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# AN ALGORITHM FOR CHECKING STRONG REGULARITY OF MATRICES IN BOTTLENECK ALGEBRAS*

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**Abstract**

Let $(B, \leq)$ be a dense, linearly ordered set without maximum and minimum and $(\oplus, \otimes) = (\max, \min)$. An $n \times n$ matrix $A = (a_{ij})$ over $B$ is called

(a) strongly regular if for some $b$ the system $A \otimes x = b$ is uniquely solvable;

(b) trapezoidal if the inequality

$$a_{ii} > \sum_{k=1}^{i}{}^{\oplus} \sum_{l=k+1}^{n}{}^{\otimes} a_{kl}$$

holds for all $i = 1, .., n$.

We show that a square matrix is strongly regular if and only if it can be transformed to a trapezoidal matrix using permutations of the rows and columns. Moreover, an $O(n^{3.5})$ method for checking the strong regularity is proved.

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**Keywords:** bottleneck algebra, regular matrix, trapezoidal matrix, uniquely solvable system.

*This paper was prepared in 1985 and submitted to the proceedings of an international conference on numerical linear algebra. The paper was formally accepted for publication but for reasons not known to the authors it was never published. Based on a preprint the results of the paper have been generalised in a number of subsequent papers of other authors. As these are based on the results of this paper the authors feel that it is in the interest of the research community that it be published, although with a big delay. Following the developments on the bottleneck assignment problem [1], the computational complexity of the Algorithm has been updated.

## 1 INTRODUCTION

The quadruple $\mathcal{B} = (B, \oplus, \otimes, \leq)$ is called a bottleneck algebra (in short BA) if $(B, \leq)$ is a nonempty, linearly ordered set without maximum and minimum and $\oplus, \otimes$ are binary operations on $B$ defined by the formulas

$$a \oplus b = \max\ \{a, b\}$$
$$a \otimes b = \min\ \{a, b\}$$

Among the most important interpretations of BA are those based on the following linearly ordered sets ($\leq$ is everywhere the natural order and $-\infty \leq l < u \leq \infty$ ):

$$((l, u), \leq), \tag{1}$$

$$((l, u) \cap Q, \leq), \tag{2}$$

$$(Z, \leq), \tag{3}$$

$$((l, u) \cap P(\alpha), \leq) \tag{4}$$

where $Q$ is the set of rationals, $Z$ is the set of integers and

$$P(\alpha) = \{\sum_{i=0}^{r} p_i \alpha^i\ ;\ p_0, ..., p_r\ integers,\ r = 0, 1, 2, ...\},$$

$\alpha$ being any fixed transcendental number (cf.[6]). We denote by $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4$ the BA based on (1) - (4) successively.

Some practical problems lead to computations in a bottleneck algebra. For example, the permanent (known also as the bottleneck assignment problem) of an $n \times n$ matrix $A = (a_{ij})$ in $\mathcal{B}_1$, i.e.

$$per(A) = \sum_{\pi}^{\oplus} \prod_{i}^{\otimes} a_{i,\pi(i)}$$

corresponds to a weighted matching in a complete bipartite graph with the maximal possible lowest score. This corresponds to those situations where the overall performance of a team is measured by the worst performance of it's individual member(s) - e.g. if each of $n$ workers performs one of $n$ tasks on an assembly line then the speed of the line is equal to the speed of the slowest worker (see [3]). An $O(n^{2.5})$ method for solving this problem is known [1].

As an other example, consider the transportation capacity (transmittance) problem. If the transportation route consists of two parts $UV$ and $VW$ (say V is a transship point) then the total route capacity is the minimum of the capacities of $UV$ and $VW$. Similarly, in a transportation network with $U_1 \ldots, U_l$ as dispatching points, $V_1, \ldots, V_m$ as transship points and $W_1, \ldots, W_n$ as destination points denoting the capacities of $U_iV_j$ resp. $V_jW_k$ by $a_{ij}$ and $b_{jk}$, respectively $(i = 1, \ldots, l; j = 1, \ldots, m; k = 1, \ldots, n)$ we have that the total transportation capacity between $U_i$ and $W_k$ is equal to

$$c_{ik} = \max_{j=1,\ldots,m} \min\{a_{ij}, b_{jk}\}$$

for all $i = 1, ..., l$ and $k = 1, ..., n$ (see Figure 1). This expression becomes more formidable using the usual extensions of $\oplus$ and $\otimes$ to matrices in $\mathcal{B}_1$ :

$$C = A \otimes B$$

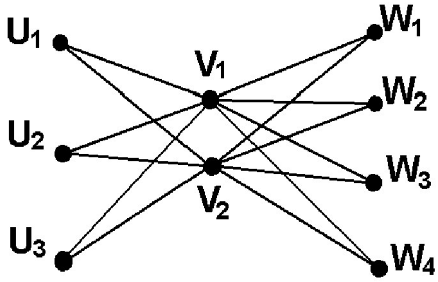


Figure 1: $l = 3, m = 2, n = 4$

having denoted by $A, B, C$ the matrices $(a_{ij}), (b_{jk}), (c_{ik})$.

