On integer eigenvectors and subeigenvectors in the max-plus algebra

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Abstract

Let $a \oplus b = \max(a, b)$ and $a \otimes b = a + b$ for $a, b \in \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$ and extend these operations to matrices and vectors as in conventional algebra. We study the problems of existence and description of integer subeigenvectors (P1) and eigenvectors (P2) of a given square matrix, that is integer solutions to $Ax \leq \lambda x$ and $Ax = \lambda x$. It is proved that P1 can be solved as easily as the corresponding question without the integrality requirement (that is in polynomial time).

An algorithm is presented for finding an integer point in the max-column space of a rectangular matrix or deciding that no such vector exists. We use this algorithm to solve P2 for any matrix over $\overline{\mathbb{R}}$. The algorithm is shown to be pseudopolynomial for finite matrices which implies that P2 can be solved in pseudopolynomial time for any irreducible matrix. We also discuss classes of matrices for which P2 can be solved in polynomial time.

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1. Introduction

This paper deals with the task of finding integer solutions to max-linear systems. For $a, b \in \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$ we define $a \oplus b = \max(a, b)$, $a \otimes b = a + b$ and extend the pair $(\oplus, \otimes)$ to matrices and vectors in the same way as in linear

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algebra, that is (assuming compatibility of sizes)

$$
\begin{aligned}
(\alpha \otimes A)_{ij} &= \alpha \otimes a_{ij}, \\
(A \oplus B)_{ij} &= a_{ij} \oplus b_{ij}, \\
(A \otimes B)_{ij} &= \bigoplus_k a_{ik} \otimes b_{kj}.
\end{aligned}
$$

All multiplications in this paper are in max-algebra and we will usually omit the $\otimes$ symbol. Note that α^{-1} stands for $-\alpha$, and we will use ϵ to denote $-\infty$ as well as any vector or matrix whose every entry is $-\infty$. A vector/matrix whose every entry belongs to $\mathbb{R}$ is called *finite*. If a matrix has no ϵ rows (columns) then it is called *row* (*column*) $\mathbb{R}$*-astic* and it is called *doubly* $\mathbb{R}$*-astic* if it is both row and column $\mathbb{R}$-astic. Note that the vector Ax is sometimes called a *max combination* of the columns of A. The following observation is easily seen.

Lemma 1.1. *If $A \in \overline{\mathbb{R}}^{m\times n}$ is row $\mathbb{R}$-astic and $x \in \mathbb{R}^n$ then Ax is finite.*

For $a \in \mathbb{R}$ the *fractional part* of a is $fr(a) := a - \lfloor a \rfloor$. For a matrix $A \in \overline{\mathbb{R}}^{m\times n}$ we use $\lfloor A \rfloor$ ($\lceil A \rceil$) to denote the matrix with (i,j) entry equal to $\lfloor a_{ij} \rfloor$ ($\lceil a_{ij} \rceil$) and similarly for vectors.

The problems of finding solutions to

$$Ax \le b \tag{1.1}$$

$$Ax = b \tag{1.2}$$

$$Ax = \lambda x \tag{1.3}$$

$$Ax \le \lambda x \tag{1.4}$$

are well known [3] and can be solved in low-order polynomial time. However, the question of finding integer solutions to these problems has, to our knowledge, not been studied yet. Integer solutions to (1.1) and (1.2) can easily be found and the aim of this paper is to discuss existence criteria and solution methods for (1.3) and (1.4) with integrality constraints. As usual, a vector $x \neq \epsilon$ satisfying (1.3)/(1.4) will be called an *eigenvector/subeigenvector* of A with respect to *eigenvalue* λ.

Max-algebraic systems of equations and inequalities and also the eigenproblem have been used to model a range of practical problems from job-shop scheduling [5], railway scheduling [7] to cellular protein production [2]. Solutions to (1.1)-(1.2) typically represent starting times of processes that have to meet specified delivery times. Solutions to (1.3) guarantee a stable run of certain systems, for instance a multiprocessor interactive system [5]. Since the time restrictions are usually expresssed in discrete terms (for instance minutes, hours or days), it may be necessary to find integer rather than real solutions to (1.1)-(1.4).

In Section 2 we summarise the existing theory necessary for the presentation of our results. In Section 3 we show that the question of existance of integer subeigenvectors can be answered in polynomial time and we give an efficient description of all such vectors. This is then used to determine a class of matrices for which the integer eigenproblem can be solved efficiently. In Section 4 we propose a solution method for finding integer points in the column space of a matrix. It will follow that integer solutions to $Ax = \lambda x$ can be found in pseudopolynomial time when A is irreducible. In Section 5 we present additional special cases of (1.3) which are solvable in polynomial time.

2. Preliminaries

We will use the following standard notation. For positive integers m, n we denote $M = \{1, ..., m\}$ and $N = \{1, ..., n\}$. If $A = (a_{ij}) \in \overline{\mathbb{R}}^{n\times n}$ then $A^{\#} = -A^T$, $\lambda(A)$ denotes the *maximum cycle mean*, that is,

$$\lambda(A) = \max\left\{\frac{a_{i_1 i_2} + ... + a_{i_k i_1}}{k} : (i_1, ..., i_k) \text{ is a cycle}, k = 1, ..., n\right\}$$

where $\max(\emptyset) = \epsilon$ by definition.

It is easily seen that $\lambda(\alpha \otimes A) = \alpha \otimes \lambda(A)$ and in particular $\lambda(\lambda(A)^{-1}A) = 0$ if $\lambda(A) > \epsilon$. The matrix $(\lambda(A)^{-1}A)$ will be denoted A_λ. If $\lambda(A) = 0$ then we say that A is *definite*. If moreover $a_{ii} = 0$ for all $i \in N$ then A is called *strongly definite*.

The *identity matrix* $I \in \overline{\mathbb{R}}^{n\times n}$ is the matrix with diagonal entries equal to zero and off diagonal entries equal to ϵ. A matrix is called *diagonal* if its diagonal entries are finite and its off diagonal entries are ϵ. A matrix Q is called a *generalised permutation matrix* if it can be obtained from a diagonal matrix by permuting the rows and/or columns. Generalised permutation matrices are the only invertible matrices in max-algebra [3], [5].

For definite matrices we define

$$A^+ = A \oplus A^2 \oplus ... \oplus A^n$$

and

$$A^* = I \oplus A \oplus ... \oplus A^{n-1}.$$

Further if A is definite at least one column in A^+ is the same as the corresponding column in A^* and we define $\tilde{A}$ to be the matrix consisting of columns identical in A^+ and A^*. The matrices $\tilde{B}$, B^+ where $B = A_\lambda$ will be denoted $\tilde{A}_\lambda$ and A_λ^+ respectively.

By D_A we mean the digraph (N, E) where $E = \{(i, j) : a_{ij} > \epsilon\}$. A is called *irreducible* if D_A is strongly connected (that is, if there is an $i - j$ path in D_A for any i and j).

