Results on Integer and Extended Integer Solutions to Max-linear Systems

Peter Butkovič*
p.butkovic@bham.ac.uk

Marie MacCaig
m.maccaig.maths@outlook.com

Abstract

This paper follows on from, and complements the material in, the thesis [8], which focuses on integer solutions to systems of equations in max-algebra. In this paper we present some results regarding the descriptions of the set of integer solutions to the integer eigenproblem, integer image problem and the two-sided system. Additionally, we extend a number of the results in [8] to deal with the set of extended integer solutions; specifically we show that many of the special cases found for integer solutions can be modified to allow a description of all extended integer solutions to be obtained in strongly polynomial time.

1 Introduction

In the max-algebra, for $a, b \in \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\}$ we define $a \oplus b = \max(a, b)$, $a \otimes b = a + b$ and extend the pair $(\oplus, \otimes)$ to matrices and vectors in the same way as in linear algebra, that is (assuming compatibility of sizes)

$$\begin{aligned}(\alpha \otimes A)_{ij} &= \alpha \otimes a_{ij},\\ (A \oplus B)_{ij} &= a_{ij} \oplus b_{ij}, \text{ and}\\ (A \otimes B)_{ij} &= \bigoplus_k a_{ik} \otimes b_{kj}.\end{aligned}$$

Except for complexity arguments, all multiplications in this paper are in max-algebra and we will usually omit the $\otimes$ symbol. Note that ε is used to denote $-\infty$ as well as any vector or matrix whose every entry is $-\infty$. A vector/matrix whose every entry belongs to $\mathbb{R}$ is called *finite*. By *integer*

*Supported by EPSRC grant EP/J00829X/1.

solution we mean a finite integer vector. An *extended integer solution*, often written $\overline{\mathbb{Z}}$-solution, is a vector with entries from $\overline{\mathbb{Z}} := \mathbb{Z} \cup \{\varepsilon\}$.

We first detail the various max-algebraic systems which will be considered in this paper.

One-sided systems in max-algebra include the one-sided inequality, $A \otimes x \leq b$, and the one-sided equality, $A \otimes x = b$. Both are defined for $A \in \overline{\mathbb{R}}^{m\times n}$ and $b \in \overline{\mathbb{R}}^{m}$. The max-algebraic eigenproblem and subeigenproblem are $A \otimes x = \lambda \otimes x$ and $A \otimes x \leq \lambda \otimes x$ respectively where $A \in \overline{\mathbb{R}}^{n\times n}$ and $\lambda \in \mathbb{R}$. As usual, a vector $x \neq \varepsilon$ satisfying $A \otimes x = \lambda \otimes x$ $[A \otimes x \leq \lambda \otimes x]$ will be called an *eigenvector [subeigenvector]* of A with respect to *eigenvalue [subeigenvalue]* λ.

The integer image, and extended integer image, problems are the problems of determining whether there exists an integer/extended integer point in the column span of a matrix.

A *two-sided max-linear system* (TSS) is of the form,

$$A \otimes x \oplus c = B \otimes x \oplus d$$

where $A, B \in \overline{\mathbb{R}}^{m\times n}$ and $c, d \in \overline{\mathbb{R}}^{m}$. If $c = d = \varepsilon$, then we say the system is *homogeneous*, otherwise it is called *nonhomogeneous*. Nonhomogeneous systems can be transformed to homogeneous systems [1]. If $B \in \overline{\mathbb{R}}^{m\times k}$, a system of the form

$$A \otimes x = B \otimes y$$

is called a *system with separated variables*.

If $f \in \overline{\mathbb{R}}^{n}$ then the function $f(x) = f^T \otimes x$ is called *max-linear*. *Max-linear programming problems* seek to minimise or maximise a max-linear function subject to constraints given by one or two sided systems. Note that, unlike in linear programming, there is no obvious way of converting maximisation of max-linear functions to minimisation of the same type of functions and vice versa.

The thesis [8] and related papers [3, 4, 5] deal with finding integer solutions to systems of equations in the max-algebra. In particular one-sided systems and the integer subeigenproblem are proved to be strongly polynomially solvable, and pseudopolynomial algorithms are found for the existence of an integer solution to the eigenproblem, the integer image problem and the two-sided system. Additionally these previous papers present special cases for which a full description of all integer solutions to each of these problems can be described in strongly polynomial time. At the time of writing it is unknown whether any of these problems can be solved in polynomial time.

In Section 2 we present the necessary background results. Section 3 contains a description of the integer eigenvectors of a matrix based on the non-critical components. In Section 4 we present a description of the integer image set as the intersection of a number of sets of integer subeigenvectors. Section 5 contains a description of the set of integer solutions to a two-sided system as the intersection of a number of sets of integer solutions to one sided systems.

Sections 6-11 deal with extended integer solutions. A full description of all $\overline{\mathbb{Z}}$-solutions to the one sided systems and the subeigenproblem can be achieved in strongly polynomial time by the results in sections 6 and 8. In Sections 7, 9, 10 and 11 we present special cases for which we can find extended integer solutions to the eigenproblem, image problem, two-sided systems and the max-linear programming problem in strongly polynomial time.

2 Preliminaries

We will use the following standard notation. For positive integers m, n, k we denote $M = \{1, ..., m\}$, $N = \{1, ..., n\}$ and $K = \{1, ..., k\}$. A vector whose j^{th} component is zero and every other component is ε will be called a *unit vector* and denoted e_j. The zero vector, of appropriate size, is denoted $\mathbf{0}$. If a matrix has no ε rows (columns) then it is called *row* (*column*) $\mathbb{R}$-*astic* and it is called *doubly* $\mathbb{R}$-*astic* if it is both row and column $\mathbb{R}$-astic. Note that the vector Ax is sometimes called a *max combination* of the columns of A.

Given a solution x to $Ax = b$, we say that a position (i, j) is *active* with respect to x if and only if $a_{ij} + x_j = b_i$, it is called *inactive* otherwise. Further, an element/entry a_{ij} of A is *active* if and only if the position (i, j) is active. Related to this definition, we call a column A_j *active* exactly when it contains an active entry. We also say that a component x_j of x is *active* in the equation $Ax = Bx$ if and only if there exists i such that either $a_{ij} + x_j = (Bx)_i$ or $(Ax)_i = b_{ij} + x_j$.