Several problems similar to those in linear algebra have been studied in BA or, in closely related structures. To mention a few of them, recall [2], [4, 5, 6]. In the case when $\otimes$ is a group operation, the concept of strong regularity of matrices was introduced in [4] and an efficient method for checking this property was derived in [2]. In this paper our aim is to do the same as in the case of BA.

## 2 DEFINITIONS AND BASIC PROPERTIES

Clearly, a bottleneck algebra $(B, \oplus, \otimes, \leq)$ is a distributive (infinite) lattice. Among many basic properties we have that $a \leq b$ and $c \leq d$ imply

$$a \oplus c \leq b \oplus d$$
$$a \otimes c \leq b \otimes d$$

for all $a, b, c, d \in B$.

The set of all $m \times n$ matrices over $B$ will be denoted by $B(m, n)$ and $B(m, 1)$ by $B_m$. Elements of $B_m$ will be called vectors. Extend $\oplus, \otimes$ and $\leq$ to matrices over $B$ in the same way as in linear algebra, that is if $A = (a_{ij})$, $B = (b_{ij})$ are matrices of compatible sizes, then

$$A \oplus B = (a_{ij} \oplus b_{ij}) \text{ and } A \otimes B = (\sum_k^{\oplus} a_{ik} \otimes b_{kj})$$

Many properties of these operations can be found in [6]. Let us mention here the following one:

$$\text{if } C \leq D \text{ then } A \otimes C \leq A \otimes D \text{ and } C \otimes A \leq D \otimes A$$

whenever the indicated products exist.

The set of all permutations of the set $\{1, 2, ..., n\}$ is denoted by $P_n$; $id$ means the identity permutation. If $A = (a_{ij}) \in B(m, n)$, $\sigma \in P_m$, $\pi \in P_n$ then $A(\sigma, \pi)$ denotes the matrix $C = (c_{i,j})$ such that

$$c_{ij} = a_{\sigma(i),\pi(j)}.$$

If $\sigma \in P_n$, $A = (a_{ij}) \in B(n, n)$ then the weight of $\sigma$ with respect to $A$, i.e.

$$a_{1,\sigma(1)} \otimes a_{2,\sigma(2)} \otimes \ldots \otimes a_{n,\sigma(n)}$$

is denoted by $w(A,\sigma)$. Thus

$$per(A) = \sum_{\sigma \in P_n}^{\oplus} w(A,\sigma)$$

and we put

$$\max(A) = \{\sigma \in P_n \ ; \ w(A,\sigma) = per(A)\}$$

For any set $H$ the symbol $|H|$ will mean the number of its elements.

Systems of simultaneous linear equations (shortly linear systems) of the form

$$A \otimes x = b \tag{5}$$

where $A \in B(m,n)$, $b \in B_m$ were studied in [5] and [6]. The solution set of (5) will be denoted by $S(A,b)$ and

$$T(A) = \ \{|S(A,b)| \ ; \ b \in B_m\}.$$

It is not difficult to verify that

$$\{0, \ \infty\} \subseteq T(A) \subseteq \{0, \ 1, \ \infty\}$$

for every $A \in B(m,n)$. A square matrix $A$ is called strongly regular if $1 \in T(A)$, i.e. if there exists a vector $b$ such that (5) is uniquely solvable. The purpose of this paper is

(i) to characterize matrices which are strongly regular (Theorems 1 and 2),

(ii) to develop an efficient algorithm for checking this property (Theorems 3 and 4).

The main results are proved under the assumption of density of $\leq$, i.e.

$$(\forall x, y \in B)(\exists z \in B)(x < y \Rightarrow x < z < y).$$

Thus $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_4$ are dense while $\mathcal{B}_3$ is not.

In what follows we assume that $A = (a_{ij}) \in B(m,n)$, $b = (b_1, ..., b_m)^T \in B_m$; $m, n \geq 1$ are integers. For convenience put

$$\begin{aligned} M &= \{1, 2, \ldots, m\}, \\ N &= \{1, 2, \ldots, n\}, \\ M_j(A,b) &= \{i \in M \ ; \ a_{ij} > b_i\}, \\ \tilde{M}_j(A,b) &= \{i \in M \ ; \ a_{ij} \geq b_i\} \end{aligned}$$

for all $j \in N$. The symbol $A_i$ for $i \in M$ denotes the $i$-th row of $A$.

## *3 UNIQUELY SOLVABLE LINEAR SYSTEMS*

Necessary and sufficient conditions for the solvability of linear systems were found in [5] but this work does not provide any criterion for such a system to be uniquely solvable. This problem will be solved in the present section.