If $A \in \overline{\mathbb{R}}^{n\times n}$ is interpreted as a matrix of direct-distances in D_A then A^k (where k is a positive integer) is the matrix of the lengths of longest paths with k arcs. Following this observation it is not difficult to deduce:

Lemma 2.1. *[3] Let $A \in \overline{\mathbb{R}}^{m\times n}$ and $\lambda(A) > \epsilon$.*

(a) $\tilde{A}_\lambda$ is column $\mathbb{R}$-astic.

(b) If A is irreducible then A_λ^+, and hence also $\tilde{A}_\lambda$, are finite.

If $a, b \in \overline{\overline{\mathbb{R}}} = \overline{\mathbb{R}} \cup \{+\infty\}$ then we define $a \oplus' b = \min(a, b)$ and $a \otimes' b = a + b$ if at least one of a, b is finite, $(-\infty) \otimes (+\infty) = (+\infty) \otimes (-\infty) = -\infty$ and $(-\infty) \otimes' (+\infty) = (+\infty) \otimes' (-\infty) = +\infty$.

Recall that our aim is to discuss integer solutions to (1.1)-(1.4). Note that if $Ax \leq b$, $x \in \mathbb{Z}^n$ and $b_i = \epsilon$ then the i^{th} row of A is ϵ. In such a case the i^{th} inequality is redundant and can be removed. We may therefore assume without loss of generality that b is finite when dealing with integer solutions to (1.1) and (1.2). Further we only summarise here the existing theory of finite eigenvectors and subeigenvectors. A full description of all solutions to (1.1)-(1.4) can be found e.g. in [3].

If $A \in \overline{\mathbb{R}}^{m \times n}$ and $b \in \mathbb{R}^m$ then for all $j \in N$ define

$$M_j(A, b) = \{k \in M : a_{kj} \otimes b_k^{-1} = \max_i a_{ij} \otimes b_i^{-1}\}.$$

We use P_n to denote the set of permutations on N. For $A \in \overline{\mathbb{R}}^{n \times n}$ the *max-algebraic permanent* is given by

$$maper(A) = \bigoplus_{\pi \in P_n} \bigotimes_{i \in N} a_{i,\pi(i)}.$$

For a given $\pi \in P_n$ its *weight* with respect to A is

$$w(\pi, A) = \bigotimes_{i \in N} a_{i,\pi(i)}$$

and the set of permutations whose weight is maximum is

$$ap(A) = \{\pi \in P_n : w(\pi, A) = maper(A)\}.$$

Propositions 2.2-2.6 below are standard results.

Proposition 2.2. *[3] If $A \in \overline{\mathbb{R}}^{m \times n}$ and $x, y \in \overline{\mathbb{R}}^n$ then*

$$x \leq y \Rightarrow Ax \leq Ay.$$

Proposition 2.3. *[3], [4], [5] Let $A \in \overline{\mathbb{R}}^{m \times n}$, $b \in \mathbb{R}^m$ and $\bar{x} = A^{\#} \otimes' b$.*
(a) $Ax \leq b \Leftrightarrow x \leq \bar{x}$
(b) $Ax = b \Leftrightarrow x \leq \bar{x}$ and

$$\bigcup_{j : x_j = \bar{x}_j} M_j(A, b) = M.$$

By Proposition 2.2 we have

Corollary 2.4. *Let $A \in \overline{\mathbb{R}}^{m \times n}$, $b \in \mathbb{R}^m$ and $\bar{x} = A^{\#} \otimes' b$.*
(a) $\bar{x}$ is always a solution to $Ax \leq b$
(b) $Ax = b$ has a solution $\Leftrightarrow \bar{x}$ is a solution $\Leftrightarrow A \otimes (A^{\#} \otimes' b) = b$.

It is known [3] that if $\lambda(A) = \epsilon$ then A has no finite eigenvectors unless $A = \epsilon$. We may therefore assume without loss of generality that $\lambda(A) > \epsilon$ when discussing integer eigenvectors.

For $A \in \overline{\mathbb{R}}^{n\times n}$ and $\lambda \in \mathbb{R}$ we denote

$$V(A, \lambda) = \{x \in \mathbb{R}^n : Ax = \lambda x\}$$

and

$$V^*(A, \lambda) = \{x \in \mathbb{R}^n : Ax \leq \lambda x\}.$$

Proposition 2.5. *[5] Let $A \in \overline{\mathbb{R}}^{n\times n}$, $\lambda(A) > \epsilon$. Then $V(A, \lambda) \neq \emptyset$ if and only if $\lambda = \lambda(A)$ and $\tilde{A}_\lambda$ is row $\mathbb{R}$-astic (and hence doubly $\mathbb{R}$-astic).*

If $V(A, \lambda(A)) \neq \emptyset$ then

$$V(A, \lambda(A)) = \{\tilde{A}_\lambda u : u \in \mathbb{R}^k\}$$

where $\tilde{A}_\lambda$ is $n \times k$.

Proposition 2.6. *[3] Let $A \in \overline{\mathbb{R}}^{n\times n}$, $A \neq \epsilon$. Then $V^*(A, \lambda) \neq \emptyset$ if and only if $\lambda \geq \lambda(A)$, $\lambda > \epsilon$.*

If $V^*(A, \lambda) \neq \emptyset$ then

$$V^*(A, \lambda) = \{(\lambda^{-1}A)^* u : u \in \mathbb{R}^n\}.$$

If $\lambda = 0$ and A is integer then $(\lambda^{-1}A)^*$ is integer and hence we deduce:

Corollary 2.7. *If $A \in \overline{\mathbb{Z}}^{n\times n}$ then $Ax \leq x$ has a finite solution if and only if it has an integer solution.*

$A \in \overline{\mathbb{R}}^{n\times n}$ is called *increasing* if $a_{ii} \geq 0$ for all $i \in N$. Since $(Ax)_i \geq a_{ii}x_i$ we immediately see that A is increasing if and only if $Ax \geq x$ for all $x \in \overline{\mathbb{R}}^n$. It follows from the definition of a definite matrix that $a_{ii} \leq 0$ for all $i \in N$. Therefore a matrix is strongly definite if and only if it is definite and increasing. It is easily seen [3] that all diagonal entries of all powers of a strongly definite matrix are zero and thus in this case

$$A^+ = A^* = \tilde{A}_\lambda.$$

Hence we have

Proposition 2.8. *If A is strongly definite then $V(A, 0) = V^*(A, 0)$.*

3. Integer subeigenvectors and eigenvectors

Proposition 2.3(a) provides an immediate answer to the task of finding integer solutions to (1.1), namely all integer vectors not exceeding $A^\# \otimes' b$. Integer solutions to (1.2) can also be straightforwardly deduced from Proposition 2.3(b) and we summarise this in the next result.

Proposition 3.1. *Let $A \in \overline{\mathbb{R}}^{m\times n}$, $b \in \mathbb{R}^m$ and $\bar{x} = A^{\#} \otimes' b$.*

(a) An integer solution to $Ax \le b$ exists if and only if $\bar{x}$ is finite. If an integer solution exists then all integer solutions can be described as the integer vectors x satisfying $x \le \bar{x}$.

