The alternating method for finding integer solutions to two-sided max-linear systems

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Abstract

For general matrices we show that we can adapt the Alternating Method of [4] for finding real solutions to two-sided max-linear systems to obtain algorithms for finding integer solutions.

1 Introduction

In max-algebra, for $a, b \in \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$, we define $a \oplus b = \max(a, b)$, $a \otimes b = a + b$ and extend the pair $(\oplus, \otimes)$ to matrices and vectors in the same way as in linear algebra, that is (assuming compatibility of sizes)

$$
\begin{aligned}
(\alpha \otimes A)_{ij} &= \alpha \otimes a_{ij}, \\
(A \oplus B)_{ij} &= a_{ij} \oplus b_{ij}, \\
(A \otimes B)_{ij} &= \bigoplus_k a_{ik} \otimes b_{kj}.
\end{aligned}
$$

Except for complexity arguments all multiplications in this paper are in max-algebra and where appropriate we will omit the $\otimes$ symbol. Note that α^{-1} stands for $-\alpha$, and for a vector γ we use γ^{-1} to mean the vector with entries γ_i^{-1}. We will use ϵ to denote $-\infty$ as well as any vector or matrix whose every entry is $-\infty$. A vector/matrix whose every entry belongs to $\mathbb{R}$ is called *finite* as is any scalar from $\mathbb{R}$. An *integer vector/matrix* is a vector/matrix with all entries from $\mathbb{Z}$. If a matrix has no ϵ rows (columns) then it is called *row* (*column*) $\mathbb{R}$-*astic* and it is called *doubly* $\mathbb{R}$-*astic* if it is both row and column $\mathbb{R}$-astic [1, 3].

For $a \in \mathbb{R}$ the *fractional part* of a is $fr(a) := a - \lfloor a \rfloor$. For a matrix $A \in \overline{\mathbb{R}}^{m\times n}$ we use $\lfloor A \rfloor$ ($\lceil A \rceil$) to denote the matrix with (i, j) entry equal to $\lfloor a_{ij} \rfloor$ ($\lceil a_{ij} \rceil$) and similarly for vectors. We define $\lfloor \epsilon \rfloor = \epsilon = \lceil \epsilon \rceil$.

A *two-sided max-linear system* (TSS) is of the form

$$Ax \oplus c = Bx \oplus d$$

where $A, B \in \overline{\mathbb{R}}^{m\times n}$ and $c, d \in \overline{\mathbb{R}}^m$. If $c = d = \epsilon$ then we say the system is *homogeneous*, otherwise it is called *nonhomogeneous*. Nonhomogeneous systems

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can be transformed to homogeneous systems [1]. If $B \in \overline{\mathbb{R}}^{m \times k}$ a system of the form

$$Ax = By$$

is called a *system with separated variables.*

The problem of finding solutions to

$$Ax = By \tag{1}$$

and

$$Ax = Bx \tag{2}$$

have been previously studied and one solution method is the Alternating Method [1, 4]. We show that we can adapt the Alternating Method in order to obtain algorithms which determine whether integer solutions to these problems exist, and find one if it exists.

2 Preliminaries

We will use the following standard notation. For positive integers m, n we denote $M = \{1, ..., m\}$ and $N = \{1, ..., n\}$. If $a, b \in \overline{\overline{\mathbb{R}}} = \overline{\mathbb{R}} \cup \{+\infty\}$ then we define $a \oplus' b = \min(a, b)$ and $a \otimes' b = a + b$ if at least one of a, b is finite, $(-\infty) \otimes (+\infty) = (+\infty) \otimes (-\infty) = -\infty$ and $(-\infty) \otimes' (+\infty) = (+\infty) \otimes' (-\infty) = +\infty$. The pair of operations $(\oplus', \otimes')$ is extended to matrices and vectors similarly as $(\oplus, \otimes)$. For $A \in \overline{\overline{\mathbb{R}}}^{m \times n}$ we define $A^{\#} = -A^T \in \overline{\overline{\mathbb{R}}}^{n \times m}$. It can be shown [1, 3] that $(A \otimes B)^{\#} = B^{\#} \otimes' A^{\#}$.