Basic information is offered by the first lemma. In what follows we suppose that a (not necessarily dense) bottleneck algebra is fixed.

**Lemma 1** *If* $|S(A,b)| = 1$ *then* $M_j(A,b) \neq \emptyset$ *for all* $j \in N$.

**Proof.** Suppose $x = (x_1, ..., x_n)^T \in S(A, b)$ and $M_k(A, b) = \emptyset$.
Take

$$x' = (x_1, ..., x_{k-1}, \alpha, x_{k+1}, ..., x_n)^T$$

where $\alpha > x_k$. Then $x' \in S(A, b)$ because $A \otimes x' \geq A \otimes x = b$ and $A_i \otimes x' > b_i$ would yield $a_{ik} \otimes \alpha > b_i$, which implies $i \in M_k(A, b)$. Hence, $x'$ is another element of $S(A, b)$, a contradiction. ■

If $|S(A, b)| = 1$ we denote for all $j \in N$

(i) $\min\{b_i \ ; \ i \in M_j(A, b)\}$ by $\overline{x}_j$ ;

(ii) $\{i \in M_j(A, b) \ ; \ b_i = \overline{x}_j \}$ by $I_j(A, b)$;

(iii) $\{i \in \tilde{M}_j(A, b) \ ; \ a_{ij} = b_i = \overline{x}_j\}$ by $K_j(A, b)$;

(iv) $I_j(A, b) \cup K_j(A, b)$ by $L_j(A, b)$ or, shortly $L_j$.

**Lemma 2** *If $|S(A, b)| = \{x\}$, $x = (x_1, ..., x_n)^T$ then*

$$x_j = \overline{x}_j \textit{ for all } j \in N$$

*and the system $\{L_1, ..., L_n\}$ is a minimal covering of the set $L = \bigcup_{j \in N} L_j$, i.e. for every $N' \subseteq N$, $N' \neq N$ we have*

$$\bigcup_{j \in N'} L_j \neq L.$$

**Proof.** Clearly

$$x_j \leq \overline{x}_j \quad \text{for all} \quad j \in N \tag{6}$$

for otherwise the relations

$$x_j > \overline{x}_j = b_i \text{ and } a_{ij} > b_i$$

would hold for some $j \in N$ and $i \in M_j(A, b)$, implying $A_i \otimes x > b_i$. To prove equality in (6), suppose $x_k < \overline{x}_k$ for some $k \in N$. We show then that

$$x' = (x_1, ..., x_{k-1}, \overline{x}_k, x_{k+1}, ..., x_n)^T$$

is also an element of $S(A, b)$. Clearly,

$$A \otimes x' \geq A \otimes x \geq b.$$

At the same time the inequality

$$A_i \otimes x' > b_i$$

for some $i \in M$ would imply

$$\min\{a_{ik}, \overline{x}_k\} > b_i \tag{7}$$

because $\sum^{\oplus}_{j \in N, j \neq k} a_{ij} \otimes x_j \leq b_i$. But (7) can hold neither for $i \notin M_k(A, b)$ (by the definition of $M_k(A, b)$) nor for $i \in M_k(A, b)$ since otherwise

$$\overline{x}_k > b_i \geq \min\{b_i\ ;\ i \in M_k(A,b)\} = \overline{x}_k.$$

In order to prove the second part of the lemma take $N' \subseteq N$, $N' \neq N$ and suppose that

$$\bigcup_{j \in N'} L_j = L.$$

Let $k \in N \setminus N'$. To get a contradiction it is sufficient to show that

$$x' = (x_1, ..., x_{k-1}, \alpha, x_{k+1}, ..., x_n)^T$$

is in $S(A,b)$ where $\alpha = \max H$ if

$$H = \{b_i;\ a_{ik} = b_i < \overline{x}_k\}$$

is nonempty and $\alpha < \overline{x}_k$ is arbitrary if $H = \emptyset$. Obviously, $A \otimes x' \leq b$. To prove the equality take $i \in M$ and distinguish the following cases.

(a) If $i \in L$ then $i \in L_r$ for some $r \in N \setminus \{k\}$. Thus either $a_{ir} > b_i = \overline{x}_r$ or $a_{ir} = b_i = \overline{x}_r$ and hence $A_i \otimes x' = b_i$.

(b) If $i \in M \setminus L$ and $a_{ik} > b_i$ then $\overline{x}_k < b_i$ ( for otherwise $i \in L$).
Thus

$$a_{ir} \otimes \overline{x}_r = b_i \tag{8}$$

is fulfilled by some $r \in N \setminus \{k\}$.