(b) An integer solution to $Ax = b$ exists if and only if

$$\bigcup_{j:\bar{x}_j \in \mathbb{Z}} M_j(A,b) = M.$$

If an integer solution exists then all integer solutions can be described as the integer vectors x satisfying $x \le \bar{x}$ with

$$\bigcup_{j:x_j=\bar{x}_j} M_j(A,b) = M.$$

Proposition 2.6 enables us to deduce an answer to integer solubility of (1.4). For $A \in \overline{\mathbb{R}}^{n\times n}$ we define

$$IV(A,\lambda) = V(A,\lambda) \cap \mathbb{Z}^n$$

and

$$IV^*(A,\lambda) = V^*(A,\lambda) \cap \mathbb{Z}^n.$$

Theorem 3.2. *Let $A \in \overline{\mathbb{R}}^{n\times n}$, $\lambda \in \mathbb{R}$.*

(i) $IV^*(A,\lambda) \neq \emptyset$ if and only if

$$\lambda(\lceil \lambda^{-1} A \rceil) \le 0.$$

(ii) If $IV^*(A,\lambda) \neq \emptyset$ then

$$IV^*(A,\lambda) = \{\lceil \lambda^{-1} A \rceil^* z : z \in \mathbb{Z}^n\}.$$

Proof. For both (i) and (ii) we will need the following. Assume that $x \in IV^*(A,\lambda)$. Using the fact that $x_i \in \mathbb{Z}$ for every i we get the equivalences:

$$\begin{aligned}
&Ax \le \lambda x \\
\Leftrightarrow &(\lambda^{-1}A)x \le x \\
\Leftrightarrow &(\forall i,j \in N)\ x_i - x_j \ge \lambda^{-1} a_{ij} \\
\Leftrightarrow &(\forall i,j \in N)\ x_i - x_j \ge \lceil \lambda^{-1} a_{ij} \rceil \\
\Leftrightarrow &\lceil \lambda^{-1} A \rceil x \le x.
\end{aligned}$$

Thus integer subeigenvectors of A with respect to λ are exactly the integer subeigenvectors of $\lceil \lambda^{-1} A \rceil \in \overline{\mathbb{Z}}^{n\times n}$ with respect to 0.

(i) Now from Proposition 2.6 we see that a finite subeigenvector of $\lceil \lambda^{-1} A \rceil$ with respect to $\lambda = 0$ exists if and only if $\lambda(\lceil \lambda^{-1} A \rceil) \le 0$.

Further $\lceil \lambda^{-1} A \rceil$ is integer so by Corollary 2.7 we have that a finite subeigenvector exists if and only if an integer subeigenvector exists.

(ii) If a finite subeigenvector exists then, again from Proposition 2.6, we know that

$$V^*(\lceil\lambda^{-1}A\rceil, 0) = \{\lceil\lambda^{-1}A\rceil^* u : u \in \mathbb{R}\}.$$

But $\lceil\lambda^{-1}A\rceil$ and therefore $\lceil\lambda^{-1}A\rceil^*$ are integer matrices, meaning that we can describe all integer subeigenvectors by taking max combinations of the columns of $\lceil\lambda^{-1}A\rceil^*$ with integer coefficients.

Observe that it is possible to obtain an integer vector from a max combination of the integer columns of the matrix with real coefficients but only if the real coefficients correspond to inactive columns. However any integer vectors obtained in this way can also be obtained by using integer coefficients, for example by taking the lower integer part of the coefficients, and thus it is sufficient to only take integer coefficients. □

It follows from Proposition 2.5 that $IV(A,\lambda) \neq \emptyset$ only if $\lambda = \lambda(A)$. We will therefore also denote $IV(A, \lambda(A))$ by $IV(A)$. Theorem 3.2 and Proposition 2.8 allow us to present a solution to the problem of integer eigenvectors for strongly definite matrices. Since $\lambda(\cdot)$ is monotone on $\overline{\mathbb{R}}^{n\times n}$ we have that for strongly definite matrices A the inequality $\lambda(\lceil A\rceil) \leq 0$ is equivalent to $\lambda(\lceil A\rceil) = 0$. Hence we have:

Corollary 3.3. *Let $A \in \overline{\mathbb{R}}^{n\times n}$ be strongly definite.*

(i) $IV(A) \neq \emptyset$ if and only if

$$\lambda(\lceil A\rceil) = 0.$$

(ii) If $IV(A) \neq \emptyset$ then

$$IV(A) = \{\lceil A\rceil^* z : z \in \mathbb{Z}^n\}.$$

Unfortunately no simple answer for the question of integer eigenvectors seems to exist in general. However Proposition 2.5 shows that it would be solved by finding a criterion for existence and a method for finding an integer point in a finitely generated subspace (namely the column space of the doubly $\mathbb{R}$-astic matrix $\tilde{A}_\lambda$). In Section 4 we present an algorithm for finding such a point. The algorithm is pseudopolynomial for finite matrices which, in light of Lemma 2.1, solves the question of integer eigenvectors for any irreducible matrix. In Section 5 we describe a number of polynomially solvable cases.

Before we finish this section we observe that the problem of integer eigenvectors can easily be solved for martices over $\overline{\mathbb{Z}}$:

Proposition 3.4. *Let $A \in \overline{\mathbb{Z}}^{n\times n}$. Then A has an integer eigenvector if and only if $\lambda(A) \in \mathbb{Z}$ and $\tilde{A}_\lambda$ is row $\mathbb{R}$-astic.*

Proof. First assume that $x \in IV(A)$. From Proposition 2.5 we know the only eigenvalue corresponding to x is $\lambda(A)$. Then $Ax = \lambda(A)x$ where the product on the left hand side is integer. To ensure that the right hand side is also integer we clearly need $\lambda(A) \in \mathbb{Z}$. Further any integer eigenvector is finite and so $\tilde{A}_\lambda$ is row $\mathbb{R}$-astic by Proposition 2.5.

Now assume that $\lambda(A) \in \mathbb{Z}$ and $\tilde{A}_\lambda$ is row $\mathbb{R}$-astic. Then $A_\lambda \in \overline{\mathbb{Z}}^{n\times n}$ thus all entries of A_λ^+, A_λ^* and $\tilde{A}_\lambda$ belong to $\overline{\mathbb{Z}}$. Again from Proposition 2.5 we know that all finite eigenvectors are described by max combinations of the columns of $\tilde{A}_\lambda$. Thus we can pick integer coefficients to obtain an integer eigenvector of A by Lemma 1.1. $\square$

Corollary 3.5. *Let $A \in \overline{\mathbb{Z}}^{n\times n}$ be irreducible. A has an integer eigenvector if and only if $\lambda(A) \in \mathbb{Z}$.*

We cannot assume that this result holds for a general matrix $A \in \overline{\mathbb{R}}^{n\times n}$ as the following examples show.

Example 3.1. *$A \in \mathbb{R}^{n\times n}$ has an integer eigenvector $\not\Rightarrow \lambda(A) \in \mathbb{Z}$.*

$$A = \begin{pmatrix} 1.1 & 1.1 \\ 1.1 & 1.1 \end{pmatrix}.$$

Let $x = (1,1)^T \in \mathbb{Z}^n$. Then $x \in IV(A, 1.1)$ but $\lambda(A) = 1.1 \notin \mathbb{Z}$.

