Max-linear programming

P.Butkovič* Abdulhadi Aminu†

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Abstract

Let $a \oplus b = \max(a, b)$ and $a \otimes b = a + b$ for $a, b \in \mathbb{R}$ and extend this pair of operations to matrices and vectors in the same way as in linear algebra. We introduce max-linear programs of the form $f^T \otimes x \to \min$ (or max) subject to $A \otimes x \oplus c = B \otimes x \oplus d$ and develop solution methods for both of them. We prove that these methods are pseudopolynomial if all entries are integer. This result is based on a published pseudopolynomial algorithm for solving the systems of the form $A \otimes x = B \otimes y$.

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1 Problem formulation

Consider the following *multi-machine interactive production process* (MMIPP): Products $P_1, ..., P_m$ are prepared using n machines (or processors), every machine contributing to the completion of each product by producing a partial product. It is assumed that every machine can work for all products simultaneously and that all these actions on a machine start as soon as the machine starts to work. Let a_{ij} be the duration of the work of the j^{th} machine needed to complete the partial product for P_i $(i = 1, ..., m; j = 1, ..., n)$. Let us denote by x_j the starting time of the j^{th} machine $(j = 1, ..., n)$. Then all partial products for P_i $(i = 1, ..., m)$ will be ready at time $\max(x_1 + a_{i1}, ..., x_n + a_{in})$. Now suppose that independently, k other machines prepare partial products for products $Q_1, ..., Q_m$ and the duration and starting times are b_{ij} and y_j, respectively. Then the *synchronisation problem* is to find starting times of all $n + k$ workshops so that each pair (P_i, Q_i) $(i = 1, ..., m)$ is completed at the same time. This task is equivalent to solving the system of equations

$$\max(x_1 + a_{i1}, ..., x_n + a_{in}) = \max(y_1 + b_{i1}, ..., y_k + b_{ik}) \quad (i = 1, ..., m).$$

*Corresponding author, School of Mathematics, University of Birmingham, Edgbaston, Birmingham B15 2TT, United Kingdom, p.butkovic@bham.ac.uk

†Department of Mathematical Sciences, Kano University of Science and Technology, Wudil, P.M.B 3244, Kano, Nigeria

It may also be required that P_i is not completed before a particular time c_i and similarly Q_i not before time d_i. Then the equations are

$$\max(x_1+a_{i1},...,x_n+a_{in},c_i)=\max(y_1+b_{i1},...,y_k+b_{ik},d_i) \quad (i=1,...,m). \quad (1)$$

If we denote $a\oplus b=\max(a,b)$ and $a\otimes b=a+b$ for $a,b\in\mathbb{R}$ then this system gets the form

$$\sum_{j=1,...,n}{}^{\oplus} a_{ij}\otimes x_j\oplus c_i=\sum_{j=1,...,k}{}^{\oplus} b_{ij}\otimes y_j \ \oplus d_i \quad (i=1,...,m). \quad (2)$$

Therefore (1) (and also (2)) is called a *two-sided system of max-linear equations* (or briefly a *two-sided max-linear system*, or just *max-linear system*).

Lemma 1.1 *(Cancellation Law) Let $v,w,a,b\in\mathbb{R}, a>b$. Then for any real x we have*

$$v\oplus a\otimes x=w\oplus b\otimes x \quad (3)$$

if and only if

$$v\oplus a\otimes x=w. \quad (4)$$

Proof. If x satisfies (3) then LHS $\geq a\otimes x>b\otimes x$. Hence RHS $=w$ and (4) follows. If (4) holds then $w\geq a\otimes x>b\otimes x$ and thus $w=w\oplus b\otimes x$. ■

Lemma 1.1 shows that in a two-sided max-linear system variables missing on one side of an equation can be artificially introduced using suitably taken small coefficients. We may therefore assume without loss of generality that (2) has the same variables on both sides, that is in the matrix-vector notation it has the form

$$A\otimes x\oplus c=B\otimes x\oplus d$$

where the pair of operations $(\oplus,\otimes)$ is extended to matrices and vectors in the same way as in linear algebra.

In applications it may be required that the starting times are optimised with respect to a given criterion. In this paper we consider the case when the objective function is also *max-linear*, that is

$$f(x)=f^T\otimes x=\max(f_1+x_1,...,f_n+x_n)$$

and it has to be either minimised or maximised. Thus the problems are

$$f^T\otimes x\longrightarrow \min \text{ or } \max$$

s.t.

$$A\otimes x\oplus c=B\otimes x\oplus d$$

Optimisation problems of this type will be called *max-linear programming problems* or, briefly, *max-linear programs (MLP).*

Systems of max-linear equations were investigated already in the first publications dealing with the algebraic structure called max-algebra (sometimes also

extremal algebra, path algebra or tropical algebra). In these publications, systems of equations with all variables on one side were considered [6], [12], [14], [3]. Other systems with a special structure were studied in the context of solving eigenvalue problems in the corresponding algebraic structures or synchronization in discrete event systems [2]. Using the $(\oplus, \otimes)$-notation, the studied systems had one of the following forms: $A \otimes x = b$, $A \otimes x = x$ or $A \otimes x = x \oplus b$, where A is a given matrix and b is a given vector.

General two-sided max-linear systems have also been studied [4], [8], [9], [13]. A general solution method was presented in [13], however, no complexity bound was given. In [8] a pseudopolynomial algorithm, called the Alternating Method, has been developed. In [4] it was shown that the solution set is generated by a finite number of vectors and an elimination method was suggested. A general iterative approach suggested in [9] assumes that finite upper and lower bounds for all variables are given. We make a substantial use of the Alternating Method for solving the two-sided max-linear systems in the present paper and derive a bisection method for the MLP that repeatedly checks solvability of systems of the form $A \otimes x = B \otimes x$. To our knowledge this problem was not studied before. We prove that the number of calls of a subroutine for checking the feasibility is polynomial when applied to max-linear programs with integer entries, yielding a pseudopolynomial computational complexity overall. Note that the problem of minimising a conventional linear function subject to max-linear constraints is known to be NP-complete (more information about this special case can be found at the end of this paper).