(c) If $i \in M \setminus L$ and $a_{ik} = b_i$ then $\overline{x}_k \neq b_i$ ( for otherwise $i \in L$). The inequality $\overline{x}_k > b_i$ yields $\alpha \geq b_i$ and thus $a_{ik} \otimes \alpha = b_i$, while $\overline{x}_k < b_i$ implies that an $r \in N \setminus \{k\}$ satisfying (8) exists.

(d) If $i \in M \setminus L$ and $a_{ik} < b_i$ then again an $r \in N \setminus \{k\}$ satisfying (8) exists.

■

The following lemma provides easily proved basic combinatorial properties.

**Lemma 3** *Let $H_1, ..., H_k$ be arbitrary finite sets and*

$$H = \bigcup_{j=1}^{k} H_j,\ \ |H| = l.$$

*(a) If $\{H_1, ..., H_k\}$ is a minimal covering of $H$ then $k \leq l$.*

*(b) If $k = l$ then $\{H_1, ..., H_k\}$ is a minimal covering of $H$ if and only if $H_1, ..., H_k$ are one-element and pairwise disjoint sets.*

**Lemma 4** *If $m = n$ and $|S(A,b)| = 1$ Then*

$$I_1(A,b), I_2(A,b), \ldots, I_n(A,b)$$

*are one-element disjoint sets.*

**Proof.** It follows from Lemma 2 that $\{L_1, ..., L_n\}$ is a minimal covering of the set $L \subseteq M = N$. Part (a) of Lemma 3 yields now that $L = M$ and hence from (b) we get that $L_1, \ldots, L_n$ are one-element and pairwise disjoint. It remains to recall that $I_1(A,b), I_2(A,b), \ldots, I_n(A,b)$ are nonempty (Lemma 1). ■

**Corollary 1** *If $m = n$ and $S(A, b) = \{x\}$ then there exists a permutation $\pi \in P_n$ satisfying*

$$a_{i,\pi(i)} > b_i = x_{\pi(i)}$$

**Proof.** It is sufficient to set $\pi(i) = j$ such that

$$I_j(A, b) = \{i\}.$$

■

**Lemma 5** *If $A, C \in B(m, n)$ and $A$ is obtained from $C$ by permuting the columns, then*

$$|S(A, b)| = |S(C, b)|$$

*for every $b \in B_m$.*

**Theorem 2** *Let $m = n$, $|S(A, b)| = 1$ if and only if the inequalities*

$$a_{i,\pi(i)} > b_i > \sum_{j\in N j\neq i}^{\oplus} a_{i,\pi(j)} \otimes b_j, \;\; i = 1, \dots, n \tag{9}$$

*are satisfied by at least one $\pi \in P_n$.*

**Proof.** For the "only if" statement it remains to show that the permutation $\pi$ in Corollary 1 satisfies

$$a_{i,\pi(j)} \otimes b_j \neq b_i \tag{10}$$

for all $i, j \in N, i \neq j$. But equality would imply

$$a_{i,\pi(j)} \otimes x_{\pi(j)} = b_i$$

and then

$$x' = (x_1, ..., x_{\pi(i)-1}, \alpha, x_{\pi(i)+1}, ..., x_n)^T$$

with $\alpha < x_{\pi(i)}$ is also in $S(A, b)$ because

$$A \otimes x' \leq A \otimes x = b,$$
$$A_r \otimes x' = a_{r,\pi(r)} \otimes x_{\pi(r)} = b_r$$

for $r \neq i$ and

$$A_i \otimes x' = a_{i,\pi(j)} \otimes x_{\pi(j)} = b_i,$$

a contradiction.

To prove the converse implication let us assume without loss of generality (Lemma 5) that $\pi = id$. Then we have for all $i \in N$

$$a_{ii} > b_i > \sum_{j\in N j\neq i}^{\oplus} a_{ij} \otimes b_j$$

and hence $b \in S(A, b)$. Clearly, for every $x = (x_1, \dots, x_n)^T \in S(A, b)$ all inequalities

$$x_i \leq b_i; \;\; i = 1, \dots, n$$

hold because otherwise

$$A_i \otimes x > b_i$$

for some $i \in N$. At the same time whenever one of these inequalities is strict (say $k$-th) then

$$A_k \otimes x \leq a_{kk} \otimes x_k \oplus \sum_{j\in N j\neq k}^{\oplus} a_{kj} \otimes b_j < b_k,$$

a contradiction. Therefore $S(A, b) = \{b\}$. ■

## 4 TRAPEZOIDAL MATRICES

**Definition 1** *A matrix $A \in B(n,n)$ will be called trapezoidal (see Fig.2) if for all $r \in N$*

$$a_{rr} > \sum_{i=1}^{r}{}^{\oplus} \sum_{j=i+1}^{n}{}^{\oplus} a_{ij} \tag{11}$$

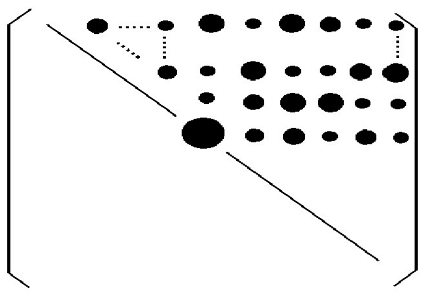

Figure 2: a trapezoidal matrix scheme

Matrices $A, C \in B(m,n)$ are said to be equivalent ($A \sim C$) if one of them can be obtained from the other using only permutations of the rows and columns. The relation $\sim$ is evidently an equivalence relation and the following assertion can easily be verified.