Example 3.2. *$A \in \overline{\mathbb{R}}^{n\times n}$ with $\lambda(A) \in \mathbb{Z} \not\Rightarrow A$ has an integer eigenvector.*

$$A = \begin{pmatrix} 2.9 & 3.5 \\ 2.5 & 2.7 \end{pmatrix}.$$

Then $\lambda(A) = 3 \in \mathbb{Z}$ but Ax is clearly not integer for any integer vector x.

Further, a matrix does not have to be integer to have an integer eigenvalue or eigenvector, and integer matrices need not have integer eigenvectors.

Example 3.3. *$A \in \mathbb{Z}^{n\times n} \not\Rightarrow A$ has an integer eigenvector and an integer eigenvalue.*

$$A = \begin{pmatrix} -1 & 2 \\ 3 & -1 \end{pmatrix}.$$

Then $\lambda(A) = \frac{5}{2} \notin \mathbb{Z}$. By Proposition 3.5 A cannot have an integer eigenvector.

Example 3.4. *A has an integer eigenvector and an integer eigenvalue $\not\Rightarrow A \in \mathbb{Z}^{n\times n}$.*

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0.2 \end{pmatrix} \notin \mathbb{Z}^{n\times n}.$$

Then $A(1,1)^T = 1(1,1)^T$ and thus A has an integer eigenvector and an integer eigenvalue.

In the above counterexample we see that the matrix A has a large number of integer entries, so the question arises whether a real matrix with no integer entries can have both integer eigenvectors and eigenvalues.

Proposition 3.6. *Let $A \in \overline{\mathbb{R}}^{n\times n}$ be a matrix such that it has an integer eigenvector corresponding to an integer eigenvalue, then A has an integer entry in every row.*

Proof. We know that the only eigenvalue corresponding to integer eigenvectors is $\lambda(A)$ and hence by assumption $\lambda(A) \in \mathbb{Z}$. Now let $x \in IV(A)$. Then $Ax = \lambda(A)x$ where the right hand side is integer and hence $(\forall i \in N)\ \max(a_{ij}+x_j) \in \mathbb{Z}$ which implies that for every $i \in N$ there exists an index j for which $a_{ij} \in \mathbb{Z}$. $\square$

4. Integer points in the column space

In this section we are concerned with the question of whether, for a given matrix $A \in \overline{\mathbb{R}}^{m\times n}$, there exists an integer vector z in the column space of A, which we will call the *image* of A. We denote

$$Im(A) = \{y \in \overline{\mathbb{R}}^m : \exists x \in \overline{\mathbb{R}}^n \text{ with } Ax = y\}.$$

and

$$IIm(A) = \{z \in \mathbb{Z}^m : \exists x \in \overline{\mathbb{R}}^n \text{ with } Ax = z\}.$$

Observe that if $A \in \overline{\mathbb{R}}^{m\times n}$ has an ϵ row then $IIm(A) = \emptyset$, and if A has an ϵ column then $IIm(A)$=$IIm(A')$ where A' is obtained from A by removing the ϵ column. Hence it is sufficient to only consider doubly $\mathbb{R}$-astic matrices for the rest of this section.

We propose the following algorithm:

Algorithm: INT-IMAGE

Input: $A \in \overline{\mathbb{R}}^{m\times n}$ doubly $\mathbb{R}$-astic, any starting vector $x^{(0)} \in \mathbb{Z}^m$.
Output: A vector $x \in IIm(A)$ or indication that no such vector exists.
(1) $r := 1$.
(2) $z := A^{\#} \otimes' x^{(r-1)}, y := A \otimes z$.
(3) If $y \in \mathbb{Z}^m$ STOP: $y \in IIm(A)$.
(4) $x_i^{(r)} := \lfloor y_i \rfloor$ for all $i \in M$.
(5) If $x_i^{(r)} < x_i^{(0)}$ for all $i \in M$ STOP: No integer image.
(6) $r := r+1$. Go to (2).

Observe that all vectors produced by Algorithm INT-IMAGE are finite due to Lemma 1.1 and the fact that $A^{\#} \otimes' u$ is finite if u is finite since $A^{\#}$ is doubly $\mathbb{R}$-astic.

Theorem 4.1. *The doubly $\mathbb{R}$-astic input matrix $A \in \overline{\mathbb{R}}^{m\times n}$ has an integer image if and only if the sequence $\{x^{(r)}\}_{r=0,1,\ldots}$ produced by Algorithm INT-IMAGE finitely converges.*

To prove this theorem on the correctness of the algorithm we first prove a number of claims and we will also need the following two results. The first follows from Corollary 2.4(b) and the second from Proposition 2.2.

Lemma 4.2. *[5] Assume that $u \in \mathbb{R}^m$ is in the image of $A \in \overline{\mathbb{R}}^{m\times n}$. Then*

$$A \otimes (A^{\#} \otimes' u) = u$$

Lemma 4.3. *[5] Let $A \in \overline{\mathbb{R}}^{m\times n}$, $x, y \in \overline{\mathbb{R}}^m$. If $x \geq y$ then*

$$A \otimes (A^{\#} \otimes' x) \geq A \otimes (A^{\#} \otimes' y).$$

Claim 4.4. *The sequence $\{x^{(r)}\}_{r=0,1,\ldots}$ is nonincreasing.*

Proof. Note that for each $x^{(r)}$ the algorithm attempts to solve $Av = x^{(r)}$ by finding $z = \bar{v} = A^{\#} \otimes' x^{(r)}$ which by Corollary 2.4 satisfies $Az \leq x^{(r)}$. If we have equality then the algorithm halts, otherwise the algorithm calculates $x^{(r+1)} = \lfloor Az \rfloor \leq Az \leq x^{(r)}$. □

Claim 4.5. *If A has an integer image then the sequence $\{x^{(r)}\}_{r=0,1,\ldots}$ is bounded below by a vector in $IIm(A)$.*

Proof. Assume $u \in IIm(A)$. Then also $\gamma \otimes u \in IIm(A)$ for all $\gamma \in \mathbb{Z}$. Pick γ small enough so that $\gamma \otimes u \leq x^{(0)}$.

Now assume that $x^{(r)} \geq v$ for some $v \in IIm(A)$. Then, using Lemmas 4.2 and 4.3, we have

$$x^{(r+1)} = \lfloor A \otimes (A^{\#} \otimes' x^{(r)}) \rfloor \geq \lfloor A \otimes (A^{\#} \otimes' v) \rfloor = \lfloor v \rfloor = v$$

and thus our claim holds by induction. □

Claim 4.6. *If $x_i^{(r)} < x_i^{(0)}$ for some r and all i then A has no integer image.*

Proof. If $u \in IIm(A)$ then by Claims 4.4 and 4.5 the sequence $\{x^{(r)}\}_{r=0,1,\ldots}$ is nonincreasing and bounded below. But further, from the proof of Claim 4.5 we can see that we can choose $\gamma \in \mathbb{Z}$ such that:

(i) $\gamma \otimes u \in IIm(A)$,

(ii) $\gamma \otimes u \leq x^{(r)}$ for all r, and

(iii) there exists i such that $(\gamma \otimes u)_i = x_i^{(0)}$.