For $a \in \mathbb{R}$ the *fractional part* of a is $fr(a) := a - \lfloor a \rfloor$. For a matrix $A \in \overline{\mathbb{R}}^{m\times n}$ we use $\lfloor A \rfloor$ ($\lceil A \rceil$) to denote the matrix with (i, j) entry equal to $\lfloor a_{ij} \rfloor$ ($\lceil a_{ij} \rceil$) and similarly for vectors. We define $\lfloor \varepsilon \rfloor = \varepsilon = \lceil \varepsilon \rceil = fr(\varepsilon)$.

If $A = (a_{ij}) \in \overline{\mathbb{R}}^{n\times n}$, then $\lambda(A)$ denotes the *maximum cycle mean*, that is,

$$\lambda(A) = \max \left\{ \frac{a_{i_1 i_2} + ... + a_{i_k i_1}}{k} : (i_1, ..., i_k) \text{ is a cycle}, k = 1, ..., n \right\}$$

where $\max(\emptyset) = \varepsilon$ by definition. Note that this definition is independent of whether we allow cycles to contain repeated nodes [1]. The maximum cycle mean can be calculated in $\mathcal{O}(n^3)$ time [1].

By D_A we mean the weighted digraph (N, E, w) where $E = \{(i,j) : a_{ij} > \varepsilon\}$ and the weight of the edge (i,j) is a_{ij}. A is called *irreducible* if D_A is strongly connected (that is, if there is an $i - j$ path in D_A for any i and j). If σ is a cycle in D_A then we denote its weight by $w(\sigma, A)$, and its length by $l(\sigma)$. A cycle is called *critical* if

$$\frac{w(\sigma, A)}{l(\sigma)} = \lambda(A).$$

We denote by $N_C(A)$ the set of *critical nodes*, that is any node $i \in N$ which is on a critical cycle.

If $\lambda(A) = 0$, then we say that A is *definite*. If moreover $a_{ii} = 0$ for all $i \in N$ then A is called *strongly definite*.

For matrices with $\lambda(A) \leq 0$ we define

$$A^+ = A \oplus A^2 \oplus ... \oplus A^n \text{ and}$$
$$A^* = I \oplus A \oplus ... \oplus A^{n-1}.$$

. Using the Floyd-Warshall algorithm; see, e.g., [1], A^* can be calculated in $\mathcal{O}(n^3)$ time.

An $n \times n$ matrix is called *diagonal*, written $diag(d_1, ..., d_n) = diag(d)$, if its diagonal entries are $d_1, ..., d_n \in \mathbb{R}$ and off diagonal entries are ε. We use I to denote the identity matrix, $I = diag(0, ..., 0)$, of appropriate size.

If $a, b \in \overline{\overline{\mathbb{R}}} := \overline{\mathbb{R}} \cup \{+\infty\}$, then we define $a \oplus' b := \min(a,b)$. Moreover $a \otimes' b := a + b$ exactly when at least one of a, b is finite, otherwise

$$(-\infty) \otimes' (+\infty) := +\infty \text{ and } (+\infty) \otimes' (-\infty) := +\infty.$$

This differs from max-multiplication where

$$(-\infty) \otimes (+\infty) := -\infty \text{ and } (+\infty) \otimes (-\infty) := -\infty.$$

The pair of operations $(\oplus', \otimes')$ is extended to matrices and vectors similarly as $(\oplus, \otimes)$.

Note that, for $\alpha \in \mathbb{R}$, α^{-1} is simply $-\alpha$ in conventional notation. For a vector γ we use $\gamma^{(-1)}$ to mean the vector with entries γ_i^{-1}. Similarly, for $A \in \overline{\overline{\mathbb{R}}}^{m \times n}$, $A^{(-1)} = (a_{ij}^{-1})$. For $A \in \overline{\overline{\mathbb{R}}}^{m \times n}$ we define $A^{\#} = -A^T \in \overline{\overline{\mathbb{R}}}^{n \times m}$. If $A = (a_{ij}) \in \overline{\mathbb{R}}^{m \times n}$ then A_j stands for the j^{th} column of A.

A description of all solutions to the one-sided equality and inequality was one of the first results proved in max-algebra. If $A \in \overline{\mathbb{R}}^{m\times n}$ and $b \in \mathbb{R}^m$, then, for all $j \in N$, define

$$M_j(A,b) = \{t \in M : a_{tj} \otimes b_t^{-1} = \max_i a_{ij} \otimes b_i^{-1}\}.$$

Proposition 2.1 *[1, 6, 7] Let $A \in \overline{\mathbb{R}}^{m\times n}$, $b \in \mathbb{R}^m$ and $\bar{x} = A^{\#} \otimes' b$.*

(i) $Ax \le b \Leftrightarrow x \le \bar{x}$

(ii) $Ax = b \Leftrightarrow x \le \bar{x}$ and

$$\bigcup_{j: x_j = \bar{x}_j} M_j(A,b) = M.$$

This can immediately be adapted to provide a simple description of all integer solutions to these systems, as stated below.

Proposition 2.2 *[3, 8] Let $A \in \overline{\mathbb{R}}^{m\times n}$, $b \in \mathbb{R}^m$ and $\bar{x} = A^{\#} \otimes' b$.*

(i) An integer solution to $Ax \le b$ exists if and only if $\bar{x}$ is finite. If an integer solution exists, then all integer solutions can be described as the integer vectors x satisfying $x \le \bar{x}$.

(ii) An integer solution to $Ax = b$ exists if and only if

$$\bigcup_{j: \bar{x}_j \in \mathbb{Z}} M_j(A,b) = M.$$

If an integer solution exists, then all integer solutions can be described as the integer vectors x satisfying $x \le \bar{x}$ with

$$\bigcup_{j: x_j = \bar{x}_j} M_j(A,b) = M.$$

The integer subeigenproblem can also be solved completely in strongly polynomial time. We denote the set of integer subeigenvectors by $IV^*(A,\lambda) = \{x \in \mathbb{Z}^n : Ax \le \lambda x\}$.