2 Max-algebraic prerequisites

Let $a \oplus b = \max(a, b)$ and $a \otimes b = a + b$ for $a, b \in \mathbb{R}$. If $a \in \mathbb{R}$ then the symbol a^{-1} stands in this paper for $-a$.

By *max-algebra* we understand the analogue of linear algebra developed for the pair of operations $(\oplus, \otimes)$, extended to matrices and vectors in the same way as in linear algebra. That is if $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$ are matrices of compatible sizes with entries from $\mathbb{R}$, we write $C = A \oplus B$ if $c_{ij} = a_{ij} \oplus b_{ij}$ for all i, j and $C = A \otimes B$ if $c_{ij} = \sum_k^{\oplus} a_{ik} \otimes b_{kj} = \max_k(a_{ik} + b_{kj})$ for all i, j. If $\alpha \in \mathbb{R}$ then $\alpha \otimes A = A \otimes \alpha = (\alpha \otimes a_{ij})$. The main advantage of using max-algebra is the possibility of dealing with a class of non-linear problems in a linear-like way. This is due to the fact that basic rules (commutative, associative and distributive laws) hold in max-algebra to the same extent as in linear algebra.

Max algebra has been studied by many authors and the reader is referred to [6], [7], [10], [2] or [3] for more information, see also [5], [12], [14]. A chapter in [11] provides an excellent state of the art overview of the field.

We will now summarize some standard properties that will be used later on. The following hold for $a, b, c \in \mathbb{R}$:

$$a \oplus b \geq a$$
$$a \geq b \Longrightarrow a \oplus c \geq b \oplus c$$

$$a \geq b \Longleftrightarrow a \otimes c \geq b \otimes c$$

For matrices (including vectors) A, B, C of compatible sizes over $\mathbb{R}$ and $a \in \mathbb{R}$ we have:

$$\begin{gathered} A \oplus B \geq A \\ A \geq B \Longrightarrow A \oplus C \geq B \oplus C \\ A \geq B \Longrightarrow A \otimes C \geq B \otimes C \\ A \geq B \Longrightarrow C \otimes A \geq C \otimes B \\ A \geq B \Longrightarrow c \otimes A \geq c \otimes B \\ (c \otimes A) \otimes B = A \otimes (c \otimes B) \end{gathered}$$

The next statement readily follows from the above mentioned relations.

Lemma 2.1 *Suppose $f \in \mathbb{R}^n$ and let $f(x) = f^T \otimes x$ be defined on $\mathbb{R}^n$. Then*

(a) $f(x)$ is max-linear that is $f(\lambda \otimes x \oplus \mu \otimes y) = \lambda \otimes f(x) \oplus \mu \otimes f(y)$ for every $x, y \in \mathbb{R}^n$ and $\lambda, \mu \in \mathbb{R}$.

(b) $f(x)$ is isotone that is $f(x) \leq f(y)$ for every $x, y \in \mathbb{R}^n, x \leq y$.

3 Max-linear programming problem and its basic properties

The aim of this paper is to develop methods for finding an $x \in \mathbb{R}^n$ that minimises [maximises] the function $f(x) = f^T \otimes x$ subject to

$$A \otimes x \oplus c = B \otimes x \oplus d \tag{5}$$

where $f = (f_1, ..., f_n)^T \in \mathbb{R}^n, c = (c_1, ..., c_m)^T, d = (d_1, ..., d_m)^T \in \mathbb{R}^m, A = (a_{ij}), B = (b_{ij}) \in \mathbb{R}^{m \times n}$ are given matrices and vectors. These problems will be denoted by $\text{MLP}^{\min}$ [$\text{MLP}^{\max}$] and we also denote everywhere $M = \{1, ..., m\}$, $N = \{1, ..., n\}$. Note that it is not possible to convert $\text{MLP}^{\min}$ to $\text{MLP}^{\max}$ or vice versa.

Any system of the form (5) is called a *non-homogenous max-linear system* and the set of solutions of this system will be denoted by S. The set of optimal solutions for $\text{MLP}^{\min}$ [$\text{MLP}^{\max}$] will be denoted by $S^{\min}$ [$S^{\max}$]. Any system of the form

$$E \otimes z = F \otimes z \tag{6}$$

is called a *homogenous max-linear system* and the solution set to this system will be denoted by S_h. In the next proposition we show that any non-homogenous max-linear system can easily be converted to a homogenous one. Here and elsewhere the symbol 0 will be used to denote both the real number zero and the zero vector of an appropriate dimension.

Proposition 3.1 *Let $E = (A|0)$ and $F = (B|0)$ be matrices arising from A and B respectively by adding a zero column. If $x \in S$ then $(x|0) \in S_h$ and conversely, if $z = (z_1, ..., z_{n+1})^T \in S_h$ then $z_{n+1}^{-1} \otimes (z_1, ..., z_n)^T \in S$.*

Proof. The statement follows straightforwardly from the definitions. ■

Given $\text{MLP}^{\min}$ [$\text{MLP}^{\max}$] we denote

$$K = \max\left\{|a_{ij}|, |b_{ij}|, |c_i|, |d_j|, |f_j|; i \in M, j \in N\right\}. \tag{7}$$

Theorem 3.1 *[8] Let $E = (e_{ij}), F = (f_{ij}) \in \mathbb{Z}^{m \times n}$ and K' be the greatest of the values $|e_{ij}|, |f_{ij}|, i \in M, j \in N$. There is an algorithm of computational complexity $O(mn(m+n)K')$ that finds an x satisfying (6) or decides that no such x exists.*

Proposition 3.1 and Theorem 3.1 show that the feasibility question for $\text{MLP}^{\max}$ and $\text{MLP}^{\min}$ can be solved in pseudopolynomial time. We will use this result to develop bisection methods for solving $\text{MLP}^{\min}$ and $\text{MLP}^{\max}$. We will prove that these methods need a polynomial number of feasibility checks if all entries are integer and hence are also of pseudopolynomial complexity.