So we have that $x_i^{(0)} = (\gamma \otimes u)_i \leq x_i^{(r)} \leq x_i^{(0)}$. This implies that the i^{th} component of every $x^{(r)}$ is the same, and so there is never an iteration where all components of $x^{(r)}$ properly decrease. □

Proof of Theorem 4.1. If the matrix has an integer image then the above results imply that $\{x^{(r)}\}_{r=0,1,\ldots}$ is nonincreasing and bounded below by some integer image z of A. Clearly this implies that the sequence $\{x^{(r)}\}_{r=0,1,\ldots}$ will converge. Further, since it is a sequence of integer vectors, at each step at least one component must decrease in value by at least one until, at the latest, it reaches the corresponding value of z, and thus the convergence must be finite.

If instead the sequence finitely converges then there exists an s such that for all $r \geq s$, $x^{(r)} = x^{(r+1)}$. It follows that $y = A \otimes (A^{\#} \otimes' x^{(s)}) \in \mathbb{Z}^m$. To see this assume not, then there exists a component i of y which is not an integer, and thus $y_i < x_i^{(s)}$. But then $x_i^{(s+1)} = \lfloor y_i \rfloor < x_i^{(s)}$ which is a contradiction.

Thus $y \in IIm(A)$. □

It should be observed that Algorithm INT-IMAGE will always terminate in a finite number of steps. But for finite matrices we can give an explicit bound. In order to analyse the performance of Algorithm INT-IMAGE for finite matrices we will use a pseudonorm on $\mathbb{R}^n$. For a vector $x \in \mathbb{R}^n$ we define

$$\Delta(x) = \max_{j\in N} x_j - \min_{j\in N} x_j.$$

Lemma 4.7. *[6] For vectors $x, y \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ the following hold:*

$$(i)\Delta(x \oplus y) \leq \Delta(x) \oplus \Delta(y) \text{ and}$$
$$(ii)\Delta(\alpha \otimes x) = \Delta(x).$$

Proposition 4.8. *Let $y \in \mathbb{R}^m$ be a vector in the image of $A \in \mathbb{R}^{m\times n}$. Then*

$$\Delta(y) \leq \bigoplus_{j=1}^{n} \Delta(A_j).$$

PROOF. Since y is in the image of A there exists a vector $x \in \mathbb{R}^n$ such that $y = Ax$. Then, using Lemma 4.7, we have that

$$y = \bigoplus_{j\in N} x_j A_j$$
$$\Rightarrow \Delta(y) = \Delta(\bigoplus_{j\in N} x_j A_j) \leq \bigoplus_{j\in N} \Delta(x_j A_j) = \bigoplus_{j=1}^{n} \Delta(A_j).$$

□

Proposition 4.9. *Let $x^{(r)}$, with $r \geq 1$, be a vector calculated in the run of Algorithm INT-IMAGE. Then $\Delta(x^{(r)}) < \bigoplus_{j=1}^{n} \Delta(A_j) + 1$.*

PROOF. We know that $x^{(r)} = \lfloor y \rfloor$ where $y \in Im(A)$. So by Proposition 4.8 we have

$$\Delta(y) \leq \bigoplus_{j=1}^{n} \Delta(A_j).$$

To complete the proof it remains to show that $\Delta(x^{(r)}) < \Delta(y) + 1$. But this is true since

$$\Delta(x^{(r)}) - \Delta(y) = \max_{j=1,\ldots,n} \lfloor y_j \rfloor - \min_{j=1,\ldots,n} \lfloor y_j \rfloor - \max_{j=1,\ldots,n} y_j + \min_{j=1,\ldots,n} y_j < 1.$$

□

We can now prove a bound on the runtime of Algorithm INT-IMAGE for finite input matrices.

Theorem 4.10. *For $A \in \mathbb{R}^{m\times n}$ and starting vector $x^{(0)} \in \mathbb{Z}^m$ Algorithm INT-IMAGE will terminate after at most*

$$D = (m-1)\left(2\bigoplus_{j=1}^{n}\Delta(A_j)+1\right)+1$$

iterations.

PROOF. First suppose that A has an integer image. It follows from Claim 4.6 that there exists an index, k say, such that the algorithm will find an integer image y of A satisfying $y_k = x_k^{(r)}$ for all r.

Let $C = \bigoplus_{j=1}^{n}\Delta(A_j)$. By Proposition 4.8 we know that $\Delta(y) \leq C$. Thus, for all i, $|y_i - y_k| \leq C$. Similarly, using Proposition 4.9 we know that, for all i, $|x_i^{(1)} - x_k^{(1)}| < C+1$. But then, since $y_k = x_k^{(1)}$ we get that

$$x_i^{(1)} - y_i < 2C+1.$$

Now in every iteration where an integer image is not found we have that there exists at least one index $i \neq k$ such that $x_i^{(r)} - x_i^{(r+1)} \geq 1$. This is since if no change occurred then we would have found an integer image.

There are at most $m-1$ components of $x^{(1)}$ that will decrease in the run of the algorithm and none will decrease by more than $2C+1$, further in every iteration at least one of these components decreases by at least 1. Thus the maximum number of iteration needed for the algorithm to get from $x^{(1)}$ to y is

$$(m-1)(2C+1)$$

and we need to add one iteration to get from $x^{(0)}$ to $x^{(1)}$.

Now, if the input matrix has no integer image and after D iterations the sequence $\{x^{(r)}\}_{r=0,1,\ldots}$ has not stabilised then there would have been an iteration where the k^{th} component decreased, and so the algorithm would have halted and concluded that A has no integer image. □

Remark 1. Each iteration requires $\mathcal{O}(mn)$ operations and so by Theorem 4.10 INT-IMAGE is a pseudopolynomial algorithm requiring $\mathcal{O}(Cm^2n)$ operations if applied to finite matrices.

Remark 2. Since $|(\tilde{A}_\lambda)_{ij}| \leq n \max|a_{ij}|$ Algorithm INT-IMAGE can be used to determine whether $IV(A) \neq \emptyset$ for irreducible matrices in pseudopolynomial time.

Example 4.1. *The algorithm INT-IMAGE is not a polynomial algorithm. This can be seen by considering the matrix*

$$A = \begin{pmatrix} 12.5 & 7.3-k & 16.9 \\ 1.8 & 7.3 & -7.2 \\ -2.6 & 0.1 & 0.9 \end{pmatrix}$$

and starting vector $x^{(0)} = (-k, 0, 0)^T$*. For any* $k \geq 0$ *the algorithm first computes* $x^{(1)} = (-k, 0, -8)^T$ *and then in each subsequent iteration either the second entry of* $x^{(r)}$ *decreases by* 1 *or the third entry of* $x^{(r)}$ *decreases by* 1 *until the algorithm reaches the vector* $(-k,\ -k-9,\ -k-16)^T \in IIm(A)$*. So the number of iterations is equal to* $1 + |-k-9| + |-k-8| + 1 = 2k + 19$*.*

In the case that $m = 2$ however it can be shown that the algorithm INT-IMAGE will terminate after at most 2 iterations. In fact a simple necessary and sufficient condition in this case is given by Theorem 5.5 in the next section.