Theorem 2.3 *[3, 8] Let $A \in \overline{\mathbb{R}}^{n\times n}$, $\lambda \in \mathbb{R}$.*

(i) $IV^*(A,\lambda) \ne \emptyset$ if and only if

$$\lambda(\lceil \lambda^{-1} A \rceil) \le 0.$$

(ii) If $IV^*(A,\lambda) \ne \emptyset$, then

$$IV^*(A,\lambda) = \{\lceil \lambda^{-1} A \rceil^* z : z \in \mathbb{Z}^n\}.$$

We will need the following known results.

Proposition 2.4 *[7] Let $A \in \overline{\mathbb{R}}^{n\times n}$, $\lambda(A) > \varepsilon$. Then $(\exists x \in \mathbb{R}^n) Ax = \lambda x \Rightarrow \lambda = \lambda(A)$.*

Lemma 2.5 *[1] If $N_C(A) = N$ then, for all $\lambda \in \mathbb{R}$,*

$$Ax = \lambda x \Leftrightarrow Ax \leq \lambda x.$$

Lemma 2.6 *[1] Let $A, B \in \overline{\mathbb{R}}^{n\times n}$ with $B = P^{-1}AP$ where P is a permutation matrix. Then A and B have the same subeigenvalues.*

3 Describing Integer Eigenvectors Using Non-critical Indices

The set of integer eigenvectors is $IV(A, \lambda) = \{x \in \mathbb{Z}^n : Ax = \lambda x\}$. From Proposition 2.4 the only eigenvalue corresponding to finite eigenvectors is $\lambda(A)$. Thus when considering integer eigenvectors, we only have to find solutions to $Ax = \lambda(A)x$. Observe that this means that we can assume without loss of generality that the matrix is definite.

Given any definite matrix $A \in \overline{\mathbb{R}}^{n\times n}$, we can perform a series of simultaneous permutations of the rows/columns so that A has the form

$$A = \begin{pmatrix} A[C] & B \\ D & A[N-C] \end{pmatrix} \tag{1}$$

where $C = \{i \in N : i \text{ is critical}\}$. Then $\lambda(A[C]) = 0$ and every node in $A[C]$ is critical, so we know that $IV(A[C], 0) = IV^*(A[C], 0)$ by Lemma 2.5. Also $\lambda(A[N-C]) < 0$ and hence $IV(A[N-C], 0) = \emptyset$ by Proposition 2.4.

Proposition 3.1 *Let $A \in \overline{\mathbb{R}}^{n\times n}$ be a definite matrix written as in* (1)*. If $x \in IV(A, 0)$ then*

(i) $x[C] \in IV(A[C], 0) = IV^*(A[C], 0)$ and

(ii) $x[N-C] \in IV^*(A[N-C], 0) - IV(A[N-C], 0) = IV^*(A[N-C], 0)$.

Proof. Suppose $Ax = x, x \in \mathbb{Z}^n$. Clearly $A[C]x[C] \leq x[C]$ and $A[N-C]x[N-C] \leq x[N-C]$. Now (i) follows from Lemma 2.5 and (ii) holds since $IV(A[N-C], 0) = \emptyset$. $\square$

Suppose $z \in IV^*(A[C],0)$. We would like to determine whether z can be extended into an integer eigenvector of A.

Proposition 3.2 *Let A be as defined in* (1) *and $z \in IV^*(A[C],0)$. Then there exists an integer vector $y \in \mathbb{Z}^{n-c}$ such that $A(z^T, y^T)^T = (z^T, y^T)$ if and only if*

$$\begin{aligned} &BA[N-C]^*Dz \le z \text{ and} \\ &A[N-C]^*Dz \in \mathbb{Z}^{n-c}. \end{aligned}$$

Proof. There exists such a y if and only if

$$\begin{aligned} &By \le z \text{ and} \\ &Dz \oplus A[N-C]y = y. \end{aligned}$$

Now, for $Y \in \overline{\mathbb{R}}^{n\times n}, \omega \in \overline{\mathbb{R}}^n$ it is known [2] that if $\lambda(Y) < 0$ then $u = Y^*\omega$ is the unique solution to $\omega \oplus Yu = u$. Together these give the result. $\square$

We construct some necessary conditions for when z can be extended.

Proposition 3.3 *Let $A \in \overline{\mathbb{R}}^{n\times n}$ be a definite matrix written as in* (1)*. If $IV(A,0) \neq \emptyset$, then $\lambda(\lceil BA[N-C]^*D\rceil) \le 0$.*

Proof. For any $x \in IV(A,0)$ we have $BA[N-C]^*Dx[C] \le x[C]$. Therefore $BA[N-C]^*D$ has an integer subeigenvector. The result follows from Theorem 2.3. $\square$

Using this we can describe an (inefficient) method to search for integer eigenvectors.

Proposition 3.4 *Fix $u \in \mathbb{Z}^n$. Let $z = \lceil A[C]\rceil^*u$ and $y = A[N-C]^*Dz$. Then either $x = (z^T, y^T)^T \in IV(A,0)$ or $(\forall x \in IV(A,0))x[C] \neq z$. Further, all integer eigenvectors can be found in this way.*

Corollary 3.5 *Let $A \in \overline{\mathbb{R}}^{n\times n}$ be split into critical and non critical blocks as in* (1)*. Then*

$$IV(A,0) = IIm^*\left(\begin{pmatrix} \lceil A[C]\rceil^* \\ A[N-C]^*D\lceil A[C]\rceil^* \end{pmatrix}\right).$$

Proof. Any $x \in IV(A,0)$ has the form $(z^T, y^T)^T$ where $z = \lceil A[C]\rceil^*u$ and $y = A[N-C]^*Dz = A[N-C]^*D\lceil A[C]\rceil^*u$ for some $u \in \mathbb{Z}^c$. $\square$

Corollary 3.5 therefore implies that the integer eigenproblem for an $n \times n$ matrix can be solved by finding the integer image of the $(n-c) \times c$ matrix $A[N-C]^* D \lceil A[C] \rceil$.

4 Using Subeigenvectors to Describe the Integer Image Set

We consider the set of integer images in the max algebra, denoted

$$IIm(A) := \{z \in \mathbb{Z}^m : (\exists x \in \overline{\mathbb{R}}^n) Ax = z\}.$$

We show that, for $A \in \mathbb{R}^{m \times n}$,

$$z \in IIm(A) \Leftrightarrow z^{(-1)} \in \bigcup_i IV^*(M^{(i)}, 0)$$

where the matrices $M^{(i)}$ are of size $m \times m$ and can be determined from A. Recall that $z^{(-1)} = (z_j^{-1})$.

