ldots, +\infty)^T$, iteration := 0
3. Repeat until iteration := $U$
   \begin{align*}
   &\text{begin} \\
   &\quad \text{for } r = 1 \text{ to } n \text{ do} \\
   &\quad \quad \text{begin} \\
   &\quad \quad \quad \text{for } s = r + 1 \text{ to } n \text{ do} \\
   &\quad \quad \quad \quad \text{begin} \\
   &\quad \quad \quad \quad \quad \text{for } j = 1 \text{ to } n \text{ do} \\
   &\quad \quad \quad \quad \quad \quad \text{begin} \\
   &\quad \quad \quad \quad \quad \quad y_j := x_j \text{ if } j \neq r \text{ and } j \neq s \\
   &\quad \quad \quad \quad \quad \text{end} \\
   &\quad \quad \quad \text{end} \\
   &\quad \quad \text{end} \\
   &\quad \text{end} \\
   &\text{end}
   \end{align*}
end

$A' := (A_r, A_s)$

$B' := (B_r, B_s)$

for all $i \in M$ do

begin

$c_i' := \max(c_i, a_{ij} + y_j), \ j \neq r, \ j \neq s$

$d_i' := \max(d_i, b_{ij} + y_j), \ j \neq r, \ j \neq s$

if $c_i < d_i$ then swap the sides of the $i^{th}$ equation

$M_{new} := \{r \in M; c_i > d_i\}$

end

Apply Algorithm MAXLINMINm2 to $A', B', c'$ and $d'$ to find $y_r$ and $y_s$

if $f(y) < f(T)$ then $T := y$

end

end

$x := T$

$x^0 := x$

iteration = iteration + 1

end

5.3 Hill-descending method

In both methods described previously, we do not have the guarantee of escaping from local optima. In an attempt to overcoming this problem, we consider modifying those methods.

5.3.1 1-optimum local search

Here we propose a 1-optimum hill descending method (HDM) for solving NLPM. The method will start with a feasible solution $x$, then chooses an immediate solution in the neighborhood, $N_1(x)$, of $x$ with better objective function value than the current solution and repeats the process until no further improvement is possible or the maximum number of time/iterations are reached. The method is formulated in the following algorithm:

Algorithm 5.5. ONEOPT-HDM (One-optimum hill-descending method for the NLPM)

Input: $A = (a_{ij}), B = (b_{ij}) \in \mathbb{R}^{m \times n}, c = (c_1, \ldots, c_m)^T, d = (d_1, \ldots, d_m)^T \in \mathbb{R}^m$, a non-linear isotone function $f$ and a maximum number of iterations $U$.

Output: An $x \in S$.

1. Find $x^0 \in S$ (for example using the Alternating Method)

2. Set $x := x^0$ and $flag := true, iteration := 0$

3. Repeat until $flag = false$ or $iteration = U$

begin

for $r = 1$ to $n$ do

begin

for $j = 1$ to $n$ do

begin

for

end

end

end

end
\( y_j := x_j \text{ if } r \neq j \)
\[
A' := A_r \\
B' := B_r
\]
for all \( i \in M \) do
\[
c_i' := \max(c_i, a_{ij} + y_j), \ j \neq r \\
d_i' := \max(d_i, b_{ij} + y_j), \ j \neq r
\]
if \( c_i' < d_i' \) then swap sides of the \( i^{th} \) equation
\[
M_{\text{new}} := \{ i \in M; c_i' > d_i' \}
\]
end
Apply Algorithm MAXLINMINm1 to \( A', B', c' \) and \( d' \)
to find \( y_r \)
if \( f(y) < f(x) \) then
begin
\( x := y \)
break
end
end
if \( x = x^0 \) then \( \text{flag} := \text{false} \ (x \text{ is the best solution found}) \)
else
begin
\( x^0 := x \)
\( \text{iteration} := \text{iteration} + 1 \)
end
end

5.3.2 2-optimum local search

The hill-descending method (HDM) will consider the neighborhood \( N_2(x) \) of a current solution \( x \). Then chooses immediately the first feasible solution in the neighborhood with smaller objective function value than the current feasible solution. The chosen solution will become the current feasible and the procedure starts again. If there is no solution in the neighborhood with better objective function than the current feasible solution the method stops.