Next we give an overview of some basic properties.

Lemma 2.1 *Let $A \in \overline{\mathbb{R}}^{m \times n}$, $x \in \mathbb{R}^n$.*

(a) If A is row $\mathbb{R}$-astic then $A \otimes x$ is finite.

(b) If A is column $\mathbb{R}$-astic then $A^{\#} \otimes' x$ is finite.

Proof. Straighforward from the definitions. □

Lemma 2.2 *[1, 3] If $A \in \overline{\mathbb{R}}^{m \times n}$ and $x, y \in \overline{\mathbb{R}}^n$ then*

$$x \leq y \Rightarrow A \otimes x \leq A \otimes y \text{ and } A \otimes' x \leq A \otimes' y.$$

Corollary 2.3 *[1, 3] If $A, B \in \overline{\mathbb{R}}^{m \times n}$ and $x \leq y$ then*

$$B^{\#} \otimes' (A \otimes x) \leq B^{\#} \otimes' (A \otimes y).$$

Lemma 2.4 *[1] Let $A, B \in \overline{\mathbb{R}}^{m \times n}$, $c, d \in \overline{\mathbb{R}}^m$. Then there exists $x \in \mathbb{R}^n$ satisfying $Ax \oplus c = Bx \oplus d$ if and only if there exists $z \in \mathbb{R}^{n+1}$ satisfying $(A|c)z = (B|d)z$.*

For any matrices of compatible sizes [1, 3],

$$X \otimes (X^{\#} \otimes' Y) \leq Y, \tag{3}$$

$$X \otimes (X^{\#} \otimes' (X \otimes Z)) = X \otimes Z, \tag{4}$$

If $A \in \overline{\mathbb{R}}^{m\times n}$ and $b \in \mathbb{R}^m$ then for all $j \in N$ define

$$M_j(A,b) = \{k \in M : a_{kj} \otimes b_k^{-1} = \max_i a_{ij} \otimes b_i^{-1}\}.$$

Proposition 2.5 *[2] Let $A \in \overline{\mathbb{R}}^{m\times n}$, $b \in \mathbb{R}^m$ and $\bar{x} = A^{\#} \otimes' b$.*

(a) An integer solution to $Ax \leq b$ exists if and only if $\bar{x}$ is finite. If an integer solution exists then all integer solutions can be described as the integer vectors x satisfying $x \leq \bar{x}$.

(b) An integer solution to $Ax = b$ exists if and only if

$$\bigcup_{j:\bar{x}_j \in \mathbb{Z}} M_j(A,b) = M.$$

If an integer solution exists then all integer solutions can be described as the integer vectors x satisfying $x \leq \bar{x}$ with

$$\bigcup_{j:x_j = \bar{x}_j} M_j(A,b) = M.$$

We define $\hat{x} = \lfloor A^{\#} \otimes' b \rfloor$. Then from Proposition 2.5 and (4) we conclude:

Corollary 2.6 *Let $A \in \overline{\mathbb{R}}^{m\times n}, b \in \overline{\mathbb{R}}^m, c \in \mathbb{Z}^n$. Then the following hold:*

(a) $\hat{x}$ is the greatest integer solution to $Ax \leq b$ (provided $\hat{x}$ is finite).

(b) $Ax = b$ has an integer solution if and only if $\hat{x}$ is an integer solution.

(c) $A \otimes \lfloor A^{\#} \otimes' (A \otimes c) \rfloor = A \otimes c$.

Consider the matrix inequality $AX \leq B$ where $A \in \overline{\mathbb{R}}^{m\times n}, B \in \overline{\mathbb{R}}^{m\times k}$, $X \in \overline{\mathbb{R}}^{n\times k}$ and let $\hat{X} = \lfloor A^{\#} \otimes' B \rfloor$. This system can be written as a set of inequalities of the form $Ax \leq b$ in the following way using the notation X_r, B_r to denote the r^{th} column of X and B respectively:

$$AX_r \leq B_r,\ r = 1, ..., k.$$

This allows us to state the following result.