The algorithm in [8] is an iterative procedure that starts with an arbitrary vector and then only uses the operations of $+, -, max$ and min applied to the starting vector and the entries of E, F. Hence using Proposition 3.1 we deduce:

Theorem 3.2 *If all entries in a homogenous max-linear system are integer and the system has a solution then this system has an integer solution. The same is true for non-homogenous max-linear systems.*

As a corollary to Lemma 1.1 we have:

Lemma 3.1 *Let $\alpha, \alpha' \in \mathbb{R}, \alpha' < \alpha$ and $f(x) = f^T \otimes x, f'(x) = f'^T \otimes x$ where*

$f'_j < f_j$ for every $j \in N$. Then the following holds for every $x \in \mathbb{R}$: $f(x) = \alpha$ if and only if $f(x) \oplus \alpha' = f'(x) \oplus \alpha$.

The following proposition shows that the problem of attainment of a value for a max-linear program can be converted to a feasibility question.

Proposition 3.2 *$f(x) = \alpha$ for some $x \in S$ if and only if the following non-homogenous max-linear system has a solution:*

$$\begin{aligned} A \otimes x \oplus c &= B \otimes x \oplus d \\ f(x) \oplus \alpha' &= f'(x) \oplus \alpha \end{aligned}$$

where $\alpha' < \alpha$ and $f'(x) = f'^T \otimes x$ where $f'_j < f_j$ for every $j \in N$.

Proof. The statement follows from Lemma 1.1 and Lemma 3.1. ■

Corollary 3.1 *If all entries in MLP$^{\max}$ or MLP$^{\min}$ are integer then an integer objective function value is attained by a real feasible solution if and only if it is attained by an integer feasible solution.*

Proof. It follows immediately from Theorem 3.2 and Proposition 3.2. ■

Corollary 3.2 *If all entries in MLP$^{\max}$ or MLP$^{\min}$ and α are integer then the decision problem whether $f(x) = \alpha$ for some $x \in S \cap \mathbb{Z}^n$ can be solved by using $O\left(mn\left(m+n\right)K'\right)$ operations where $K' = \max\left(K+1, |\alpha|\right)$.*

Proof. For α' and f'_j in Proposition 3.2 we can take $\alpha - 1$ and $f_j - 1$ respectively. Using Theorem 3.1 and Proposition 3.2 the computational complexity then is

$$O\left(\left(m+1\right)\left(n+1\right)\left(m+n+2\right)K'\right) = O\left(mn\left(m+n\right)K'\right).$$

■

A set $C \subseteq \mathbb{R}^n$ is said to be *max-convex* if $\lambda \otimes x \oplus \mu \otimes y \in C$ for every $x, y \in C, \lambda, \mu \in \mathbb{R}$ with $\lambda \oplus \mu = 0$.

Proposition 3.3 *S and S_h are max-convex.*

Proof. $A \otimes (\lambda \otimes x \oplus \mu \otimes y) \oplus c =$
$= A \otimes (\lambda \otimes x \oplus \mu \otimes y) \oplus \lambda \otimes c \oplus \mu \otimes c =$
$= \lambda \otimes (A \otimes x \oplus c) \oplus \mu \otimes (A \otimes y \oplus c) =$
$= \lambda \otimes (B \otimes x \oplus d) \oplus \mu \otimes (B \otimes y \oplus d) =$
$= B \otimes (\lambda \otimes x \oplus \mu \otimes y) \oplus \lambda \otimes d \oplus \mu \otimes d =$
$= B \otimes (\lambda \otimes x \oplus \mu \otimes y) \oplus d.$

Hence S is max-convex and S_h is max-convex for similar reasons. ■

Proposition 3.4 *If $x, y \in S, f(x) = \alpha < \beta = f(y)$ then for every $\gamma \in (\alpha, \beta)$ there is a $z \in S$ satisfying $f(z) = \gamma$.*

Proof. Let $\lambda = 0, \mu = \beta^{-1} \otimes \gamma, z = \lambda \otimes x \oplus \beta \otimes y$. Then $\lambda \oplus \mu = 0, z \in S$ by Proposition 3.3 and by Lemma 2.1 we have

$$f(z) = \lambda \otimes f(x) \oplus \mu \otimes f(y) = \alpha \oplus \beta^{-1} \otimes \gamma \otimes \beta = \gamma.$$

■

Before we develop solutions methods for solving the optimisation problems MLP$^{\min}$ and MLP$^{\max}$ we need to find and prove criteria for the existence of optimal solutions. For simplicity we denote $\inf_{x \in S} f(x)$ by $f^{\min}$, similarly $\sup_{x \in S} f(x)$ by $f^{\max}$.

We start with the lower bound. We may assume without loss of generality that in (5) we have $c \geq d$. Let $M^{>} = \{i \in M; c_i > d_i\}$. For $r \in M^{>}$ we denote

$$L_r = \min_{k \in N} f_k \otimes c_r \otimes b_{rk}^{-1}$$

and

$$L = \max_{r \in M^{>}} L_r.$$

As usual $\max \emptyset = -\infty$ by definition.

Lemma 3.2 *If $c \geq d$ then $f(x) \geq L$ for evey $x \in S$.*

Proof. If $M^{>} = \emptyset$ then the statement follows trivially since $L = -\infty$. Let $x \in S$ and $r \in M^{>}$. Then

$$(B \otimes x)_r \geq c_r$$

and so

$$x_k \geq c_r \otimes b_{rk}^{-1}$$

for some $k \in N$. Hence $f(x) \geq f_k \otimes x_k \geq f_k \otimes c_r \otimes b_{rk}^{-1} \geq L_r$ and the theorem statement follows. ■

Theorem 3.3 *$f^{\min} = -\infty$ if and only if $c = d$.*

Proof. If $c = d$ then $\alpha \otimes x \in S$ for any $x \in \mathbb{R}^n$ and every $\alpha \in \mathbb{R}$ small enough. Hence by letting $\alpha \longrightarrow -\infty$ we have $f(\alpha \otimes x) = \alpha \otimes f(x) \longrightarrow -\infty$.

If $c \neq d$ then without loss of generality $c \geq d$ and the statement now follows by Lemma 3.2 since $L > -\infty$. ■

Now we discuss the upper bound.