**Lemma 6** *If $A \sim C$ then $A$ is strongly regular if and only if $C$ is strongly regular.*

**Theorem 3** *A necessary condition for $A \in B(n,n)$ to be strongly regular is the existence of a trapezoidal matrix equivalent to $A$. If, moreover, $\leq$ is dense, then this condition is also sufficient.*

**Proof.** Suppose that $b = (b_1, \dots, b_n)^T$ is a vector from $B_n$ satisfying $|S(A,b)| = 1$. Obviously, there exists a permutation $\sigma \in P_n$ for which

$$b_{\sigma(1)} \leq b_{\sigma(2)} \leq \dots \leq b_{\sigma(n)} \tag{12}$$

and obviously

$$|S(A(\sigma, id), d)| = 1$$

after having denoted $(b_{\sigma(1)}, b_{\sigma(2)} \dots, b_{\sigma(n)})^T$ by $(d_1, d_2, \dots, d_n)^T$. It follows from Theorem 1 that for $C = (c_{ij}) = A(\sigma, \pi)$ the inequalities

$$c_{rr} > d_r > \sum_{j \in N j \neq r}{}^{\oplus} c_{ij} \otimes d_j \tag{13}$$

hold for some $\pi \in P_n$ and for all $r \in N$. We show that $C$ is trapezoidal. For this purpose it suffices (by (13)) to prove that

$$d_r > \sum_{i=1}^{r}{}^{\oplus} \sum_{j=i+1}^{n}{}^{\oplus} c_{ij} \tag{14}$$

for all $r \in N$. We will verify this by induction. For $r = 1$ inequality (14) follows from (13) since (12) means in fact that

$$d_1 \leq d_2 \leq \dots \leq d_n. \tag{15}$$

Suppose now that

$$d_{r-1} > \sum_{i=1}^{r-1}{}^{\oplus} \sum_{j=i+1}^{n}{}^{\oplus} c_{ij}.$$

Since $d_r \geq d_{r-1}$ it is sufficient to verify the inequality $d_r > c_{rj}$ for all $j \in \{r+1, \ldots, n\}$. But these inequalities follow immediately from (13) and (15).

To prove the converse implication suppose that $A \sim C$ where $C = c_{ij}$ is trapezoidal. It is sufficient (Lemma 6) to prove that $C$ is strongly regular. Denote the sum

$$\sum_{i=1}^{r}{}^{\oplus} \sum_{j=i+1}^{n}{}^{\oplus} c_{ij}$$

by $D_r$ for all $r \in N$. Thus,

$$D_1 \leq D_2 \leq \ldots \leq D_r < c_{rr} \tag{16}$$

for all $r \in N$. Let $b = (b_1, \ldots, b_n)^T \in B_n$ be an arbitrary vector satisfying the inequalities

$$\begin{aligned} & D_n < b_n < c_{nn} \\ \text{and} \quad & D_i < b_i < c_{ii} \otimes b_{i+1} \end{aligned} \tag{17}$$

for $i = n-1, n-2, \ldots, 1$ (whose existence follows from the assumption density of $\leq$ and from (16)). Clearly $b_1 < b_2 < \ldots < b_n$ implying the inequality

$$b_i > c_{ij} \otimes b_j \tag{18}$$

for $j < i$ immediately. But (18) holds also for $j > i$ because from (17) we have

$$b_i > D_i \geq c_{ij}.$$

Hence

$$c_{ii} > b_i \geq \sum_{j \in N j \neq i}{}^{\oplus} c_{ij} \otimes b_j$$

holds for all $i \in N$ and thus by Theorem 1 we conclude that $|S(C, b)| = 1$. ■

**Remark 1** The second part of the just finished proof was constructive and the relations (17) enable us to find a vector $b$ satisfying $|S(C, b)| = 1$ using $O(n^2)$ operations.