5. Efficiently solvable special cases

In addition to being useful for finding integer eigenvectors the question of whether or not a matrix has an integer image is interesting on its own. Here we consider a few cases when this question can be solved in polynomial time as well as linking it to instances where we can find integer eigenvectors.

It follows from the definitions that $IV(A, 0) \subseteq IIm(A)$ for any $A \in \overline{\mathbb{R}}^{n\times n}$. Here we first present some types of matrices for which equality holds, and further show that in these cases we can describe the subspaces efficiently. Later we discuss matrices with two rows/columns. Throughout this section we assume without loss of generality that A is doubly $\mathbb{R}$-astic.

Let A be a square matrix. Consider a generalised permutation matrix Q. It is easily seen that $IIm(A) = IIm(A \otimes Q)$. Further, from [3] we know that for every matrix A with $maper(A) > \epsilon$ there exists a generalised permutation matrix Q such that $A \otimes Q$ is strongly definite and Q can be found in $\mathcal{O}(n^3)$ time. Therefore when considering the integer image of a matrix with $maper(A) > \epsilon$ we can assume without loss of generality that the matrix is strongly definite.

Remark 3. From Corollary 3.3 we can immediately see that $\lambda(\lceil A \otimes Q\rceil) = 0$ is a sufficient condition for a matrix A with $maper(A) > \epsilon$ to have an integer image.

We define a *column typical* matrix to be a matrix $A \in \overline{\mathbb{R}}^{m\times n}$ such that for each j we have $fr(a_{ij}) \neq fr(a_{kj})$ for any i and k, $i \neq k$.

Suppose $A \in \overline{\mathbb{R}}^{m\times n}$. For a given $x \in \overline{\mathbb{R}}^{n}$ such that $A \otimes x = z \in \mathbb{Z}^m$ we say that an element a_{ij} of A is *active* with respect to x if $a_{ij} + x_j = z_i$. Otherwise we say that a_{ij} is *inactive* with respect to x.

Theorem 5.1. *Let* $A \in \overline{\mathbb{R}}^{n\times n}$ *be a column typical matrix.*

(a) If $maper(A) = \epsilon$ *then* A *has no integer image.*

(b) If $maper(A) > \epsilon$ *and* $|ap(A)| > 1$ *then* A *has no integer image.*

(c) If $maper(A) > \epsilon$ *and* $|ap(A)| = 1$ *let* Q *be the unique generalised permutation matrix such that* $A \otimes Q$ *is strongly definite. Then*

$$IIm(A) = IIm(A \otimes Q) = IV(A \otimes Q) = IV^*(A \otimes Q, 0).$$

PROOF. First observe that if A is column typical and $Ax \in IIm(A)$ then no two active elements of A with respect to x can lie in the same column. This is since the vector $x_j A_j$ can have at most one integer entry. Further it is obvious that there will be one active element per row and so we deduce that there exists a permutation $\pi \in P_n$ such that the active elements of A with respect to x are $a_{i,\pi(i)}$ and no others.

(a) Assume $maper(A) = \epsilon$. Suppose $Ax \in IIm(A)$. Then $a_{i,\pi(i)} + x_{\pi(i)} \in \mathbb{Z}$ for all $i \in N$ which implies that $a_{i,\pi(i)} \neq \epsilon$ for all i which is a contradiction.

(b) Assume $maper(A) > \epsilon$. Suppose $Ax = z \in IIm(A)$.

Let $\sigma \in P_n$ be different from π. Then

$$\sum_{i=1}^{n} a_{i,\pi(i)} + x_{\pi(i)} > \sum_{i=1}^{n} a_{i,\sigma(i)} + x_{\sigma(i)}. \tag{5.1}$$

To see this note that not all $a_{i,\sigma(i)}$ can be active since there exist $i, k \in N$ with $i \neq k$ such that $\pi(i) = \sigma(k)$ and therefore if $a_{k,\sigma(k)}$ was active then $fr(a_{k,\sigma(k)}) = fr(a_{i,\pi(i)})$, which does not happen. Hence we have that

$$a_{i,\sigma(i)} + x_{\sigma(i)} \leq \max_{j} a_{ij} + x_j = a_{i,\pi(i)} + x_{\pi(i)}$$

for all $i \in N$ where there is at least one i for which equality does not hold.

Finally, from (5.1),

$$\sum_{i=1}^{n} a_{i,\pi(i)} > \sum_{i=1}^{n} a_{i,\sigma(i)}$$

and so $ap(A) = \{\pi\}$.

(c) Assume $maper(A) > \epsilon$ and $|ap(A)| = 1$. Let $B = A \otimes Q$. Since B is strongly definite we know that

$$IV^*(B, 0) = IV(B) \subseteq IIm(B)$$

so it is sufficient to prove that $IIm(B) \subseteq IV(B)$.

Suppose $z \in IIm(B)$. Then there exists $x \in \overline{\mathbb{R}}^n$ such that $Bx = z$ and the only active elements of B with respect to x are $b_{i,\pi(i)}$. Further from the proof of (b) we see that π is a permutation of maximum weight with respect to B and therefore $\pi = id$.

We conclude that $z_i = \max_j(b_{ij} + x_j) = b_{ii} + x_i = x_i$ for all $i \in N$ and therefore $z \in IV(B)$. $\square$

Using Corollary 3.3 we deduce

Corollary 5.2. *If $A \in \overline{\mathbb{R}}^{n \times n}$ is column typical then the question of whether or not A has an integer image can be solved in polynomial time.*

Above we saw that if the entries in each column of a strongly definite matrix had different fractional parts then only the integer (diagonal) entries were active. So we now consider strongly definite matrices for which the only integer entries are on the diagonal to see if the results can be generalised to this class of matrices.

We say that a strongly definite matrix $A \in \overline{\mathbb{R}}^{n\times n}$ is *nearly non-integer* (NNI) if the only integer entries appear on the diagonal.

Lemma 5.3. *Let $A \in \overline{\mathbb{R}}^{n\times n}$, $n \geq 3$, be strongly definite and NNI. Then there is no x satisfying $Ax = z \in \mathbb{Z}^n$ such that a_{ij} with $i \neq j$ is active.*

PROOF. Let A be a strongly definite, NNI matrix. Suppose that there exists a vector x satisfying $Ax \in IIm(A)$ such that there exists a row $k_1 \in N$ with an off diagonal entry active.