Lemma 3.3 *Let $c \geq d$. If $x \in S$ and $(A \otimes x)_i > c_i$ for all $i \in M$ then $x' = \alpha \otimes x \in S$ and $(A \otimes x')_i = c_i$ for some $i \in M$ where*

$$\alpha = \max_{i \in M} \left(c_i \otimes (A \otimes x)_i^{-1} \right). \tag{8}$$

Proof. Let $x \in S$. If

$$(A \otimes x)_i > c_i$$

for every $i \in M$ then $A \otimes x = B \otimes x$. For every $\alpha \in \mathbb{R}$ we also have

$$A \otimes (\alpha \otimes x) = B \otimes (\alpha \otimes x).$$

It follows from the choice of α that also

$$(A \otimes (\alpha \otimes x))_i = \alpha \otimes (A \otimes x)_i \geq c_i$$

for every $i \in M$ with equality for at least one $i \in M$. Hence $x' \in S$ and the Lemma follows. ■

Let us denote

$$U = \max_{r \in M} \max_{j \in N} f_j \otimes a_{rj}^{-1} \otimes c_r.$$

Lemma 3.4 *If $c \geq d$ then the following hold:*

(a) If $x \in S$ and $(A \otimes x)_r \leq c_r$ for some $r \in M$ then $f(x) \leq U$.

(b) If $A \otimes x = B \otimes x$ has no solution then $f(x) \leq U$ for every $x \in S$.

Proof. (a) Since

$$a_{rj} \otimes x_j \leq c_r$$

for all $j \in N$ we have

$$f(x) \leq \max_{j \in N} f_j \otimes a_{rj}^{-1} \otimes c_r \leq U.$$

(b) If $S = \emptyset$ then the statement holds trivially. Let $x \in S$. Then

$$(A \otimes x)_r \leq c_r$$

for some $r \in M$ since otherwise $A \otimes x = B \otimes x$, and the statement now follows from (a). ■

Theorem 3.4 *$f^{\max} = +\infty$ if and only if $A \otimes x = B \otimes x$ has a solution.*

Proof. We may assume without loss of generality that $c \geq d$. If $A \otimes x = B \otimes x$ has no solution then the statement follows from Lemma 3.4. If it has a solution, say z, then for all sufficiently big $\alpha \in \mathbb{R}$ we have

$$A \otimes (\alpha \otimes z) = B \otimes (\alpha \otimes z) \geq c \oplus d$$

and hence $\alpha \otimes z \in S$. The statement now follows by letting $\alpha \longrightarrow +\infty$. ■

We also need to show that the maximal [minimal] value is attained if $S \neq \emptyset$ and $f^{\max} < +\infty$ $\left[f^{\min} > -\infty\right]$. Due to continuity of f this will be proved by showing that both for minimisation and maximisation the set S can be reduced to a compact subset. To achieve this we denote for $j \in N$:

$$h_j = \min\left(\min_{r \in M} a_{rj}^{-1} \otimes c_j, \min_{r \in M} b_{rj}^{-1} \otimes d_j, f_j^{-1} \otimes L\right), \tag{9}$$

$$h'_j = \min\left(\min_{r \in M} a_{rj}^{-1} \otimes c_j, \min_{r \in M} b_{rj}^{-1} \otimes d_j\right) \tag{10}$$

and $h = (h_1, ..., h_n)^T, h' = (h'_1, ..., h'_n)^T$. Note that h is finite if and only if $f^{\min} > -\infty$.

Proposition 3.5 *For any $x \in S$ there is an $x' \in S$ such that $x' \geq h$ and $f(x) = f(x')$.*

Proof. Let $x \in S$. It is sufficient to set $x' = x \oplus h$ since if $x_j < h_j, j \in N$ then x_j is not active on any side of any equation or in the objective function and therefore changing x_j to h_j will not affect any of the equations or the objective function value. ■

Corollary 3.3 *If $f^{\min} > -\infty$ and $S \neq \emptyset$ then there is a compact set $\overline{S}$ such that*

$$f^{\min} = \min_{x \in \overline{S}} f(x).$$

Proof. Note that h is finite since $f^{\min} > -\infty$. Let $\tilde{x} \in S, \tilde{x} \geq h$, then

$$\overline{S} = S \cap \left\{ x \in \mathbb{R}^n; h_j \leq x_j \leq f_j^{-1} \otimes f(\tilde{x}), j \in N \right\}$$

is a compact subset of S and $\tilde{x} \in S$. If there was a $y \in S, f(y) < \min_{x \in \overline{S}} f(x) \leq f(\tilde{x})$ then by Proposition 3.5 there is a $y' \geq h, y' \in S, f(y') = f(y)$. Hence

$$f_j \otimes y'_j \leq f(y') = f(y) \leq f(\tilde{x})$$

for every $j \in N$ and thus $y' \in \overline{S}, f(y') < \min_{x \in \overline{S}} f(x)$, a contradiction. ■

Proposition 3.6 *For any $x \in S$ there is an $x' \in S$ such that $x' \geq h'$ and $f(x) \leq f(x')$.*

Proof. Let $x \in S$ and $j \in N$. It is sufficient to set $x' = x \oplus h'$ since if $x_j < h'_j$ then x_j is not active on any side of any equation and therefore changing x_j to h'_j does not violate any of the equations. The rest follows from isotonicity of $f(x)$. ■

Let $\overline{S}' = S \cap \left\{ x \in \mathbb{R}^n; h'_j \leq x_j \leq f_j^{-1} \otimes U, j \in N \right\}$.

Corollary 3.4 *If $f^{\max} < +\infty$ then*

$$f^{\max} = \max_{x \in \overline{S}'} f(x).$$

Proof. The statement follows immediately from Lemma 3.4, Theorem 3.4 and Proposition 3.6. ■

Corollary 3.5 *If $S \neq \emptyset$ and $f^{\min} > -\infty$ [$f^{\max} < +\infty$] then $S^{\min} \neq \emptyset$ [$S^{\max} \neq \emptyset$].*

It follows from Lemma 3.2 that $f^{\max} > L$. However this information is not useful if $c = d$ since then $L = -\infty$. Since we will need a lower bound for $f^{\max}$ even when $c = d$ we define $L' = f(h')$ and formulate the following.