**Remark 2** One can easily see that a necessary and sufficient condition for a matrix $A = (a_{ij}) \in B(2, 2)$ to be equivalent to a trapezoidal matrix is

$$a_{11} \otimes a_{22} \neq a_{12} \otimes a_{21}$$

or, equivalently, $|\max(A)| = 1$. This result corresponds to the one in the group case proved in [2] but, unfortunately, it is not true in general in bottleneck algebras for matrices of order $n > 2$. To see this, consider the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 2 & 2 \\ 6 & 4 & 3 \\ 6 & 6 & 4 \end{pmatrix}$$

in $\mathcal{B}_1$.
Here we have $|\max(A)| = 2$ but $A$ is trapezoidal (and hence strongly regular by Theorem 2).

Nevertheless, it is possible to prove that $|\max(A)| = 1$ implies the strong regularity of $A$ but the proof is beyond the scope of this paper.

**Remark 3** The condition in Theorem 2 is in general not sufficient without the assumption of density. This is demonstrated by the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

in $\mathcal{B}_3$ which is trapezoidal but one can verify by an elementary use of Theorem 1 that A is not strongly regular.
The following lemma shows that the permanent of a trapezoidal matrix can be computed using only $O(n)$ operations.

**Lemma 7** *If $A = (a_{ij}) \in B(n, n)$ is trapezoidal then*

$$per(A) = \prod_{i \in N}^{\otimes} a_{ii}$$

*(and hence $id \in \max(A)$).*

**Proof.** Let $a_{rr} = \prod_{i \in N}^{\otimes} a_{ii}$ and take an arbitrary $\pi \in P_n$. We show that

$$a_{i,\pi(i)} \leq a_{rr} \tag{19}$$

for at least one $i \in N$. If $\pi(i) > i$ for some $i \in \{1, \ldots, r\}$ then $a_{i,\pi(i)} < a_{rr}$ since $A$ is trapezoidal. If $\pi(i) \leq i$ for all $i \in \{1, \ldots, r\}$ then, of course, $\pi(i) = i$ for all $i \in \{1, \ldots, r\}$ yielding equality in (19) for $i = r$. ■

## 5 AN ALGORITHM FOR CHECKING STRONG REGULARITY

Theorem 2 provides a possibility to check the strong regularity by testing all $(n!)^2$ pairs of rows and column permutations. But, this has, of course, no practical meaning except for very small values of $n$. Therefore we now develop an efficient method for solving this problem.

In what follows we denote for $A = (a_{ij}) \in B(n, n)$ by $p(A)$ the set

$$\{i \in N;\ (\exists k \in N)(\forall j \in N - \{k\}) \quad a_{ik} \geq per(A) > a_{ij}\}$$

and $A_i$ will be called a permanent row whenever $i \in p(A)$. The element $a_{ik}$ will be called the leading entry of $A_i$. Similarly, the $k - th$ column is said to be permanent if

$$a_{ik} \geq per(A)$$

for some $i \in p(A)$. Evidently, $A \sim C$ implies $|p(A)| = |p(C)|$ because, by permuting the rows, we only cause indices to change of permanent rows and a permutation of columns does not lead to any change of $p(A)$ at all.

**Theorem 4** *(a) If $A \in B(n, n)$ is strongly regular then $p(A) \neq \emptyset$.*

*(b) If $\leq$ is dense and $p(A) = N$ then $A$ is strongly regular.*

**Proof.** (a) Let $C = (c_{ij})$ be a trapezoidal matrix equivalent to $A$. It follows from Lemma 7 that for some $r \in N$

$$c_{11} \geq per(C) = c_{rr} > c_{1j}$$

for all $j \in N \setminus \{1\}$. Hence $1 \in p(C)$ and it remains to recall that $|p(A)| = |p(C)|$.

(b) If $r, s \in p(A)$, $r \neq s$ and $k, l \in N$ satisfy the relations

$$a_{rk} \geq per(A),\; a_{sl} \geq per(A)$$

then $k \neq l$ for, otherwise, $a_{r,\pi(r)} < per(A)$ or $a_{s,\pi(s)} < per(A)$ for arbitrary $\pi \in P_n$, yielding $w(A, \pi) < per(A)$ for every $\pi \in P_n$, a contradiction. Therefore, we can permute the columns of $A$ in such a way that in the obtained matrix $C = (c_{ij})$ the inequality

$$c_{ii} \geq per(C) > c_{ij} \tag{20}$$

holds for all $i, j \in N$, $i \neq j$. According to Theorem 2 it now suffices to show that C is equivalent to a trapezoidal matrix. But this follows from (20) since it is sufficient to simultaneously permute the rows and columns so that the diagonal entries form a non-decreasing sequence. ∎

In the following we denote for $A = a_{ij} \in B(n, n)$ the set

$$\{(i, j);\, a_{ij} < per(A)\}$$

by $P(A)$. If $P(A) \neq \emptyset$ then $b(A)$ denotes $\sum\limits_{(i,j)\in P(A)}^{\oplus} a_{ij}$ and thus $b(A) < per(A)$.