So $\exists k_2 \in N,\ k_2 \neq k_1$ such that a_{k_1,k_2} is active. Then

$$a_{k_1,k_2} + x_{k_2} \geq a_{k_1,k_1} + x_{k_1} = x_{k_1}. \tag{5.2}$$

There is an active element in every row so consider row k_2. Then a_{k_2,k_2} is inactive because $fr(x_{k_2}) = 1 - fr(a_{k_1,k_2}) > 0$ so $a_{k_2,k_2} + x_{k_2} \notin \mathbb{Z}$. Further a_{k_2,k_1} is inactive since if it wasn't then it would hold that $a_{k_2,k_1} + x_{k_1} > a_{k_2,k_2} + x_{k_2} = x_{k_2}$ which together with (5.2) would imply that the cycle (k_1, k_2) has strictly positive weight, which is a contradiction with the definiteness of A.

Thus $\exists k_3 \in N,\ k_3 \neq k_1, k_2$ such that a_{k_2,k_3} is active and similarly as before

$$a_{k_2,k_3} + x_{k_3} > a_{k_2,k_2} + x_{k_2} = x_{k_2}. \tag{5.3}$$

Consider row k_3. Again it can be seen that both a_{k_3,k_3} and a_{k_3,k_2} are inactive. Further we show that a_{k_3,k_1} is inactive. If it was active then we would have $a_{k_3,k_1} + x_{k_3} > x_{k_1}$ which together with (5.2) and (5.3) would imply that cycle (k_1, k_2, k_3) has strictly positive weight, a contradiction.

Thus $\exists k_4 \in N,\ k_4 \neq k_1, k_2, k_3$ such that a_{k_3,k_4} is active.

Continuing in this way we see that,

$$(\forall i \in N)(\forall j \in \{1, 2, ..., i\})\ a_{k_i,k_j} \text{ is inactive.}$$

But this means that no element in row k_n can be active, a contradiction.□

Theorem 5.4. *Let $A \in \overline{\mathbb{R}}^{n\times n}$ be a strongly definite, NNI matrix. Then*

$$IIm(A) = IV(A) = IV^*(A, 0).$$

PROOF. If $n = 2$ then A is column typical and the statement follows from Theorem 5.1. Hence we assume $n \geq 3$.

$IV(A) \subseteq IIm(A)$ holds trivially. To prove the converse let $A \in \overline{\mathbb{R}}^{n\times n}$, $n \geq 3$, be strongly definite and NNI. Then by Lemma 5.3 there is no x satisfying $Ax = z \in \mathbb{Z}^n$ such that a_{ij} with $i \neq j$ is active. Thus only the diagonal elements can be active. Hence for any $z \in IIm(A)$ we have $Ax = z$ for some x with $a_{ii} = 0$ active for all $i \in N$. Therefore $x = z$ and so $z \in IV(A)$. □

We now show that if either m or n is equal to 2 we can straightforwardly decide whether $IIm(A) = \emptyset$.

Theorem 5.5. *Let $A = (a_{ij}) \in \overline{\mathbb{R}}^{2\times n}$ be doubly $\mathbb{R}$-astic, and $d_j := a_{1j} - a_{2j}$ for all $j \in N$.*

(a) If any d_j is an integer then A has an integer image.

(b) If no d_j is integer then A has an integer image if and only if

$$(\exists i, j)\lfloor d_i \rfloor \neq \lfloor d_j \rfloor.$$

PROOF. (a) Without loss of generality assume $d_1 \in \mathbb{Z}$. Then

$$A \otimes (-a_{11}, \epsilon, ..., \epsilon)^T = (0, -d_1)^T \in \mathbb{Z}^2.$$

(b) First assume without loss of generality that $\lfloor d_1 \rfloor \neq \lfloor d_2 \rfloor$, $d_1 < d_2$ and that $d_1, d_2 \notin \mathbb{Z}$.

Case 1 $d_1, d_2 \in \mathbb{R}$.

Let $d = \lceil d_1 \rceil$ so that $a_{21} + d > a_{11}$ and $a_{22} + d < a_{12}$. Then

$$A \otimes (-a_{21} - d,\ -a_{12},\ \epsilon, ..., \epsilon)^T = (0, -d)^T \in \mathbb{Z}^2.$$

Case 2 $d_1 \in \mathbb{R}$, $d_2 = +\infty$.

Then $a_{22} = \epsilon$ and for $k \in \mathbb{Z}$ big enough,

$$A \otimes (-fr(a_{21}),\ -fr(a_{12}) + k,\ \epsilon, ..., \epsilon) = (\lfloor a_{12} \rfloor + k, \lfloor a_{21} \rfloor)^T \in \mathbb{Z}^2.$$

Case 3 $d_1 = -\infty$, $d_2 \in \mathbb{R}$.

Then $a_{11} = \epsilon$ and for $k \in \mathbb{Z}$ big enough,

$$A \otimes (-fr(a_{21}) + k,\ -fr(a_{12}),\ \epsilon, ..., \epsilon) = (\lfloor a_{12} \rfloor, \lfloor a_{21} \rfloor + k)^T \in \mathbb{Z}^2.$$

Case 4 $d_1 = -\infty$, $d_2 = +\infty$.

Here $a_{11} = a_{22} = \epsilon$ and

$$A \otimes (-fr(a_{21}),\ -fr(a_{12}),\ \epsilon, ..., \epsilon)^T = (\lfloor a_{12} \rfloor, \lfloor a_{21} \rfloor)^T \in \mathbb{Z}^2.$$

For the other direction assume that $d = \lfloor d_j \rfloor < d_j$ for all $j \in N$ and suppose, for a contradiction, that there exists $x \in \overline{\mathbb{R}}^n$ such that $Ax = b \in \mathbb{Z}^2$. Without loss of generality we may assume $b = (0, b')^T$ for some $b' \in \mathbb{Z}$.

If $-b' \leq d$ then

$$\begin{aligned}
&(\forall j \in N)\ \ -b' < d_j = a_{1j} - a_{2j} \\
\therefore &(\forall j \in N)\ a_{1j} > a_{2j} - b' \\
\therefore &(\forall j \in N)\ M_j(A, b) = \{1\} \\
\therefore &\bigcup_{j \in N} M_j(A, b) = \{1\}
\end{aligned}$$

Thus by Proposition 3.1 no such x exists.

If instead $-b' > d$ then since $b' \in \mathbb{Z}$ we have $b' \geq \lfloor d_i \rfloor + 1 > d_i$. Then a similar argument as above shows $M_j(A, b) = \{2\}$ for all j and again we conclude that no such x exists. $\square$

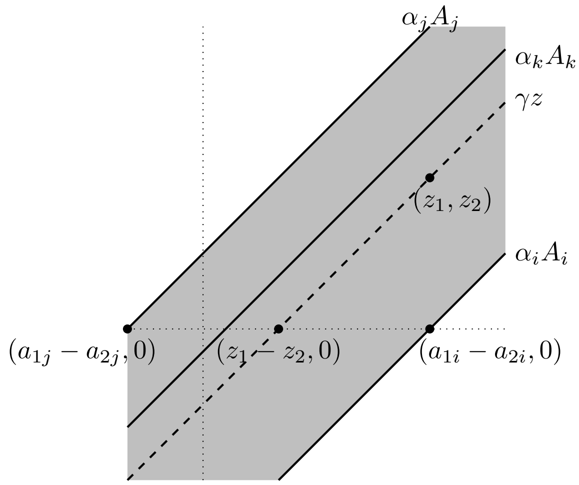

Figure 1: Graphical representation for a finite $2 \times n$ matrix to have an integer image.