Our problem is to find $z \in \mathbb{Z}^m$ such that there exists $x \in \overline{\mathbb{R}}^n$ with $A \otimes x = z^{(-1)}$. Equivalently, such that $diag(z_1, ..., z_m) \otimes A \otimes x = 0$. Now, from Proposition 2.2, we know that a solution to $B \otimes x = 0$ exists if and only if $\bar{x} = B^{\#} \otimes' 0$ is a solution. Therefore we can reinterpret our problem as trying to find integers $z_1, ..., z_m$ such that

$$(diag(z_1, ..., z_m) \otimes A) \otimes [(diag(z_1, ..., z_m) \otimes A)^{\#} \otimes' 0] = 0. \quad (2)$$

Rewriting this we get that z satisfies,

$$(\forall i \in M) \max_{t \in N} \left(\min_{j \in M} (a_{it} - a_{jt} + z_i - z_j) \right) = 0. \quad (3)$$

4.1 Description of the subeigenspaces

Proposition 4.1 *Let $A \in \mathbb{R}^{m \times n}$. Then $z^{(-1)} \in IIm(A)$ if and only if*

$$(\forall i \in M)(\exists t \in N)(\forall j \in M) \ z_i - z_j \geq a_{jt} - a_{it}.$$

Proof. We have that, $z^{(-1)} \in IIm(A)$ is equivalent to (3).

Observe that, for any $i \in M, t \in N$ we have

$$\min_{j \in M} (a_{it} - a_{jt} + z_i - z_j) \leq 0$$

since when $j = i$ the term $a_{it} - a_{jt} + z_i - z_j = 0$. To ensure that the maximum of a set of non positive numbers is equal to zero we require that at least one of them is equal to zero. Thus (3) is equivalent to

$$
\begin{aligned}
&(\forall i \in M)(\exists t \in N)\ \min_{j \in M}(a_{it} - a_{jt} + z_i - z_j) = 0 \\
\Leftrightarrow &(\forall i \in M)(\exists t \in N)(\forall j \in M)\ a_{it} - a_{jt} + z_i - z_j \geq 0.
\end{aligned}
$$

□

Definition 4.2 *For $k_1, k_2, ..., k_m \in N$ define $M^{(k_1,...,k_m)} \in \mathbb{R}^{m \times m}$ by*

$$
(M^{(k_1,...,k_m)})_{ij} = \begin{cases} 0, & \text{if } i = j; \\ a_{jk_i} - a_{ik_i}, & \text{if } i \neq j. \end{cases}
$$

Theorem 4.3 *Let $A \in \mathbb{R}^{m \times n}$.*

$$
IIm(A) = \left\{ z : z^{(-1)} \in \bigcup IV^*(M^{(k_1,...,k_m)}, 0) \text{ where } (k_1, k_2, ..., k_m) \in N^m \right\}.
$$

Proof. From Proposition 4.1 we know that $z^{(-1)} \in IIm(A)$ if and only if there exists a set $(k_1, ..., k_m)$ of indices such that k_i satisfies

$$
(\forall j \in M)\ z_i - z_j \geq a_{jk_i} - a_{ik_i}.
$$

These are exactly the conditions for z to be a subeigenvector of the matrix $M^{(k_1,...,k_m)}$. □

Remark 4.4 *Currently the number of matrices $M^{(k_1,...,k_m)}$ we are constructing is n^m. It can be shown that, for each i, we only need to consider indices k_i such that there exists j with $a_{jk_i} - a_{ik_i} = \min_{t \in N}(a_{jt} - a_{it})$. However, for each i, there could be up to $n - 1$ of these indices so we are still left with $(n-1)^m$ matrices to check.*

We can construct a matrix B such that $IIm(A) \subseteq \{z : z^{(-1)} \in IV^*(B, 0)\}$. We detail this in the proposition below.

Proposition 4.5 *Let $A \in \mathbb{R}^{m \times n}$. Define $B = (b_{ij}) \in \mathbb{R}^{m \times m}$ by*

$$
b_{ij} = \begin{cases} 0, & \text{if } i = j; \\ \min\limits_{t \in N}(a_{jt} - a_{it}), & \text{if } i \neq j. \end{cases} \tag{4}
$$

Then

$$
\lambda(\lceil B \rceil) > 0 \Rightarrow IIm(A) = \emptyset.
$$

Proof. From Theorem 4.3, it is sufficient to show that, for any set $(k_1, ..., k_m) \in N^m$,

$$IV^*(M^{(k_1,...,k_m)}, 0) \subseteq IV^*(B, 0).$$

Then, by Theorem 2.3,

$$\lambda(\lceil B \rceil) > 0 \Rightarrow IV^*(B, 0) = \emptyset.$$

Take $z \in IV^*(M^{(k_1,...,k_m)}, 0)$. For each $i, j \in M,\ i \neq j$,

$$z_i - z_j \geq a_{jk_i} - a_{ik_i} \geq \min_{t \in N}(a_{jt} - a_{it}) = b_{ij}.$$

Thus $z \in IV^*(B, 0)$. $\square$

Example 4.6

$$A = \begin{pmatrix} 0 & -1.2 \\ 0.9 & 0 \\ 2.2 & -1.1 \end{pmatrix}.$$

Observe that $IIm(A) = \emptyset$ since each column of the matrix contains entries with different fractional parts, and therefore at most one entry per column can be active with respect to any integer image, meaning there are not enough candidates for active entries.

The 3×2 table below describes all the pairs of possible conditions that z must satisfy to be in the image of A. In each row of the table we require the equations in at least one cell to be satisfied in order to have an integer image.

$z_1 - z_2 \geq 0.9$ $z_1 - z_3 \geq 2.2$	$z_1 - z_2 \geq 1.2$ $z_1 - z_3 \geq 0.1$
$z_2 - z_1 \geq -0.9$ $z_2 - z_3 \geq 1.3$	$z_2 - z_1 \geq -1.2$ $z_2 - z_3 \geq -1.1$
$z_3 - z_1 \geq -2.2$ $z_3 - z_2 \geq -1.3$	$z_3 - z_1 \geq -0.1$ $z_3 - z_2 \geq 1.1$

The table is related to the matrices $M^{(r,s,t)}$ as follows. Suppose we choose the first cell in row 1, the second cell in row 2, and the first cell in row 3. Then we are looking for z that satisfies $z_1 - z_2 \geq 0.9$, $z_1 - z_3 \geq 2.2$, $z_2 - z_1 \geq -1.2$, $z_2 - z_3 \geq -1.1$, $z_3 - z_1 \geq -2.2$ and $z_3 - z_2 \geq -1.3$. These are exactly the conditions for $M^{(1,2,1)} z \leq z$.