**Lemma 8** *If $A \sim C$ then*

*(a) $|P(A)| = |P(C)|$ and*

*(b) if, moreover, $P(A) \neq \emptyset$ then $b(A) = b(C)$.*

**Proof.** Trivial. ∎

**Lemma 9** *If $n > 1$ and $A \in B(n, n)$ is strongly regular then $P(A) \neq \emptyset$. Moreover, if $\leq$ is dense then there exists a vector $b = (b_1, \ldots, b_n)^T \in B_n$ such that $|S(A, b)| = 1$ and $b_i > b(A)$ for all $i \in N$.*

**Proof.** Let $C = (c_{ij})$ be a trapezoidal matrix equivalent to $A$. It follows from Lemma 7 and 8 that to prove $P(A) \neq \emptyset$ one only has to realize that

$$(1, j) \in P(C) \text{ for all } j \in \{2, 3, \ldots, n\} \neq \emptyset.$$

Take arbitrary $d = (d_1, \ldots, d_n)^T \in B_n$ defined by the formulas

$$D_n \oplus b(A) < d_n < c_{nn}$$

and

$$D_i \oplus b(A) < d_i < c_{ii} \otimes d_{i+1}$$

for all $i = n-1, n-2, \ldots, 1$ where $D_1, \ldots, D_n$ have the same meaning as in the proof of Theorem 2. The existence of $d_1, \ldots, d_n$ follows from (16), from

$$c_{ii} \geq per(A) > b(A)$$

(cf. Lemma 7) and from the assumption of density. The equality $|S(C,d)| = 1$ can now be verified in the same way as at the end of the proof of Theorem 2. If $A = C(\sigma, \pi)$ then $|S(A,b)| = 1$ where $b = (b_1, \ldots, b_n)^T = (d_{\pi(1)}, \ldots, d_{\pi(n)})^T$. ∎

**Theorem 5** *Let $\leq$ be dense. Suppose that $A = (a_{ij}) \in B(n,n)$ can be written blockwise in the form*

$$\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

*where $A_{11} \in B(r,r)$, $1 \leq r < n$ and*

$$per(A) > \sum_{i=1}^{r}{}^{\oplus} \sum_{j=i+1}^{n}{}^{\oplus} a_{ij}. \tag{21}$$

*Then $A$ is strongly regular if and only if $A_{22}$ is strongly regular.*

**Proof.** Let $A$ be strongly regular. Then by Theorem 1 and Lemma 9 there is a vector $b = (b_1, \ldots, b_n)^T$ and $\pi \in P_n$ satisfying the following conditions for all $i \in N$:

$$b_i > b(A) \tag{22}$$

$$a_{i,\pi(i)} > b_i > \sum_{j \in N j \neq i}{}^{\oplus} a_{i,\pi(j)} \otimes b_j. \tag{23}$$

Assumption (21) yields that

$$\sum_{i=1}^{r}{}^{\oplus} \sum_{j=i+1}^{n}{}^{\oplus} a_{ij} \leq b(A). \tag{24}$$

From (22), (23), (24) we get $\pi(i) \leq i$ for all $i \in R = \{1, 2, \ldots, r\}$. Thus $\pi$ is the identity on $R$ and

$$\pi' = \pi \upharpoonright (N \setminus R)$$

is a permutation of the set $N \setminus R$. But then (23) implies for all $i \in N \setminus R$

$$a_{i,\pi'(i)} = a_{i,\pi(i)} > b_i > \sum_{j \in N j \neq i}{}^{\oplus} a_{i,\pi(j)} \otimes b_j \geq$$
$$\sum_{j \in N \setminus R, j \neq i}{}^{\oplus} a_{i,\pi(j)} \otimes b_j = \sum_{j \in N \setminus R, j \neq i}{}^{\oplus} a_{i,\pi'(j)} \otimes b_j$$

Hence, by Theorem 1 $A_{22}$ is strongly regular.

Now let us suppose that $A_{22}$ is strongly regular. Since permuting the last $n-r$ rows and columns of $A$ does not change the validity of the assumptions of Theorem 4, without loss of generality, we may assume that $A_{22}$ is trapezoidal (Theorem 2), i.e.

$$a_{kk} > \sum_{i=r+1}^{k}{}^{\oplus} \sum_{j=i+1}^{n}{}^{\oplus} a_{ij} \tag{25}$$

for all $k \in N \setminus R$. It now suffices to show that $A$ is trapezoidal. If $\pi \in \max(A)$ then it follows from (21) that $\pi$ is identity on $R$. Thus, if $id \notin \max(A)$, i.e.

$$w(A, id) < w(A, \sigma)$$

for some $\sigma \in \max(A)$ then $\sigma \restriction R$ is the identity and $\sigma \restriction N \setminus R$ is a permutation $\sigma'$ satisfying

$$w(A_{22}, id) < w(A_{22}, \sigma')$$

which contradicts Lemma 7. Hence

$$a_{kk} \geq per(A) > \sum_{i=1}^{r}{}^{\oplus} \sum_{j=i+1}^{n}{}^{\oplus} a_{ij}$$

for all $k \in N$. By this (and (25)) the proof is completed. ■

Consequently, we are ready to formulate the algorithm for checking the strong regularity (SR) of a given matrix in the bottleneck algebra $(B, \oplus, \otimes, \leq)$ assuming that $\leq$ is dense. Note that every matrix in $B(1,1)$ is strongly regular.