Corollary 2.7 *Let $A \in \overline{\mathbb{R}}^{m\times n}, B \in \overline{\mathbb{R}}^{m\times k}, C \in \mathbb{Z}^{n\times k}$. Then the following hold:*

(a) $\hat{X}$ is the greatest integer solution to $AX \leq B$ (provided $\hat{X}$ is finite), that is $A \otimes \lfloor A^{\#} \otimes' B \rfloor \leq B$.

(b) $AX = B$ has an integer solution if and only if $\hat{X}$ is an integer solution.

(c) $A \otimes \lfloor A^{\#} \otimes' (A \otimes C) \rfloor = A \otimes C$.

3 Alternating Method for Integer Solutions

In this section we show that the Alternating Method [1, 4] can be easily adapted to design algorithms that determine whether integer solutions to (1) or (2) exist, and if so find one. For the rest of this section we follow similar arguments as in [1] and [4].

If the i^{th} row of either A or B is ϵ then we have $(Ax)_i = \epsilon = (By)_i$ which, since x and y are finite, means that the i^{th} row of the other matrix is also ϵ. Thus we may remove the redundant i^{th} equation from the equality. If instead either of A or B has an ϵ column then this column may be removed without affecting the solution. Hence we assume without loss of generality that A, B are doubly $\mathbb{R}$-astic.

3.1 Systems with separable variables

We propose the following algorithm to find integer solutions to the system with separated variables (1):

Algorithm: SEP-INT-TSS

Input: $A \in \overline{\mathbb{R}}^{m \times n}, B \in \overline{\mathbb{R}}^{m \times k}$ doubly $\mathbb{R}$-astic, any starting vector $x(0) \in \mathbb{Z}^n$.

Output: An integer solution (x, y) to $Ax = By$ or indication that no such solution exists.

1. $r := 0$.
2. $y(r) := \lfloor B^{\#} \otimes' (A \otimes x(r)) \rfloor$.
3. $x(r+1) := \lfloor A^{\#} \otimes' (B \otimes y(r)) \rfloor$.
4. If $x_i(r+1) < x_i(0)$ for all $i \in N$ then STOP (no solution).
5. If $A \otimes x(r+1) = B \otimes y(r)$ then STOP (solution found).
6. Go to 2.

In order to prove the correctness of this algorithm we will first prove a number of lemmas. Note that during the run of the algorithm x, y will always be finite vectors by Lemma 2.1. Recall that, from Corollary 2.6, for matrices X, Y, Z of compatible sizes, where Z is an integer matrix we have that:

$$X \otimes \lfloor X^{\#} \otimes' Y \rfloor \leq Y, \tag{5}$$

$$X \otimes \lfloor X^{\#} \otimes' (X \otimes Z) \rfloor = X \otimes Z, \tag{6}$$

As in [4] we define operators π, ψ as follows:

$$\pi : y \to \lfloor A^{\#} \otimes' (B \otimes y) \rfloor$$
$$\psi : x \to \lfloor B^{\#} \otimes' (A \otimes x) \rfloor$$

so that

$$x(r+1) = \pi(y(r)) \tag{7}$$

and

$$y(r) = \psi(x(r)). \tag{8}$$

Note that both π and ψ are monotone functions by Corollary 2.3.

For $x \in \mathbb{R}^n, y \in \mathbb{R}^k$ we define a pair (x, y) to be *stable* if $(x, y) = (\pi(y), \psi(x))$.

Lemma 3.1 *Every stable pair (x, y) is a solution*

Proof. Let (x, y) be stable, clearly x, y are integer vectors. Using (5) twice we get

$$A \otimes x = A \otimes \pi(y) = A \otimes \lfloor A^{\#} \otimes' (B \otimes y) \rfloor \leq B \otimes y = B \otimes \psi(x) \leq A \otimes x.$$

Thus $A \otimes x = B \otimes y$. $\square$

A *stable integer solution* is a stable pair of integer vectors (x, y) satisfying (1).