**Algorithm 5.6.** TWOOPT-HDM(Two-optimum hill-descending method for the NLPM)

**Input:** \( A = (a_{ij}), B = (b_{ij}) \in \mathbb{R}^{m \times n}, c = (c_1, \ldots, c_m)^T, d = (d_1, \ldots, d_m)^T \in \mathbb{R}^m \), a non-linear isotone function \( f \) and a maximum number of iterations \( U \).

**Output:** An \( x \in S \).

1. Find \( x^0 \in S \) (for example using the Alternating Method)
2. Set \( x := x^0, \text{flag} := \text{true}, T := 0, \text{iteration} := 0 \)
3. Repeat until \( \text{flag} = \text{false} \) or \( \text{iteration} = U \)
begin
   for \( r = 1 \) to \( n \) do
   begin
      for \( s = r + 1 \) to \( n \) do
begin
for $j = 1$ to $n$ do 
begin
\[ y_j := x_j \text{ if } j \neq r \text{ and } j \neq s \]
end
\[ A' := (A_r, A_s) \]
\[ B' := (B_r, B_s) \]
for all $i \in M$ do 
begin
\[ c'_{ij} := \max(c_i, a_{ij} + y_j), j \neq r, j \neq s \]
\[ d'_{ij} := \max(d_i, b_{ij} + y_j), j \neq r, j \neq s \]
if $c_i < d_i$ then swap the sides
\[ M'_{new} := \{r \in M; c_i > d_i\} \]
end
Apply Algorithm MAXLINMINm2 to $A', B', c'$ and $d'$ to find $y_r$ and $y_s$
if $f(y) < f(x)$ then 
begin
\[ x := y \]
\[ T := 1 \]
break 
end
end
if $T = 1$ then $T := 0$ and break 
if $x = x^0$ then $flag := false$ (x is the best solution found)
else 
begin
\[ x^0 := x \]
\[ \text{iteration} := \text{iteration} + 1 \]
end
end

5.4 The Multi-start heuristic

The multi-start heuristic method will consider all the 1-optimum and 2-optimum local search methods proposed in the previous sections and repeat each method for a certain number of times.

Since the method (Alternating Method) we are using to find the initial feasible solution depends on the starting vector we consider a positive integer say $R > 1$ and repeat the 1-optimum local search methods $R$ times. For each run, we generate a new starting vector which may in return give a different feasible solution to the problem and then apply the 1-optimum or 2-optimum local search methods. After repeating the method $R$ times we find among the $R$ solutions the best solution to our problem. This method is summed up in the following algorithms.

Algorithm 5.7. MULTI-START (Multi-start heuristic for the NLPM)
Input: $A = (a_{ij}), B = (b_{ij}) \in \mathbb{R}^{m \times n}$, $c = (c_1, \ldots, c_m)^T, d = (d_1, \ldots, d_m)^T \in \mathbb{R}^n$, a non-linear isotone function $f$ and a maximum number of iterations $R$. 

Output: A solution $x^{\text{best}} \in S$.

1. Set $\text{run} := 0$ and $x^{\text{best}} := (+\infty, \ldots, +\infty)^T$

2. Randomly find a starting vector $x(\text{run})$

3. Use $x(\text{run})$ and apply the Alternating Method to find a solution $x^0$ to $A \otimes x \oplus c = B \otimes x \oplus d$

4. Apply a 1-optimum (or 2-optimum) local search method starting from $x^0$ to find a solution $x(\text{run})$

5. if $f(x(\text{run})) < f(x^{\text{best}})$ then $x^{\text{best}} := x(\text{run})$

6. if $\text{run} = R$ then stop ($x^{\text{best}}$ is the best feasible solution found)
   else $\text{run} := \text{run} + 1$ and go to 3.