Now

$$B = \begin{pmatrix} 0 & 0.9 & 0.1 \\ -1.2 & 0 & -1.1 \\ -2.2 & -1.3 & 0 \end{pmatrix}, \ \lceil B \rceil = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & -1 \\ -2 & -1 & 0 \end{pmatrix}.$$

We calculate that $\lambda(\lceil B \rceil) = 0$ *and can conclude that* $IV^*(B,0) \neq \emptyset$. *Hence* $IIm(A) \neq IV^*(B,0)$.

Further in this example we claim that the only sets of indices (k_1, k_2, k_3) *we have to consider are* $(1,2,1)$ *and* $(2,2,1)$. *This is since for all* $i, t \in \{1,2\}$ *any integer subeigenvectors of the matrix* $M^{(i,1,t)}$ *will also be subeigenvectors of the matrix* $M^{(i,2,t)}$. *To see this consider the second row of cells in the table. If* z *satisfies* $M^{(i,1,t)}z \leq z$ *then we know that* $z_2 - z_1 \geq -0.9$ *and* $z_2 - z_3 \geq 1.3$ *since we are choosing to satisfy the first cell in the second row. Clearly though we are also satisfying* $z_2 - z_1 \geq -1.2$ *and* $z_2 - z_3 \geq -1.1$ *which are the pair of conditions in the second cell of this row and hence* $M^{(i,2,t)}z \leq z$ *also. The key here was that the minimum lower bound for both* $z_2 - z_1$ *and* $z_2 - z_3$ *can be found in cell (2,2) of the table.*

A similar argument shows that $(\forall i, j \in \{1,2\})\ IV^*(M^{(i,j,2)}) \subseteq IV^*(M^{(i,j,1)})$.

Therefore we conclude that

$$\{z^{(-1)} : z \in IIm(A)\} = IV^*(M^{(1,2,1)}, 0) \cup IV^*(M^{(2,2,1)}, 0)$$

where

$$M^{(1,2,1)} = \begin{pmatrix} 0 & 0.9 & 2.2 \\ -1.2 & 0 & -1.1 \\ -2.2 & -1.3 & 0 \end{pmatrix} \text{ and } M^{(2,2,1)} = \begin{pmatrix} 0 & 1.2 & 0.1 \\ -1.2 & 0 & -1.1 \\ -2.2 & -1.3 & 0 \end{pmatrix}.$$

Finally $\lambda(\lceil M^{(1,2,1)} \rceil) > 0$ *and* $\lambda(\lceil M^{(2,2,1)} \rceil) > 0$ *by checking cycles* $(1,3,1)$ *and* $(1,2,1)$ *respectively, confirming that* $IIm(A) = \emptyset$ *as expected.*

The above example demonstrated that if there is a row i in the table containing a cell $s \in N$ such that

$$(\forall j \in M, j \neq i) a_{js} - a_{is} = \min_{t \in N} a_{jt} - a_{it}$$

then, for any fixed choice of $k_1, .., k_{i-1}, k_{i+1}, .., k_m \in N$,

$$(\forall j \in M, j \neq i) IV^*(M^{(k_1,...,k_{i-1},j,k_{i+1},...,k_m)}, 0) \subseteq IV^*(M^{(k_1,...,k_{i-1},s,k_{i+1},...,k_m)}, 0).$$

Hence in this case we only have to consider $k_i = s$ when looking at which matrices $M^{(k_1,...,k_m)}, (k_1, ..., k_m) \in N^m$ to check.

Corollary 4.7 *Let* $A \in \mathbb{R}^{2 \times n}$ *and* B *be as defined in* (4). *Then* $IIm(A) = \{z : z^{(-1)} \in IV^*(B, 0)\}$.

Proof. Since $m = 2$ we are looking for $z \in \mathbb{Z}^2$ and so the table of possible inequalities will contain 2 rows of n cells each containing only one inequality. This means that there will always be some cell containing the minimum lower bound for all the inequalities considered in that row.

Formally let $s_1, s_2 \in N$ be such that $a_{2s_1} - a_{1s_1} = \min_{t \in N} a_{2t} - a_{1t}$ and $a_{1s_2} - a_{2s_2} = \min_{t \in N} a_{1t} - a_{2t}$. Then, for fixed $k_2 \in M$,

$$(\forall k_1 \in M) IV^*(M^{(k_1,k_2)}, 0) \subseteq IV^*(M^{(s_1,k_2)}, 0)$$

and for fixed $k_1 \in M$,

$$(\forall k_2 \in M) IV^*(M^{(k_1,k_2)}, 0) \subseteq IV^*(M^{(k_1,s_2)}, 0).$$

Hence $IIm(A) = \{z : z^{(-1)} \in IV^*(M^{(s_1,s_2)}, 0)\}$ and $M^{(s_1,s_2)} = B$. □

5 A description of integer solutions to TSS

Here we investigate how to use one-sided equalities (for which we can describe integer solutions in strongly polynomial time) to describe the set of integer solutions to a two-sided system.

For $A, B \in \mathbb{R}^{m \times n}$, we write the TSS $Ax = Bx$ as a simultaneous system of one-sided equalities. Recall that $\mathbf{0}$ denotes the zero vector. We use $A_{i\bullet}$ to denote the i^{th} row of A. In a slight abuse of notation we write $x = (y_1, \leq y_2, y_3)^T$ to mean any vector x with $x_1 = y_1, x_2 \leq y_2$ and $x_3 = y_3$. For each row i we define

$$C_i := \begin{pmatrix} A_{i\bullet} \\ B_{i\bullet} \end{pmatrix} \in \mathbb{R}^{2 \times n}.$$

Proposition 5.1 *Let $A, B \in \mathbb{R}^{m \times n}$. Then $Ax = Bx$ if and only if*

$$(\forall i \in M)(\exists \alpha_i \in \mathbb{R}) C_i x = \alpha_i \mathbf{0}.$$

Proof. Clear since, for $\alpha = (\alpha_i) \in \mathbb{R}^m$, $Ax = Bx \Leftrightarrow Ax = \alpha = Bx$. □

For matrices $A, B, C \in \mathbb{R}^m \times n$ and $\alpha \in \mathbb{R}$ we define,

$$\begin{aligned} IS(C, \alpha) &:= \{x \in \mathbb{Z}^n : Cx = \alpha \mathbf{0}, \alpha \in \mathbb{R}\} \text{ and} \\ IS(A, B) &:= \{z \in \mathbb{Z}^n : Az = Bz\}. \end{aligned}$$

We now consider how to find solutions to the system of one-sided equalities described in Proposition 5.1.