Lemma 3.2 *If (x, y) is an integer solution then $(\pi(y), \psi(x))$ is a stable integer solution.*

Proof. Assume (x, y) is an integer solution. By Lemma 3.1 we need only to show that $(\pi(y), \psi(x))$ is stable. Using (6) together with the fact that x is integer we get that

$$\begin{aligned}\psi(\pi(y)) = \psi(\lfloor A^{\#} \otimes' (B \otimes y) \rfloor) &= \lfloor B^{\#} \otimes' (A \otimes \lfloor A^{\#} \otimes' (B \otimes y) \rfloor) \rfloor \\ &= \lfloor B^{\#} \otimes' (A \otimes \lfloor A^{\#} \otimes' (A \otimes x) \rfloor) \rfloor \\ &= \lfloor B^{\#} \otimes' (A \otimes x) \rfloor = \psi(x).\end{aligned}$$

We can also apply a similar argument to show that $\pi(\psi(x)) = \pi(y)$ and thus $(\pi(y), \psi(x))$ is stable as required. $\square$

Lemma 3.3 *The sequence $\{A(x(r))\}_{r=0,1,\ldots}$ produced by Algorithm SEP-INT-TSS is nonincreasing.*

Proof. Using (5) we obtain

$$\begin{aligned}A \otimes x(r+1) &= A \otimes \lfloor A^{\#} \otimes' (B \otimes y(r)) \rfloor \leq B \otimes y(r) \\ &= B \otimes \lfloor B^{\#} \otimes' (A \otimes x(r)) \rfloor \leq A \otimes x(r).\end{aligned}$$

$\square$

Lemma 3.4 *The sequence $\{x(r)\}_{r=0,1,\ldots}$produced by Algorithm SEP-INT-TSS is nonincreasing.*

Proof. $x(r+1) = \pi(\lfloor B^{\#} \otimes' (A \otimes x(r)) \rfloor)$. Now $A \otimes x(r)$ is nonincreasing by Lemma 3.3 and so $B^{\#} \otimes' (A \otimes x(r))$ is nonincreasing. Since π is monotone it holds that $x(r+1) \leq x(r)$. $\square$

Lemma 3.5 *If a solution exists then the sequence $\{x(r)\}_{r=0,1,...}$ is lower bounded for any $x(0)$.*

Proof. Firstly let (x, y) be a stable integer solution (exists by Lemma 3.2) and let $\alpha \in \mathbb{Z}$. We claim that $\alpha \otimes (x, y)$ is also a stable integer solution. Note that

$$\begin{aligned}\alpha \otimes x = \alpha \otimes \pi(y) &= \alpha \otimes \lfloor A^{\#} \otimes' (B \otimes y)\rfloor \\ &= \lfloor \alpha \otimes (A^{\#} \otimes' (B \otimes y))\rfloor \\ &= \lfloor A^{\#} \otimes' (B \otimes (\alpha \otimes y))\rfloor = \pi(\alpha \otimes y).\end{aligned}$$

Similarly $\alpha \otimes y = \psi(\alpha \otimes x)$ and hence the claim holds by Lemma 3.1.

We prove the lemma by induction on r.

Note we may choose α as above small enough so that $\alpha \otimes x \leq x(0)$ and so if $r = 0$ then the statement holds. So now assume that a solution, and thus a stable solution, exists and that $x(r) \geq u$ for some stable solution (u, v) with $u \leq x(0)$. Then

$$\begin{aligned}x(r+1) = \pi(\psi(x(r)) &= \pi(\lfloor B^{\#} \otimes' (A \otimes x(r))\rfloor) \\ &\geq \pi(\lfloor B^{\#} \otimes' (A \otimes u)\rfloor) \\ &= \pi(\psi(u)) = \pi(v) = u.\end{aligned}$$

Thus by induction the statement holds. □

Theorem 3.6 *If all components of $x(r)$ or $y(r)$ have properly decreased after a number of steps of Algorithm SEP-INT-TSS then* (1) *has no solution.*

Proof. In the proof of Lemma 3.5 we saw that if a solution exists $\{x(r)\}_{r=0,1,...}$ can be bounded below by αx where $\alpha x \leq x(0)$. Choose α so that $\alpha x \leq x(0)$ and there is equality in at least one component. Then since $\{x(r)\}_{r=0,1,...}$ is nonincreasing this component will not change during the run of the algorithm if a solution can be found.