6 Computational results

In this section we will compare the 1-optimum and 2-optimum local search methods, developed in the previous sections. We divide this section into two subsections: In the first subsection we consider the objective function to be max-linear and present results obtained when comparing exact method for solving NLPM and $r$-optimum local search methods where $r = 1, 2$. In the second subsection the objective function is $f(x) = 2x_1 + 2x_2 + \cdots + 2x_n$ and we compare the 1-optimum and 2-optimum local search methods.

All the algorithms developed for both heuristics and exact methods were implemented in Matlab 7.2. Matrices and vectors are generated randomly. The experiments were carried out on a PC with Intel Centrino Duo, 1.66 GHz of CPU and 0.99 GB of RAM.

6.1 Instances when the objective function is max-linear

In this section we consider NLPM when the objective function is max-linear. First we show the performance of multi-start heuristics then compare the exact method and $r$-optimum local search methods where $r = 1, 2$. We begin by finding a feasible solution to the given problem (using the Alternating Method). Then each of the methods will use this feasible solution in order to find an optimal or a best feasible to the given problem. For each method, we report the computing time spent up to the point where its optimal solution or best feasible solution is found for the first time. In the 1-optimum and 2-optimum local search methods, we impose a maximum number of iterations as five (5) for the SDMW.

Recall that $f^{\text{min}} = \inf_{x \in S} f(x)$ denotes the optimal value of the objective function. The size of matrices and range of entries for the problems are given at the end of the tables. Entries in each table have the following meaning:

- **FV**: Feasible objective function value;
- **OFV**: Optimal objective function value;
- **Sol**: The best feasible objective function value found;
- **# times best**: How many times the method produces the best solution among the heuristics;
times fast: How many times the method produces solution faster than all other methods in the table;

times exact: How many times the heuristic method finds an exact solution;

RE The relative error determined as follows

$$RE = \left| \frac{\text{Sol} - \text{OFV}}{\text{OFV}} \right|.$$  \hspace{1cm} (15)

Note that RE is used only for instances whose optimal objective function value is positive.
<table>
<thead>
<tr>
<th>((m, n))</th>
<th>Range</th>
<th>FV</th>
<th>EXACT INTEGER SDMI OFV Time</th>
<th>SDM Maximinmax</th>
<th>1-OPTIMUM LOCAL SEARCH SDMI Time</th>
<th>1-OPTIMUM LOCAL SEARCH SDM Time</th>
<th>1-OPTIMUM LOCAL SEARCH HDM Time</th>
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Table 1: Results on the comparison of performance of exact method, 1-optimum local search methods and multi-start heuristics on NLPM with max-linear objective function. The matrices have 20 rows and up to 300 columns.
Table 2: Results on the comparison of performance of exact method, 2-optimum local search methods and multi-start heuristics on NLPM with max-linear objective function. The matrices have 20 rows and up to 160 columns.

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<th>EXACT INTEGER SDMW</th>
<th>EXACT INTEGER HDM</th>
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</tr>
<tr>
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</tr>
<tr>
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Table 3: Results on the comparison of performance of exact method and r-optimum local search methods, $r = 1, 2$. The matrices have 10 rows and 20 columns. Range of entries is $(0, 10^5)$.
<table>
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<th>SDMW</th>
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**Table 4**: Results on the comparison of performance of exact method and r-optimum local search methods, r = 1, 2. The matrices have 10 rows and 50 columns. Range of entries is (0, 10^5).
<table>
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<th>SDMW</th>
<th>HDM</th>
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<th>SDMW</th>
<th>HDM</th>
<th>2-OPTIMUM LOCAL SEARCH</th>
<th>SDMI</th>
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<th>HDM</th>
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</table>

| # times best | 6 | 0 | 10 | 6 | 0 | 0 |
| # times fastest | 2 | 0 | 2 | 2 | 0 | 0 |
| # times exact | 0 | 0 | 0 | 0 | 0 | 0 |

Table 5: Results on the comparison of performance of exact method and r-optimum local search methods, \( r = 1, 2 \). The matrices have 10 rows and 100 columns. Range of entries is \((0, 10^5)\).
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<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 6: Results on the comparison of performance of exact method and r-optimum local search methods, \( r = 1, 2 \). The matrices have 40 rows and 100 columns. Range of entries is \((0, 10^5)\).
Tables 1 and 2 show that the multi-start heuristic is not improving the result obtained by the local search methods.