**Algorithm**

*Input:* $C \in B(n,n)$;
*Output:* $D \in B(n,n)$, $D \sim C$, *D trapezoidal, or an indication that* $C$ *is not strongly regular(SR).*

($1^0$) $A := C$; $D := C$.

($2^0$) *If* $n = 1$ *then* $C$ *is SR, stop.*

($3^0$) *Compute per(A) and p(A); r:=* $|p(A)|$.

($4^0$) *If* $r = 0$ *then* $C$ *is not SR, stop.*

($5^0$) *Permute the rows and columns of* $A$ *in such a way that the permanent rows and columns will become the first* $r$ *rows and columns and the leading entries of the permanent rows are on the diagonal.*

($6^0$) *Perform also the corresponding permutation of the rows and columns of* $D$ *and denote the obtained matrix again by* $D = (d_{ij})$.
*If* $r = n$ *then stop* ($D$ *is trapezoidal and* $C$ *is SR).*

($7^0$)

$$\mathbf{A} := \begin{pmatrix} d_{r+1,r+1} & \cdots & d_{r+1,n} \\ \vdots & \ddots & \vdots \\ d_{n,r+1} & \cdots & d_{n,n} \end{pmatrix}; \quad n := n - r; \quad goto\ (2^0)$$

To illustrate the Algorithm consider the matrix $C$ in $\mathcal{B}_1$.

$$\mathbf{C} = \begin{pmatrix} 2 & 5 & 2 & 6 & 4 & 3 \\ 3 & 2 & 1 & 3 & 8 & 4 \\ 8 & 3 & 5 & 7 & 8 & 2 \\ 7 & 6 & 4 & 5 & 6 & 5 \\ 0 & 3 & 3 & 7 & 8 & 2 \\ 1 & 2 & 0 & 3 & 2 & 0 \end{pmatrix}$$

Using the Algorithm we will get successively

$$A = C,\ per(A) = 3,\ p(A) = \{6\};$$

$$\mathbf{A} = \begin{pmatrix} 2 & 5 & 2 & 4 & 3 \\ 3 & 2 & 1 & 8 & 4 \\ 8 & 3 & 5 & 8 & 2 \\ 7 & 6 & 4 & 6 & 5 \\ 0 & 3 & 3 & 8 & 2 \end{pmatrix}; \quad per(A) = 4,\ p(A) = \{5\}\ ;$$

$$\mathbf{A} = \begin{pmatrix} 2 & 5 & 2 & 3 \\ 3 & 2 & 1 & 4 \\ 8 & 3 & 5 & 2 \\ 7 & 6 & 4 & 5 \end{pmatrix}; \quad per(A) = 4,\ p(A) = \{1, 2\}\ ;$$

$$\mathbf{A} = \begin{pmatrix} 8 & 5 \\ 7 & 4 \end{pmatrix}; \quad per(A) = 5,\ p(A) = \{2\}\ ;$$

$\mathbf{A} = (5)$.
Hence $C$ is SR and the trapezoidal matrix equivalent to $C$ is

$$\mathbf{D} = \begin{pmatrix} 3 & 2 & 2 & 0 & 1 & 0 \\ 7 & 8 & 3 & 2 & 0 & 3 \\ 6 & 4 & 5 & 3 & 2 & 2 \\ 3 & 8 & 2 & 4 & 3 & 1 \\ 5 & 6 & 6 & 5 & 7 & 4 \\ 7 & 8 & 3 & 2 & 8 & 5 \end{pmatrix}$$

Notice that after replacing, for instance $c_{56} = 2$ by 4 we get a matrix which is not SR and the Algorithm would detect this in the second loop by finding $p(A) = \emptyset$.

## 6 CONCLUSION

We conclude with some remarks on the computional complexity. The value $per(A)$ can be determined using $O(n^{2.5})$ operations [1], the set $p(A)$ obviously by $O(n^2)$. The total number of operations in all other steps does not exceed $O(n^2)$. Hence, in a single loop $((2^0)$ - $(7^0))$ not more than $O(n^{2.5})$ operations are needed. Since in each loop we reduce the order of the considered matrix by at least 1 we get that the algorithm will terminate in the worst case after $O(n^{3.5})$ operations.

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