Corollary 3.6 *If $x \in S$ then $x' = x \oplus h'$ satisfies $f(x') \geq L'$ and thus $f^{\max} \geq L'$.*

4 The Algorithms

It follows from Proposition 3.1 and Theorem 3.1 that in pseudopolynomial time either a feasible solution to (5) can be found or it can be decided that no such solution exists. Due to Theorems 3.3 and 3.4 we can also recognise the cases when the objective function is unbounded. We may therefore assume that a feasible solution exists, the objective function is bounded (from below or above depending on whether we wish to minimise or maximise) and hence an optimal solution exists (Corollary 3.5). If $x^0 \in S$ is found then using the scaling (if necessary) proposed in Lemma 3.3 or Corollary 3.6 we find (another)

x^0 satisfying $L \leq f(x^0) \leq U$ or $L' \leq f(x^0) \leq U$ (see Lemmas 3.2 and 3.4). The use of the bisection method applied to either $(L, f(x^0))$ or $(f(x^0), U)$ for finding a minimiser or maximiser of $f(x)$ is then justified by Proposition 3.4. The algorithms are based on the fact that (see Proposition 3.2) checking the existence of an $x \in S$ satisfying $f(x) = \alpha$ for a given $\alpha \in \mathbb{R}$ can be converted to a feasibility problem. They stop when the interval of uncertainty is shorter than a given precision $\varepsilon > 0$.

Algorithm 4.1 *MAXLINMIN (Max-Linear Minimisation)*

Input: $f = (f_1, ..., f_n)^T \in \mathbb{R}^n, c = (c_1, ..., c_m)^T, d = (d_1, ..., d_m)^T \in \mathbb{R}^m, c \geq d, c \neq d, A = (a_{ij}), B = (b_{ij}) \in \mathbb{R}^{m \times n}, \varepsilon > 0$.
Output: $x \in S$ such that $f(x) - f^{\min} \leq \varepsilon$

1. If $L = f(x)$ for some $x \in S$ then stop ($f^{\min} = L$).
2. Find an $x^0 \in S$. If $\left(A \otimes x^0\right)_i > c_i$ for all $i \in M$ then scale x^0 by α defined in (8).
3. $L(0) := L, U(0) := f(x^0), r := 0$
4. $\alpha := \frac{1}{2}(L(r) + U(r))$
5. Check whether $f(x) = \alpha$ is satisfied by some $x \in S$ and in the positive case find one.

   If yes then $U(r+1) := \alpha, L(r+1) := L(r)$

   If not then $U(r+1) := U(r), L(r+1) := \alpha$
6. $r := r + 1$
7. If $U(r) - L(r) \leq \varepsilon$ then stop else go to 4.

Theorem 4.1 *Algorithm MAXLINMIN is correct and the number of iterations before termination is*

$$O\left(\log_2 \frac{U - L}{\varepsilon}\right).$$

Proof. Correctness follows from Proposition 3.4 and Lemma 3.2. Since $c \neq d$ we have at the end of step 2: $f(x^0) \geq L > -\infty$ (Lemma 3.2) and $U(0) := f(x^0) \leq U$ by Lemma 3.4. Thus the number of iterations is $O\left(\log_2 \frac{U-L}{\varepsilon}\right)$ since after every iteration the interval of uncertainty is halved. ■

Algorithm 4.2 *MAXLINMAX (Max-Linear Maximisation)*

Input: $f = (f_1, ..., f_n)^T \in \mathbb{R}^n, c = (c_1, ..., c_m)^T, d = (d_1, ..., d_m)^T \in \mathbb{R}^m$, $A = (a_{ij}), B = (b_{ij}) \in \mathbb{R}^{m \times n}, \varepsilon > 0$.
Output: $x \in S$ such that $f^{\max} - f(x) \leq \varepsilon$ or an indication that $f^{\max} = +\infty$.

1. If $U = f(x)$ for some $x \in S$ then stop ($f^{\max} = U$)

2. Check whether $A \otimes x = B \otimes x$ has a solution. If yes, stop ($f^{\max} = +\infty$)
3. Find an $x^0 \in S$ and set $x^0 := x^0 \oplus h'$ where h' is as defined in (10)
4. $L(0) := f(x^0), U(0) := U, r := 0$
5. $\alpha := \frac{1}{2}(L(r) + U(r))$
6. Check whether $f(x) = \alpha$ is satisfied by some $x \in S$ and in the positive case find one.

   If yes then $U(r+1) := U(r), L(r+1) := \alpha$

   If not then $U(r+1) := \alpha, L(r+1) := L(r)$
7. $r := r + 1$
8. If $U(r) - L(r) \leq \varepsilon$ then stop else go to 5.

Theorem 4.2 *Algorithm MAXLINMAX is correct and the number of iterations before termination is*

$$O\left(\log_2 \frac{U - L'}{\varepsilon}\right).$$

Proof. Correctness follows from Proposition 3.4 and Lemma 3.4. By Lemma 3.4 and Corollary 3.6 $U \geq f(x^0) \geq L'$ and thus the number of iterations is $O\left(\log_2 \frac{U-L'}{\varepsilon}\right)$ since after every iteration the interval of uncertainty is halved. ■

5 The integer case

The algorithms of the previous section may immediately be applied to $\text{MLP}^{\min}$ or $\text{MLP}^{\max}$ when all input data are integer. However, we show that in such a case $f^{\min}$ and $f^{\max}$ are integers and therefore the algorithms find an *exact* solution once the interval of uncertainty is of length one since then either $L(r)$ or $U(r)$ is the optimal value. Note that L and U are now integers and we will show how integrality of $L(r)$ and $U(r)$ can be maintained during the run of the algorithms. This implies that the algorithms will find exact optimal solutions in a finite number of steps and we will prove that their computational complexity is pseudopolynomial.