Table 3 reports the result on comparison of exact, 1-optimum and 2-optimum local search methods. The objective function is max-linear and the input matrices have 10 rows and 20 columns. All the methods were able to produce an exact solution for six (6) instances except HDM for 2-optimum local search which produces four (4). Each of the five methods has average RE=0.04 and HDM for 2-optimum has average RE=0.34. The fastest among all the methods is the HDM for 1-optimum local search with average computing time of 0.04 seconds. The exact method is slow with the average computing time of 216.3 seconds.

Table 4 gives the result on the comparison of exact, 1-optimum and 2-optimum local search methods. The matrices have 10 rows and 50 columns. SDMI, HDM for 1-optimum and SDMI for 2-optimum local were able to find exact solution in three instances. Each of these methods have average RE=0.78. SDMW for 1-optimum has average RE=2.90. SDMW and HDM for 2-optimum has average RE=1.57 and RE=2.56 respectively. The fastest method is SDMW for 1-optimum with average computing time of 0.1 seconds.. The exact method is once more very slow with average computing time of 218.3 seconds.

Table 5 reports the output on the comparison of exact, 1-optimum and 2-optimum local search methods. The matrices have 10 rows and 100 columns. In the 1-optimum local search SDMI and HDM were able to find exact solution twice. While in the 2-optimum only SDMI is able to find exact solution in two instances. The average RE for SDMI, HDM in 1-optimum and SDMI in 2-optimum is 0.7. The fastest method is the 1-optimum SDMW with average computing time of 1.3 seconds. The exact method is slow with average computing time of 131.7 seconds.

Table 6 gives the output on the comparison of exact, 1-optimum and 2-optimum local search methods. The matrices have 40 rows and 100 columns. All the methods produced the same solution with average RE=1.16. 1-optimum SDMI is the fastest method with average computing time of 0.5 seconds. The exact method is slow with average computing time of 6958.7 seconds.

We conclude that the 1-optimum SDMI, HDM and 2-optimum SDMI are the best solution methods that can produce a solution with minimum objective function value. The method that can produce solution with minimum computing time is 1-optimum SDMI.

6.2 Instances when the objective function is not max-linear

In this section we will consider the objective function is \( f(x) = 2^x_1 + 2^x_2 + \cdots + 2^x_n \). We compare performance of all the local search methods presented in Section 5. For each problem we find a feasible solution using the Alternating Method. Each of the heuristic methods will use this feasible solution to find a best feasible solution. We report computing time each method used to find a best feasible solution for the first time. As with the other experiments the maximum number of iterations for SDMW for both 1-optimum and 2-optimum local search methods is fixed as five (5).
<table>
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Table 7: Results on the comparison of 1- optimum and 2-optimum local search methods. The objective function is \(f(x) = 2^{x_1} + 2^{x_2} + \ldots + 2^{x_n}\). Input matrices have dimensions \(10 \times 50\) and \(10 \times 100\).
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</table>

Table 8: Results on the comparison of 1-optimum and 2-optimum local search methods. The objective function is $f(x) = 2^{x_1} + 2^{x_2} + \cdots + 2^{x_n}$. Input matrices have dimensions $20 \times 50$ and $20 \times 100$. 
It can easily be verified from tables 7 and 8 that 2-optimum SDMI is the best method that can produce a solution with a minimal objective function value. 1-optimum SDMW is the fastest method that can produce solution in a very short period of time.

Finally, we concluded that when the objective function is max-linear 1-optimum local search methods are more efficient. When the objective function is not max-linear (that is \( f(x) = 2^{x_1} + 2^{x_2} + \cdots + 2^{x_n} \)) 2-optimum local search methods are more appropriate.

References