The sequence $\{y(r)\}_{r=0,1,...}$ satisfies the same properties as $\{x(r)\}_{r=0,1,...}$ and so an identical argument can be applied. □

We can now prove that Algorithm SEP-INT-TSS is correct.

Theorem 3.7 *The integer sequence $\{(x(r), y(r))\}_{r=0,1,...}$ generated by Algorithm SEP-INT-TSS finitely converges if and only if an integer solution exists. Convergence is monotonic, to a stable solution, for any choice of $x(0) \in \mathbb{Z}^n$.*

Proof.

If a solution exists then monotonic convergence of $\{x(r)\}_{r=0,1,...}$ follows from Lemmas 3.4 and 3.5. The convergence is finite since we are dealing with integer vectors, and at least one component decreases by at least 1 each time until the limit is reached. Similar arguments can be applied to $\{y(r)\}_{r=0,1,...}$.

If $(x(r), y(r)) \to (z_1, z_2)$ and the convergence is finite then there exists $s \in \mathbb{N}$ such that for all $r \geq s$ we have $x(r) = x(r+1) = x(r+2) = ... = z_1$ and

$y(r) = y(r+1) = y(r+2) = ... = z_2$. We show that (z_1, z_2) is stable and thus an integer solution. By (7) and (8),

$$\pi(z_2) = \pi(y(s)) = x(s+1) = z_1 \text{ and } \psi(z_1) = \psi(x(s)) = y(s) = z_2.$$

$\square$

We now calculate the complexity of Algorithm SEP-INT-TSS.

Theorem 3.8 *If $A \in \mathbb{R}^{m\times n}, B \in \overline{\mathbb{R}}^{m\times k}$ and Algorithm SEP-INT-TSS starts with $x(0) \in \mathbb{Z}^n$ then it will terminate after at most*

$$(n-1)(1 + \gamma^{\#} \otimes A^{\#} \otimes A \otimes \gamma)$$

iterations where $\gamma = x(0)$.

Proof. Suppose that a solution exists. From Theorem 3.6 we know that there exists an index, k say, such that $x_k(r) = x_k(0) = \gamma_k$ for all r. The algorithm halts when Ax does not change, which occurs at the latest when all x_j with $j \neq k$ have decreased enough so that they are no longer active in any row. This happens when

$$(\forall i \in M)(\forall j \in N)\ a_{ij} + x_j \leq a_{ik} + \gamma_k.$$

Equivalently

$$(\forall j \in N)\ x_j \leq \min_i (a_{ik} + \gamma_k - a_{ij}) = (A^{\#} \otimes' A)_{jk} + \gamma_k := u_{jk}.$$

Since the value of k is unknown we can guarantee that x_j no longer influences the product Ax if it satisfies

$$x_j \leq \min_k u_{jk} = \min_k ((A^{\#} \otimes' A)_{jk} + \gamma_k) = (A^{\#} \otimes' A \otimes' \gamma)_j := \mu_j.$$

Note that we require A to be finite here so that the value of μ_j is finite.

Finally we know that there are at most $n-1$ components of $x(0)$ that will decrease during the run of the algorithm, and in each iteration where a solution is not found at least one component will decrease by a value of at least 1. Therefore the total number of iterations possible is

$$\begin{aligned}(n-1)\max_j \gamma_j - \mu_j + 1 &= (n-1)(1 + \mu^{\#} \otimes \gamma) \\ &= (n-1)(1 + \gamma^{\#} \otimes A^{\#} \otimes A \otimes \gamma) := D.\end{aligned}$$

If instead no solution exists then after D iterations the value of Ax is still changing and the components of x have all decreased below the corresponding values of $x(0)$. Thus the algorithm halts within D iterations with the conclusion that no solution exists. $\square$

To calculate a formula for the complexity we first consider when

$$\gamma^{\#} \otimes A^{\#} \otimes A \otimes \gamma$$

is minimised. Let $C = A^{\#} \otimes A$. From [1] we have that, for any finite subeigenvector z of C corresponding to $\lambda(C)$,

$$\min_{x \in \mathbb{R}^n} x^{\#} \otimes C \otimes x = z^{\#} \otimes C \otimes z = \lambda(C).$$