Example 5.2 *Find $x \in \mathbb{Z}^3$, $p \in \mathbb{R}^3$ such that*

$$C_1 x = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix} x = \begin{pmatrix} p_1 \\ p_1 \end{pmatrix}, \tag{5}$$

$$C_2 x = \begin{pmatrix} 0 & 0.2 & 0.5 \\ -0.5 & 1 & -1.1 \end{pmatrix} x = \begin{pmatrix} p_2 \\ p_2 \end{pmatrix}, \tag{6}$$

$$C_3 x = \begin{pmatrix} 0.5 & 0 & 0.5 \\ 1 & -0.5 & 0.5 \end{pmatrix} x = \begin{pmatrix} p_3 \\ p_3 \end{pmatrix}. \tag{7}$$

From (5), it is clear that $fr(p_1) = 0$ since $x \in \mathbb{Z}^n$. Thus we look for $\alpha_1 \in \mathbb{Z}$ such that $p_1 = \alpha_1$ and

$$\begin{pmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix} (\alpha_1^{-1} x) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

From Proposition 2.2, we calculate that $\alpha_1^{-1} x \leq (0, -1, 1)^T$ with equality for either rows 1 and 2 or rows 2 and 3. Therefore,

$$x \in IS(C_1, p_1) = \{\beta(0, -1, \leq -1)^T : \beta \in \mathbb{Z}\} \cup \{\beta(\leq 0, -1, -1)^T : \beta \in \mathbb{Z}\}.$$

Now consider (6). Trivially, $fr(p_2) \in \{0, 0.2, 0.5, 0.9\}$. But from Proposition 2.2, $p_2^{-1} x \leq (0, -1, -0.5)^T$ with equality for either rows 1 and 2 or rows 2 and 3. Since $x \in \mathbb{Z}^3$ this implies $p_1 \in \mathbb{Z}$. Therefore

$$x \in IS(C_2, p_2) = \{\beta(0, -1, \leq -1)^T : \beta \in \mathbb{Z}\}.$$

From (7), using a similar method we see that $fr(p_3) = 0$ or 0.5, and that

$$\begin{aligned} x \in IS(C_3, p_3) =& IS(C_3, 0) \cup IS(C_3, 0.5) \\ =& \{\beta(-1, 0, \leq -1)^T : \beta \in \mathbb{Z}\} \cup \{\beta(\leq -1, \leq 0, 0)^T : \beta \in \mathbb{Z}\}. \end{aligned}$$

Finally, let

$$A = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 0.2 & 0.5 \\ 0.5 & 0 & 0.5 \end{pmatrix} \text{ and } B = \begin{pmatrix} -1 & 1 & 0 \\ -0.5 & 1 & -1.1 \\ 1 & -0.5 & 0.5 \end{pmatrix}.$$

Then $Ax = Bx$ if and only if

$$\begin{aligned} x \in IS(A,B) &= IS(C_1,p_1) \cap IS(C_2,p_2) \cap IS(C_3,p_3) \\ &= \left(\left\{ z_1 \begin{pmatrix} 0 \\ -1 \\ \leq -1 \end{pmatrix} : z_1 \in \mathbb{Z} \right\} \cup \left\{ z_2 \begin{pmatrix} \leq 0 \\ -1 \\ -1 \end{pmatrix} : z_2 \in \mathbb{Z} \right\} \right) \\ &\cap \left\{ z_3 \begin{pmatrix} 0 \\ -1 \\ \leq -1 \end{pmatrix} : z_3 \in \mathbb{Z} \right\} \\ &\cap \left(\left\{ z_4 \begin{pmatrix} -1 \\ 0 \\ \leq -1 \end{pmatrix} : z_4 \in \mathbb{Z} \right\} \cup \left\{ z_5 \begin{pmatrix} \leq -1 \\ \leq 0 \\ 0 \end{pmatrix} : z_5 \in \mathbb{Z} \right\} \right). \end{aligned}$$

In this case we see that $IS(A,B) = \emptyset$ since any $x \in IS(C_2,p_2)$ has $x_1 = 0$ which implies that $x \notin IS(C_3,p_3)$.

The ideas outlined in this example can be applied to any system of the form $C(\alpha^{-1}x) = \mathbf{0}$.

Proposition 5.3 *Let $A, B \in \mathbb{R}^{m\times n}$. Then, for all $i \in M$ we can describe $IS(C_i,p_i)$ for all valid choices of p_i in $\mathcal{O}(mn^3)$ time.*

Proof. For each of the m matrices there are at most n choices for the fractional part of p_i and for each choice at most n^2 pairs of active entries. □

Although the set of integer solutions to each of the one-sided systems can be described in strongly polynomial time, it remains open to determine whether we can efficiently find a point in the intersection of these one-sided solution spaces.

6 Extended integer solutions to one-sided systems

For given $A \in \overline{\mathbb{R}}^{m\times n}, b \in \overline{\mathbb{R}}^m$, we study the questions of whether there exists a vector $x \in \overline{\mathbb{Z}}^n$, $x \neq \varepsilon$, such that $Ax = b$. We call such a vector a $\overline{\mathbb{Z}}$*-solution*. We define

$$\overline{IS}(A,b) = \{x \in \overline{\mathbb{Z}}^n : Ax = b\}.$$

Note that

$$\overline{IS}(A,\varepsilon) = \{x \in \overline{\mathbb{Z}}^n : x_j = \varepsilon \text{ if } A_j \neq \varepsilon, j \in N\}$$

and, for $b \neq \varepsilon$, $\overline{IS}(\varepsilon, b) = \emptyset$. Hence we assume that $A \neq \varepsilon$ and $b \neq \varepsilon$.