Theorem 5.1 *If A, B, c, d, f are integer, $S \neq \emptyset$ and $f^{\min} > -\infty$ then $f^{\min} \in \mathbb{Z}$ (and therefore $S^{\min} \cap \mathbb{Z}^n \neq \emptyset$).*

Proof. Suppose $f^{\min} \notin \mathbb{Z}$ and let $z = (z_1, ..., z_n)^T \in S^{\min}$. We assume again without loss of generality that $c \geq d$. For any $x \in \mathbb{R}^n$ denote

$$F(x) = \{j \in N; f_j \otimes x_j = f(x)\}.$$

Hence we have

$$z_j \notin \mathbb{Z} \text{ for every } j \in F(z). \tag{11}$$

We will now show that all $z_j, j \in F(z)$ can be reduced while maintaining feasibility which will be a contradiction with optimality of z. To prove this we develop a special procedure called the Reduction Algorithm. Let us first denote for $x \in \mathbb{R}^n$:

$$Q(x) = \{i \in M; (A \otimes x)_i > c_i\}$$

and for $i \in M$ and $x \in \mathbb{R}^n$

$$\begin{aligned} T_i(x) &= \{j \in N; a_{ij} \otimes x_j = (A \otimes x)_i\}, \\ R_i(x) &= \{j \in N; b_{ij} \otimes x_j = (B \otimes x)_i\}. \end{aligned}$$

Since all entries are integer $a_{ij} \otimes z_j = c_i$ cannot hold for any $i \in M$ and $j \in F(z)$ and if $a_{ij} \otimes z_j < c_i$ for every $i \in M$ and $j \in F(z)$ then all $z_j, j \in F(x)$ could be reduced without violating any equation which contradicts the optimality of z. Hence $Q(z) \neq \emptyset$.

Reduction Algorithm

1. $P(z) := F(z)$
2. $E_1 := \{i \in Q(z); T_i(z) \subseteq P(z) \,\&\, R_i(z) \nsubseteq P(z)\}$
   $E_2 := \{i \in Q(z); T_i(z) \nsubseteq P(z) \,\&\, R_i(z) \subseteq P(z)\}$
3. If $E_1 \cup E_2 = \emptyset$ then $P(z)$ is the set of indices of variables to be reduced, STOP
4. $P(z) := P(z) \cup \bigcup_{i \in E_1} (R_i(z) \setminus P(z)) \cup \bigcup_{i \in E_2} (T_i(z) \setminus P(z))$
5. Go to 2.

Claim: Reduction Algorithm terminates after a finite number of steps and at termination

$$z_j \notin \mathbb{Z} \text{ for } j \in P(z). \tag{12}$$

Proof of Claim: Finiteness follows from the fact that the set $P(z)$ strictly increases in size at every iteration and $P(z) \subseteq N$. For the remaining part of the claim it is sufficient to prove the following for any iteration of this algorithm: If (12) holds at Step 2 then it is also true at Step 5. The statement then follows from the fact that (12) is true when Step 2 is reached for the first time due to Step 1 and assumption (11). Consider therefore a fixed iteration at the beginning of which (12) holds. Suppose without loss of generality that $E_1 \cup E_2 \neq \emptyset$ and take any $i \in E_1$. Hence $z_j \notin \mathbb{Z}$ for $j \in T_i(z)$, thus $(A \otimes z)_i \notin \mathbb{Z}$. But $i \in Q(z)$, implying $(B \otimes z)_i = (A \otimes z)_i$ and so $(B \otimes z)_i \notin \mathbb{Z}$ too. Since b_{ij} are also integer this yields that $z_j \notin \mathbb{Z}$ for $j \in R_i(z)$. Therefore $z_j \notin \mathbb{Z}$ for $j \in \bigcup_{i \in E_1} (R_i(z) \setminus P(z))$. Similarly, $z_j \notin \mathbb{Z}$ for $j \in \bigcup_{i \in E_2} (T_i(z) \setminus P(z))$ and the claim follows.

If $i \in M \setminus Q(z)$ then by integrality of the entries both $a_{ij} \otimes z_j < c_i$ and $b_{ij} \otimes z_j < c_i$ for $j \in P(z)$. We conclude that all z_j for $j \in P(z)$ can be reduced without violating any of the equations, a contradiction with optimality of z.

Hence $f^{\min} \in \mathbb{Z}$. The existence of an integer optimal solution now follows from Corollary 3.1. ■

Theorem 5.2 *If A, B, c, d, f are integer, $S \neq \emptyset$ and $f^{\max} < +\infty$ then $f^{\max} \in \mathbb{Z}$ (and therefore $S^{\max} \cap \mathbb{Z}^n \neq \emptyset$).*

Proof. (Sketch) The proof follows the ideas of the proof of Theorem 5.1. We suppose $c \geq d$, $f^{\max} \notin \mathbb{Z}$ and let $z = (z_1, ..., z_n)^T \in S^{\max}$. We take one fixed $j \in F(z)$ (hence $z_j \notin \mathbb{Z}$) and show that it is possible to increase z_j without violating equality in any of the equations. Similarly as in the proof of Theorem 5.1 it is shown that the increase of z_j only forces the non-integer components of z to increase. Due to integrality of all entries it is not possible that the equality in an equation is achieved by both integer and non-integer components of z. At the same time an equality of the form $(A \otimes z)_i = c_i$ (if any) cannot be attained by non-integer components, thus $a_{ij} \otimes z_j < c_i$ and $b_{ij} \otimes z_j < c_i$ whenever $z_j \notin \mathbb{Z}$ and hence there is always scope for an increase of $z_j \notin \mathbb{Z}$. The rest of the argument is the same as in the proof of Theorem 5.1. ■

Integer modifications of the algorithms are now straightforward since L, L' and U are also integer: we only need to ensure that the algorithms start from an integer vector (see Theorem 3.2) and that the integrality of both ends of the intervals of uncertainty is maintained, for instance by taking one of the integer parts of the middle of the interval.

We start with the minimisation. Note that

$$L, L', U \in [-3K, 3K] \tag{13}$$

where K is defined by (7).