Note that, if $d_i < d_j$, the condition $\lfloor d_i \rfloor \neq \lfloor d_j \rfloor$ means that

$$(\exists z \in \mathbb{Z})\ z \in [d_i, d_j].$$

So an equivalent condition for a finite matrix A to have an integer image is that there exists an integer between $\min_j a_{1j} - a_{2j}$ and $\max_j a_{1j} - a_{2j}$. We represent this condition graphically in Figure 1. In Figure 1 the solid lines represent points in $Im(A)$ that are multiples of a single column and the shaded area represents all the points in $Im(A)$. If there exists $z \in IIm(A)$ then also $(z_1 - z_2, 0)^T \in IIm(A)$ and the x-coordinate satisfies, for some i and j

$$z_1 - z_2 \in [a_{1j} - a_{2j},\ a_{1i} - a_{2i}].$$

Now we deal with matrices for which $n = 2$. It should be noted that these results were also independently discovered in [8]. We start with a lemma whose proof is straightforward.

Lemma 5.6. *Suppose $A \in \overline{\mathbb{R}}^{m \times 2}$.*
(a) If $\exists j \in \{1, 2\}$ such that $(\forall i, k \in M) fr(a_{ij}) = fr(a_{kj})$ then $IIm(A) \neq \emptyset$.
(b) If $\exists \gamma \in \mathbb{R}$ such that $A_1 = \gamma A_2$ then $IIm(A) \neq \emptyset$ if and only if

$$(\exists j \in \{1, 2\})(\forall i, k \in M) fr(a_{ij}) = fr(a_{kj}).$$

Theorem 5.7. *Suppose $A \in \overline{\mathbb{R}}^{m \times 2}$ is a doubly $\mathbb{R}$-astic matrix not satisfying the conditions in Lemma 5.6. Let l, r be the indices such that*

$$a_{l2} - a_{l1} = \min_{i \in M} a_{i2} - a_{i1} \text{ and } a_{r2} - a_{r1} = \max_{i \in M} a_{i2} - a_{i1}.$$

Let

$$\bar{L} = \{i \in M : fr(a_{i1}) = fr(a_{l1})\},$$
$$\bar{R} = \{i \in M : fr(a_{i2}) = fr(a_{r2})\},$$

$L = \bar{L} - \bar{R}$ and $R = \bar{R} - \bar{L}$. Denote $fr(a_{r2}) - fr(a_{l1})$ by f. Then

(1) If $\bar{L} \cup \bar{R} \neq M$ then $IIm(A) = \emptyset$.

(2) Otherwise $IIm(A) \neq \emptyset$ if and only if

$$\left\lfloor \min_{i \in L}(a_{i1} - a_{i2}) + f \right\rfloor - \left\lceil \max_{i \in R}(a_{i1} - a_{i2})) + f \right\rceil \geq 0.$$

Proof. We first prove that $fr(x_1) = 1 - fr(a_{l1})$ and $fr(x_2) = 1 - fr(a_{r2})$ for any x satisfying $Ax \in IIm(A)$. We do this by showing that both a_{l1} and a_{r2} are active for any such x.

Assume for a contradiction that $Ax \in IIm(A)$ but a_{l1} is not active. Then we have that $a_{l1} + x_1 < a_{l2} + x_2 \in \mathbb{Z}$ and therefore

$$x_1 - x_2 < a_{l2} - a_{l1} = \min_{i \in M} a_{i2} - a_{i1}.$$

Moreover there must be an active entry in the first column of A and so $\exists k \in M$ such that $a_{k1} + x_1 \geq a_{k2} + x_2$, equivalently $x_1 - x_2 \geq a_{k2} - a_{k1}$, a contradiction. A similar argument works for a_{r2}.

(1) This is now easily seen to be true since for any x with $fr(x_1) = 1 - fr(a_{l1})$ and $fr(x_2) = 1 - fr(a_{r2})$ there will be at least one index $i \in M$ such that $(Ax)_i \notin \mathbb{Z}$.

(2) $Ax \in IIm(A)$ implies that $fr(x_1) = 1 - fr(a_{l1})$ and $fr(x_2) = 1 - fr(a_{r2})$. So the set $\bar{L} \cap \bar{R}$ contains all the row indices for which we can guarantee that $(Ax)_i \in \mathbb{Z}$. So we construct a matrix A' from A by removing all rows with indices in $\bar{L} \cap \bar{R}$. We also define sets L' and R' to be the sets of row indices in A' that correspond to the sets L and R respectively. Observe that

$$IIm(A) \neq \emptyset \text{ if and only if } IIm(A') \neq \emptyset$$

and further

$$\{x \in \mathbb{R}^2 : A \otimes x \in IIm(A)\} = \{x \in \mathbb{R}^2 : A' \otimes x \in IIm(A')\} := X.$$

Since any $x \in X$ has the form

$$\begin{pmatrix} \gamma_1 + 1 - fr(a_{l1}) \\ \gamma_2 + 1 - fr(a_{r2}) \end{pmatrix}$$

for some $\gamma_1, \gamma_2 \in \mathbb{Z}$ we can decide whether $IIm(A') \neq \emptyset$ by determining whether there exists $\alpha \in \mathbb{Z}$ such that

$$x = \begin{pmatrix} -fr(a_{l1}) \\ \alpha - fr(a_{r2}) \end{pmatrix} \in X.$$

The set L' (R') is exactly the set of row indices i for which a'_{i1} (a'_{i2}) is active for any $x \in X$. So such an α exists if and only if the following sets of inequalities can be satisfied.

$$\begin{cases}(\forall i \in L')a_{i1} + x_1 > a_{i2} + x_2 \\ (\forall i \in R')a_{i2} + x_2 > a_{i1} + x_1\end{cases}$$
$$\Leftrightarrow \begin{cases}(\forall i \in L')a_{i1} - fr(a_{l1}) > a_{i2} - fr(a_{r2}) + \alpha \\ (\forall i \in R')a_{i2} - fr(a_{r2}) + \alpha > a_{i1} - fr(a_{l1})\end{cases}$$
$$\Leftrightarrow \max_{i \in R'} a_{i1} - a_{i2} + f < \alpha < \min_{i \in L'} a_{i1} - a_{i2} + f$$

Therefore $IIm(A') \neq \emptyset$ if and only if there exists an integer

$$\alpha \in \left[\left\lceil \max_{i \in R'}(a_{i1} - a_{i2}) + f \right\rceil, \left\lfloor \min_{i \in L'}(a_{i1} - a_{i2}) + f \right\rfloor\right].$$

$\square$

Remark 4. Note that the proof tells us how to describe all integer images of the matrix $A \in \mathbb{R}^{m \times 2}$ since we can easily describe all α such that

$$\begin{pmatrix} -fr(a_{l1}) \\ \alpha - fr(a_{r2}) \end{pmatrix} \in X.$$

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