Let $z \in V^*(C, \lambda(C))$. Then since we know that, for $x, y \in \mathbb{R}$,

$$\lfloor x \rfloor - \lfloor y \rfloor \leq x - y + 1,$$

we have

$$\min_{i,j}(-\lfloor z_i \rfloor + c_{ij} + \lfloor z_j \rfloor) \leq \min_{i,j}(-z_i + c_{ij} + z_j + 1) \leq \lambda(C) + 1,$$

which implies that

$$\lfloor z \rfloor^{\#} \otimes C \otimes \lfloor z \rfloor \leq \lambda(C) + 1.$$

For any matrix $Y \in \mathbb{R}^{m \times n}$ define

$$K(Y) = \left\lceil \max\{|y_{ij}| : i \in M, j \in N\} \right\rceil. \tag{9}$$

Observe that if Y is square then $|\lambda(Y)| \leq K(Y)$ and hence $\lambda(C) \leq K(C) \leq 2K(A)$.

The number of operations required for one iteration (steps 2-5 in the algorithm) is

$$\mathcal{O}((mn + mk + k) + (mn + mk + n) + n + (mn + mk + m)).$$

Therefore using the above results we get that the number of iterations is bounded by

$$(n-1)(1 + \lambda(A^{\#} \otimes A) + 1) \leq (n-1)(2 + 2K(A)).$$

Thus the complexity of the Algorithm SEP-INT-TSS is

$$2(1 + K(A))(n-1)\mathcal{O}(mn + mk + m + n + k) = \mathcal{O}(mn(n+k)K(A)).$$

3.2 General two sided systems

Lemma 2.4 allows us to write any general two-sided system as a homogeneous system, (2), so it is sufficient to develop a method to find integer solutions to homogeneous systems. The following statement is obvious.

Proposition 3.9 *Let $A, B \in \overline{\mathbb{R}}^{m \times n}$. The problem of finding $x \in \mathbb{Z}^n$ satisfying $Ax = Bx$ is equivalent to finding $x \in \mathbb{Z}^n, y \in \mathbb{R}^m$ such that*

$$\begin{pmatrix} A \\ B \end{pmatrix} x = \begin{pmatrix} I \\ I \end{pmatrix} y$$

where $I \in \overline{\mathbb{R}}^{m \times m}$.

We propose the following algorithm to find integer solutions to (2).

Algorithm: GEN-INT-TSS

Input: $A', B' \in \overline{\mathbb{R}}^{m\times n}$ doubly $\mathbb{R}$-astic, $I \in \overline{\mathbb{R}}^{m\times m}$, any starting vector $x(0) \in \mathbb{Z}^n$.

Output: A solution $x \in \mathbb{Z}^n$ to $A'x = B'x$ or indication that no such vectors exist.

1. $r := 0$, $A := \begin{pmatrix} A' \\ B' \end{pmatrix}$, $B := \begin{pmatrix} I \\ I \end{pmatrix}$.
2. $y(r) := B^{\#} \otimes' (A \otimes x(r))$.
3. $x(r+1) := \lfloor A^{\#} \otimes' (B \otimes y(r)) \rfloor$.
4. If $x_i(r+1) < x_i(0)$ for all $i \in N$ then STOP (no solution).
5. If $A \otimes x(r+1) = B \otimes y(r)$ then STOP (solution found).

6: Go to (2).

Note that $A \in \overline{\mathbb{R}}^{2m\times n}, B \in \overline{\mathbb{R}}^{2m\times m}$ and during the run of the algorithm x, y will always be finite vectors since we begin with an integer vector, and always multiply by doubly $\mathbb{R}$-astic matrices.

Similarly as before we define monotone functions

$$\pi : y \to \lfloor A^{\#} \otimes' (B \otimes y) \rfloor$$
$$\psi : x \to B^{\#} \otimes' (A \otimes x)$$

so that (7) and (8) still hold.