Suppose $b_i = \varepsilon$ for some $i \in M$. Let $F_i(A) = \{j \in N : a_{ij} > \varepsilon\}$. Then $x_j = \varepsilon$ for all $j \in F_i(A)$ and we can remove the i^{th} equation from the system as well as any column A_j with $j \in F_i(A)$. Therefore we may assume without loss of generality that b is finite. We also assume without loss of generality that A is doubly $\mathbb{R}$-astic.

Proposition 6.1 *Suppose $A \in \overline{\mathbb{R}}^{m \times n}$ is doubly $\mathbb{R}$-astic, $b \in \mathbb{R}^m$. A $\overline{\mathbb{Z}}$-solution to $Ax = b$ exists if and only if an integer solution exists.*

Proof. The sufficient direction is obvious since $\mathbb{Z} \subseteq \overline{\mathbb{Z}}$. So assume that x is a $\overline{\mathbb{Z}}$-solution to $Ax = b$. Since b is finite,

$$(\forall i \in M)(\exists j \in N) a_{ij} x_j > \varepsilon.$$

Then $A'x' = b$ where A' is obtained from A by removing columns $A_j, j \in E(x) = \{j \in N : x_j = \varepsilon\}$ and x' is obtained from x similarly. In particular, no column j of A with $j \in E(x)$ is active and therefore we can replace the ε entries in x with small enough integers to obtain a vector $x'' \in \mathbb{Z}^n$ such that $Ax = b$. $\square$

We define $\hat{x} = \lfloor A^{\#} \otimes' b \rfloor$. For each $j \in N$ set $M_j = \{i \in M : \hat{x}_j = b_i \otimes a_{ij}^{-1}\}$. The following theorem is an extension of [8, Proposition 2.1], and is immediate from Proposition 2.1.

Theorem 6.2 *Let $A \in \overline{\mathbb{R}}^{m \times n}$ be doubly $\mathbb{R}$-astic and $b \in \mathbb{R}^m$. Let $\hat{x} = \lfloor A^{\#} \otimes' b \rfloor$. Then*

(i) $A\hat{x} \leq b$,

(ii) $x \in IS(A, b)$ if and only if $x \leq \hat{x}$ and

$$\bigcup_{j : x_j = \hat{x}_j} M_j = M.$$

Corollary 6.3 *$Ax = b$ has a $\overline{\mathbb{Z}}$-solution if and only if $\hat{x}$ is a $\overline{\mathbb{Z}}$-solution.*

7 Extended Integer Eigenvectors

Recall that $V(A, \lambda)$ denotes the set of finite eigenvectors of A with respect to λ, and let $V(A) = V(A, \lambda(A))$ denote the set of all finite eigenvectors with respect to any eigenvalue. We use $\overline{V}(A, \lambda)$ to denote the set of all eigenvectors with respect to λ, that is,

$$\overline{V}(A, \lambda) := \{x \in \overline{\mathbb{R}}^n : Ax = \lambda x, x \neq \varepsilon\}.$$

Similarly we define the set of all $\overline{\mathbb{Z}}$ eigenvectors with respect to some $\lambda \in \overline{\mathbb{R}}$ as follows,

$$\overline{IV}(A, \lambda) := \{z \in \overline{\mathbb{Z}}^n : Az = \lambda z, z \neq \varepsilon\}.$$

Let $\Lambda(A) := \{\lambda \in \overline{\mathbb{R}} : (\exists x \in \overline{\mathbb{R}}^n) Ax = \lambda x, x \neq \varepsilon\}$,

$$\overline{V}(A) := \bigcup_{\lambda \in \Lambda(A)} \overline{V}(A, \lambda) \text{ and } \overline{IV}(A) := \overline{V}(A) \cap \overline{\mathbb{Z}}.$$

7.1 A description of the set of extended integer eigenvectors

In order to give a description of the set of $\overline{\mathbb{Z}}$ eigenvectors we follow the material in [1].

It is known [1] that if $\lambda(A) = \varepsilon$ then $\Lambda(A) = \{\varepsilon\}$ and $V(A, \varepsilon) = \{G \otimes u : u \in \overline{\mathbb{R}}^n\}$ where $G = (g_1, ..., g_n)$,

$$g_j = \begin{cases} e_j, & \text{if } A_j = \varepsilon; \\ \varepsilon, & \text{otherwise.} \end{cases} \tag{8}$$

It immediately follows that $\overline{IV}(A, \varepsilon) = \{G \otimes z : z \in \overline{\mathbb{Z}}^n\}$ in the case when $\lambda(A) = \varepsilon$.

We assume for the rest of this section that $\lambda(A) > \varepsilon$. Any matrix $A \in \overline{\mathbb{R}}^{n \times n}$ can be transformed to *Frobenius Normal Form* (FNF) by simultaneous permutations of the rows and columns [1]. That is

$$\begin{pmatrix} A_{11} & \varepsilon & \dots & \varepsilon \\ A_{21} & A_{22} & \dots & \varepsilon \\ \vdots & \vdots & & \vdots \\ A_{r1} & A_{r2} & \dots & A_{rr} \end{pmatrix} \tag{9}$$

where A_{ii} are irreducible. We outline the main results for finding eigenvectors $x \in \overline{\mathbb{R}}^n$ of A below.

For A in FNF we partition the node set N into $N_1, ..., N_r$. The *condensation digraph*, C_A, is defined [1] as the digraph

$$(\{N_1, ..., N_r\}, \{(N_i, N_j) : (\exists p \in N_i)(\exists q \in N_j) a_{pq} > \varepsilon\}).$$

We use $N_i \to N_j$ to mean that there is a directed path from a node in N_i to a node in N_j. The nodes of C_A with no incoming arcs are called the *initial classes*, those with no outgoing arcs are called the *final classes*.

The following result classifies when a matrix has only finite eigenvectors.