Algorithm 5.1 *INTEGER MAXLINMIN (Integer Max-Linear Minimisation)*

Input: $f = (f_1, ..., f_n)^T \in \mathbb{Z}^n, c = (c_1, ..., c_m)^T, d = (d_1, ..., d_m)^T \in \mathbb{Z}^m, c \geq d, c \neq d$,
$A = (a_{ij}), B = (b_{ij}) \in \mathbb{Z}^{m \times n}$.
Output: $x \in S^{\min} \cap \mathbb{Z}^n$

1. If $L = f(x)$ for some $x \in S \cap \mathbb{Z}^n$ then stop ($f^{\min} = L$).
2. Find $x^0 \in S \cap \mathbb{Z}^n$. If $\left(A \otimes x^0\right)_i > c_i$ for all $i \in M$ then scale x^0 by α defined in (8).
3. $L(0) := L, U(0) := f(x^0), r := 0$
4. $\alpha := \left\lceil \frac{1}{2} (L(r) + U(r)) \right\rceil$

5. Check whether $f(x) = \alpha$ is satisfied by some $x \in S \cap \mathbb{Z}^n$ and in the positive case find one.

   If x exists then $U(r+1) := \alpha, L(r+1) := L(r)$.

   If it does not then $U(r+1) := U(r), L(r+1) := \alpha$.

6. $r := r + 1$

7. If $U(r) - L(r) = 1$ then stop $(U(r) = f^{\min})$ else go to 4.

Theorem 5.3 *Algorithm INTEGER MAXLINMIN is correct and terminates after using* $O(mn(m+n)K\log K)$ *operations.*

Proof. Correctness follows from the correctness of MAXLINMIN and from Theorem 5.1. For computational complexity first note that the number of iterations is $O(\log(U-L)) \leq O(\log 6K) = O(\log K)$. The computationally prevailing part of the algorithm is the checking whether $f(x) = \alpha$ for some $x \in S \cap \mathbb{Z}^n$ when α is given. By Corollary 3.2 this can be done using $O(mn(m+n)K')$ operations where $K' = \max(K+1, |\alpha|)$. Since $\alpha \in [L, U]$, using (13) we have $K' = O(K)$. Hence the computational complexity of checking whether $f(x) = \alpha$ for some $x \in S \cap \mathbb{Z}^n$ is $O(mn(m+n)K)$ and the statement follows. ■

Algorithm 5.2 *INTEGER MAXLINMAX (Integer Max-Linear Maximisation)*

Input: $f = (f_1, ..., f_n)^T \in \mathbb{Z}^n, c = (c_1, ..., c_m)^T, d = (d_1, ..., d_m)^T \in \mathbb{Z}^m$, $A = (a_{ij}), B = (b_{ij}) \in \mathbb{Z}^{m \times n}$.
Output: $x \in S^{\max} \cap \mathbb{Z}^n$ or an indication that $f^{\max} = +\infty$.

1. If $U = f(x)$ for some $x \in S \cap \mathbb{Z}^n$then stop $(f^{\max} = U)$.

2. Check whether $A \otimes x = B \otimes x$ has a solution. If yes, stop $(f^{\max} = +\infty)$

3. Find an $x^0 \in S \cap \mathbb{Z}^n$ and set $x^0 := x^0 \oplus h'$ where is as defined in (10)

4. $L(0) := f(x^0), U(0) := U, r := 0$

5. $\alpha := \left\lfloor \frac{1}{2}(L(r) + U(r)) \right\rfloor$

6. Check whether $f(x) = \alpha$ is satisfied by some $x \in S \cap \mathbb{Z}^n$ and in the positive case find one.

   If x exists then $U(r+1) := U(r), L(r+1) := \alpha$.

   If not then $U(r+1) := \alpha, L(r+1) := L(r)$.

7. $r := r + 1$

8. If $U(r) - L(r) = 1$ then stop $(L(r) = f^{\max})$ else go to 5.

Theorem 5.4 *Algorithm INTEGER MAXLINMAX is correct and terminates after using* $O(mn(m+n)K\log K)$ *operations.*

Proof. Correctness follows from the correctness of MAXLINMAX and from Theorem 5.2. The computational complexity part follows the lines of the proof of Theorem 5.3 after replacing L by L'. ■

6 An example

Let us consider the max-linear program (minimisation) in which

$$f = (3, 1, 4, -2, 0)^T,$$

$$A = \begin{pmatrix} 17 & 12 & 9 & 4 & 9 \\ 9 & 0 & 7 & 9 & 10 \\ 19 & 4 & 3 & 7 & 11 \end{pmatrix},$$

$$B = \begin{pmatrix} 2 & 11 & 8 & 10 & 9 \\ 11 & 0 & 12 & 20 & 3 \\ 2 & 13 & 5 & 16 & 4 \end{pmatrix},$$

$$c = \begin{pmatrix} 12 \\ 15 \\ 13 \end{pmatrix}, d = \begin{pmatrix} 12 \\ 12 \\ 3 \end{pmatrix}$$

and the starting vector is

$$x^0 = (-6, 0, 3, -5, 2)^T.$$

Clearly, $f(x^0) = 7, M^> = \{2, 3\}$ and the lower bound is

$$\begin{aligned} L &= \max_{r \in M^>} \min_{k \in N} f_k \otimes c_r \otimes b_{rk}^{-1} \\ &= \max\left(\min\left(7, 16, 7, -7, 12\right), \min\left(14, 1, 12, -5, 9\right)\right) = -5. \end{aligned}$$

We now make a record of the run of INTEGER MAXLINMIN for this problem.

Iteration 1: Check whether $L = -5$ is attained by $f(x)$ for some $x \in S$ by solving the system

$$\begin{pmatrix} 17 & 12 & 9 & 4 & 9 & 12 \\ 9 & 0 & 7 & 9 & 10 & 15 \\ 19 & 4 & 3 & 7 & 11 & 13 \\ 3 & 1 & 4 & -2 & 0 & -6 \end{pmatrix} \otimes w = \begin{pmatrix} 2 & 11 & 8 & 10 & 9 & 12 \\ 11 & 0 & 12 & 20 & 3 & 12 \\ 2 & 13 & 5 & 16 & 4 & 3 \\ 2 & 0 & 3 & -3 & -1 & -5 \end{pmatrix} \otimes w.$$

There is no solution, hence $L(0) := -5, U(0) := 7, r := 0, \alpha := 1$.