For $x \in \mathbb{R}^n, y \in \mathbb{R}^m$ we define a pair (x, y) to be *stable* if $(x, y) = (\pi(y), \psi(x))$, and hence $x \in \mathbb{Z}^n$. A *stable solution* is a stable pair of vectors (x, y) that are a solution to $Ax = By$ (hence x is an integer solution to $A'x = B'x$).

Lemmas 3.10-3.14 and Theorem 3.15 below hold by almost identical arguments to their counterparts in Subsection 3.1, sometimes using (3) and (4) instead of (5) and (6).

Lemma 3.10 *Every stable pair (x, y) is a solution to $Ax = By$*

Lemma 3.11 *If (x, y) is a solution then $(\pi(y), \psi(x))$ is a stable solution.*

Lemma 3.12 *The sequence $\{A(x(r))\}_{r=0,1,...}$ produced by Algorithm GEN-INT-TSS is non increasing.*

Lemma 3.13 *The sequence $\{x(r)\}_{r=0,1,...}$ produced by Algorithm GEN-INT-TSS is non increasing.*

Lemma 3.14 *If a solution exists then the sequence $\{x(r)\}_{r=0,1,...}$ is lower bounded for any $x(0)$.*

Theorem 3.15 *If all components of $x(r)$ have properly decreased after a number of steps of Algorithm GEN-INT-TSS then neither $Ax = By$, nor $A'x = B'x$, has a solution when x is integer.*

We can now prove the correctness of Algorithm GEN-INT-TSS.

Theorem 3.16 *The integer sequence $\{(x(r)\}_{r=0,1,\ldots}$ generated by Algorithm GEN-INT-TSS finitely converges if and only if an integer solution to $A'x = B'x$ exists. Convergence is monotonic, to a stable solution (x, y) of $Ax = By$, for any choice of $x(0) \in \mathbb{Z}^n$.*

Proof.

If an integer solution to $A'x = B'x$ exists then a solution to $Ax = By$ exists where $x \in \mathbb{Z}^n$, $y \in \mathbb{R}^m$ and hence monotonic convergence of $\{x(r)\}_{r=0,1,\ldots}$ follows from Lemmas 3.13 and 3.14. The convergence is finite since we are dealing with integer vectors, meaning that at least one component decreases by at least one each time until the limit is reached.

For the other direction suppose $x(r) \to z_1$ and the convergence is finite. Then there exists $s \in \mathbb{N}$ such that

$$(\forall r \geq s)\ x(r) = x(r+1) = x(r+2) = \ldots = z_1.$$

Further, for all $r \geq s$ we have $y(r) = \psi(x(r)) = \psi(x(r+1)) = y(r+1)$ and hence the sequence $y(r)_{r=0,1,\ldots}$ finitely converges, $y(r) \to z_2$ say.

We show that (z_1, z_2) is stable and thus a solution to $Ax = By$. This is true since, by (7) and (8),

$$\pi(z_2) = \pi(y(s)) = x(s+1) = z_1 \text{ and } \psi(z_1) = \psi(x(s)) = y(s) = z_2.$$

Therefore $Az_1 = Bz_2$ which implies $A'z_1 = B'z_1$ as required. $\square$

Theorem 3.17 *If $A', B' \in \mathbb{R}^{m \times n}$ and Algorithm GEN-INT-TSS starts with $x(0) \in \mathbb{Z}^n$ then it will terminate after at most*

$$(n-1)(1 + \gamma^{\#} \otimes A^{\#} \otimes A \otimes \gamma)$$

iterations where

$$A = \begin{pmatrix} A' \\ B' \end{pmatrix}, \gamma = x(0).$$

Proof. Follows the lines of that of Theorem 3.8. $\square$

We can also argue as before that if A is finite (if A' and B' are finite) and we start Algorithm GEN-INT-TSS with a vector $\lfloor z \rfloor$ where z is a finite subeigenvector of $C = A^{\#} \otimes A$ then the complexity is

$$\mathcal{O}(m'n'(n'+k')K(A))$$

where $m' = 2m, n' = n, k' = m$.

Corollary 3.18 *Algorithm GEN-INT-TSS terminates after*

$$\mathcal{O}(K(A'|B')(mn(m+n)))$$

operations, if applied to instances where both of the matrices A', B' are finite.

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