Check whether $f(x) = 1$ is satisfied by some $x \in S$ by solving

$$\begin{pmatrix} 17 & 12 & 9 & 4 & 9 & 12 \\ 9 & 0 & 7 & 9 & 10 & 15 \\ 19 & 4 & 3 & 7 & 11 & 13 \\ 3 & 1 & 4 & -2 & 0 & 0 \end{pmatrix} \otimes w = \begin{pmatrix} 2 & 11 & 8 & 10 & 9 & 12 \\ 11 & 0 & 12 & 20 & 3 & 12 \\ 2 & 13 & 5 & 16 & 4 & 3 \\ 2 & 0 & 3 & -3 & -1 & 1 \end{pmatrix} \otimes w.$$

There is a solution $x = (-6, 0, -3, -5, 1)^T$. Hence $U(1) := 1, L(1) := -5, r := 1, U(1) - L(1) > 1$.

Iteration 2: Check whether $f(x) = -2$ is satisfied by some $x \in S$ by solving

$$\begin{pmatrix} 17 & 12 & 9 & 4 & 9 & 12 \\ 9 & 0 & 7 & 9 & 10 & 15 \\ 19 & 4 & 3 & 7 & 11 & 13 \\ 3 & 1 & 4 & -2 & 0 & -3 \end{pmatrix} \otimes w = \begin{pmatrix} 2 & 11 & 8 & 10 & 9 & 12 \\ 11 & 0 & 12 & 20 & 3 & 12 \\ 2 & 13 & 5 & 16 & 4 & 3 \\ 2 & 0 & 3 & -3 & -1 & -2 \end{pmatrix} \otimes w.$$

There is no solution. Hence $U(2) := 1, L(2) := -2, r := 2, U(2) - L(2) > 1$.

Iteration 3: Check whether $f(x) = 0$ is satisfied by some $x \in S$ by solving

$$\begin{pmatrix} 17 & 12 & 9 & 4 & 9 & 12 \\ 9 & 0 & 7 & 9 & 10 & 15 \\ 19 & 4 & 3 & 7 & 11 & 13 \\ 3 & 1 & 4 & -2 & 0 & -1 \end{pmatrix} \otimes w = \begin{pmatrix} 2 & 11 & 8 & 10 & 9 & 12 \\ 11 & 0 & 12 & 20 & 3 & 12 \\ 2 & 13 & 5 & 16 & 4 & 3 \\ 2 & 0 & 3 & -3 & -1 & 0 \end{pmatrix} \otimes w.$$

There is no solution. Hence $U(3) := 1, L(3) := 0, U(1) - L(1) = 1$, stop, $f^{\min} = 1$, an optimal solution is $x = (-6, 0, -3, -5, 1)^T$.

7 An easily solvable special case

One-sided systems of max-linear equations have been studied for many years and they are very well understood [6], [7], [14], [3]. Note that a one-sided system is a special case of a two-sided system (5) where $a_{ij} > b_{ij}$ and $c_i < d_i$ for every i and j. Not surprisingly, max-linear programs with one-sided constraints have also been known for some time [14]. Here we present this special case for the sake of completeness.

Let us consider one-sided systems of the form

$$A \otimes x = b \tag{14}$$

where $A = (a_{ij}) \in \mathbb{R}^{m \times n}$ and $b = (b_1, ..., b_m)^T \in \mathbb{R}^m$. These systems can be solved more easily than their linear-algebraic counterparts. One of the methods follows from the next theorem in which $S = \{x \in \mathbb{R}^n; A \otimes x = b\}$.

Theorem 7.1 *Let $\overline{x} = (\overline{x}_1, ..., \overline{x}_n)^T$ where $\overline{x}_j = \min_{i \in M} b_i \otimes a_{ij}^{-1}$ for $j \in N$. Then*

(a) $x \leq \overline{x}$ for every $x \in S$ and

(b) $x \in S$ if and only if $x \leq \overline{x}$ and

$$\bigcup_{j: x_j = \overline{x}_j} M_j = M$$

where for $j \in N$

$$M_j = \left\{ i \in M; \overline{x}_j = b_i \otimes a_{ij}^{-1} \right\}.$$

Proof. Can be found in standard texts on max-algebra [6], [10], [14]. ■

Suppose that $f = (f_1, ..., f_n)^T \in \mathbb{R}^n$ is given. The task of minimising [maximising] $f(x) = f^T \otimes x$ subject to (14) will be denoted by $\mathrm{MLP}_1^{\min}$ [$\mathrm{MLP}_1^{\max}$]. The sets of optimal solutions will be denoted $S_1^{\min}$ and $S_1^{\max}$ respectively. It follows from Theorem 7.1 and from isotonicity of $f(x)$ that $\overline{x} \in S_1^{\max}$. We now present a simple algorithm which solves $\mathrm{MLP}_1^{\min}$.

Algorithm 7.1 *ONEMAXLINMIN (One-sided max-linear minimisation)*
Input: $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$ *and* $c \in \mathbb{R}^n$.
Output: $x \in S_1^{\min}$

1. *Find* $\overline{x}$ *and* $M_j, j \in N$
2. *Sort* $(f_j \otimes \overline{x}_j; j \in N)$, *without loss of generality let*
$$f_1 \otimes \overline{x}_1 \leq f_2 \otimes \overline{x}_2 \leq ... \leq f_n \otimes \overline{x}_n$$
3. $J := \{1\}, r = 1$
4. *If*
$$\bigcup_{j \in J} M_j = M$$
then stop ($x_j = \overline{x}_j$ *for* $j \in J$ *and* x_j *small enough for* $j \notin J$)
5. $r := r + 1, J := J \cup \{r\}$
6. *Go to 4.*

Theorem 7.2 *Algorithm ONEMAXLINMIN is correct and its computational complexity is* $O(mn^2)$.

Proof. Correctness is obvious and computational complexity follows from the fact that the loop 4.-6. is repeated at most n times and each run is $O(mn)$. Step 1. is $O(mn)$ and Step 2. is $O(n \log n)$. ■

Note that the problem of minimising a conventional linear function subject to one-sided max-linear constraints is *NP*-complete, since the classical minimum set covering problem can be formulated as a special case of this problem with matrix A over $\{0, -1\}, b = 0$ and objective function $x_1 + ... + x_n \longrightarrow \min$.

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