Theorem 7.1 *[1] Let $A = (a_{ij}) \in \overline{\mathbb{R}}^{n\times n}$. Then $\overline{V}(A) = V(A)$ if and only if A is irreducible.*

Let $(\lambda^{-1}A)^+ = (\gamma_{ij}) = (g_1, ..., g_n)$. Let

$$I(\lambda) = \{i \in R : \lambda(N_i) = \lambda, N_i \text{ spectral}\}$$

and

$$N_C(\lambda) = \bigcup_{i\in I(\lambda)} N_C(A_{ii}) = \{j \in N : \gamma_{jj} = 0, j \in \cup_{i\in I(\lambda)} N_i\}.$$

Theorem 7.2 *[1]Let* (9) *be a FNF of $A \in \overline{\mathbb{R}}^{n\times n}$. Then*

$$\Lambda(A) = \{\lambda(A_{jj}) : \lambda(A_{jj}) = \max_{N_i\to N_j} \lambda(A_{ii})\}.$$

Theorem 7.3 *[1]*

$$V(A, \lambda) = \{(\lambda^{-1}A)^+ z : z \in \overline{\mathbb{R}}^n, z_j = \varepsilon \textit{ for all } j \notin N_C(\lambda)\}.$$

From this it is clear that $\overline{\mathbb{Z}}$-eigenvectors are vectors $x \in \overline{\mathbb{Z}}^n - \{\varepsilon\}$ which are in the image of $(\lambda^{-1}A)^+$ with specific ε components. We will consider the $\overline{\mathbb{Z}}$-image problem in Section 9.

7.2 A strongly polynomially solvable case

For a fixed choice of λ we look for $\overline{\mathbb{Z}}$-vectors x such that $(\lambda^{-1}A)x = x$. Observe that, if such a vector exists, then

$$x_i \in \mathbb{Z} \Rightarrow (\exists j \in N) a_{ij} \in \mathbb{Z} \text{ is active with respect to } x.$$

Let $A \in \overline{\mathbb{R}}^{n\times n}$. If A has exactly one integer entry per row we say that A satisfies *Property OneIR*. If a matrix has at most one integer entry per row then we say it satisfies *weak Property OneIR*, or that it *weakly satisfies Property OneIR*.

It was proved in [8] that the integer eigenproblem is strongly polynomially solvable for matrices satisfying Property OneIR. Here we show that the extended integer eigenproblem is strongly polynomially solvable for matrices weakly satisfying Property OneIR. Indeed the problem can be solved in strongly polynomial time under the assumption of weak Property OneIR using the method for TSSs under weak Property OneFP which we develop in Subsection 10.1. This is since the problem can trivially be written as $Ax = Ix$. We summarise this in the result below.

Theorem 7.4 *Fix $\lambda \in \Lambda(A)$. If $\lambda^{-1}A$ weakly satisfies Property OneIR, then we can determine whether an $\overline{\mathbb{Z}}$-solution to $Ax = \lambda x$ exists using Algorithm $\overline{\mathbb{Z}}$-SEP-TSS-P1 in $\mathcal{O}(m^2(m^2+n^2))$ time.*

In order to be able to describe the whole set of $\overline{\mathbb{Z}}$-eigenvectors for a general matrix A in strongly polynomial time using this theorem we need that, for every $\lambda \in \Lambda(A)$, $\lambda^{-1}A$ weakly satisfies Property OneIR. This would be guaranteed if A satisfied the following condition,

$$(\forall i \in N)(\forall j, t \in N, j \neq t) a_{ij}, a_{it} \in \mathbb{R} \Rightarrow fr(a_{ij}) \neq fr(a_{it}). \tag{10}$$

We say that a matrix satisfying (10) is *row typical.*

Since $|\Lambda(A)| \leq n$ the following follows from Theorem 7.4

Corollary 7.5 *If A is row typical then $\overline{IV}(A)$ can be described in strongly polynomial time.*

8 Extended Integer Subeigenvectors

The set of $\overline{\mathbb{Z}}$-subeigenvectors is,

$$\overline{IV}^*(A,\lambda) := \{x \in \overline{\mathbb{Z}}^n : Ax \leq \lambda x, x \neq \varepsilon\}.$$

In the same way as for eigenvectors, if $\lambda = \varepsilon$ then $\overline{IV}^*(A,\varepsilon) = \{G \otimes z : z \in \overline{\mathbb{Z}}^n\}$ where G is defined in (8). Thus we can assume that $\lambda > \varepsilon$.

Observe that, for $x \in \overline{\mathbb{Z}}^n$, $Ax \leq \lambda x \Leftrightarrow \lceil \lambda^{-1}A \rceil x \leq x$. The following is then immediate, and is an extension of Theorem 2.3.

Proposition 8.1 *Let $A \in \overline{\mathbb{R}}^{n\times n}$ and $\lambda > \varepsilon$. Then*

$$x \in \overline{IV}^*(A,\lambda) \Leftrightarrow x \in \overline{IV}^*(\lceil \lambda^{-1}A \rceil, 0).$$

It remains to describe all λ for which $\overline{IV}^*(A,\lambda) \neq \emptyset$). We also show that, for a $\overline{\mathbb{Z}}$ matrix C, $\overline{IV}^*(C,0)$ can be fully described in strongly polynomial time. Thus the set $\overline{IV}^*(A,\lambda)$ can be found in strongly polynomial time also by Proposition 8.1. We do this by following the known results about general subeigenvectors from [1].

Theorem 8.2 *[1] If $A \in \overline{\mathbb{R}}^{n\times n}$, then*

$$\min\{\lambda : (\exists x \in \overline{\mathbb{R}}^n, x \neq \varepsilon) Ax \leq \lambda x\} = \min \Lambda(A).$$

Corollary 8.3 $\overline{IV}^*(A,\lambda) \neq \emptyset$ *if and only if* $0 \geq \min \Lambda(\lceil \lambda^{-1} A \rceil)$.

Proof. If $\overline{IV}^*(A,\lambda) \neq \emptyset$, then $IV^*(\lceil \lambda^{-1} A \rceil, 0) \neq \emptyset$ by Proposition 8.1 and the result follows from Theorem 8.2.

For the other direction, assume that $0 \geq \min \Lambda(\lceil \lambda^{-1} A \rceil)$ and let $B = \lceil \lambda^{-1} A \rceil$. Note that $B \in \overline{\mathbb{Z}}^{n \times n}$. Now, for all $\lambda \geq \min \Lambda(B)$, there exists $x \in \overline{\mathbb{R}}^n$ such that $Bx \leq \lambda x$. Therefore $\exists x \in \overline{\mathbb{R}}^n$,

$$\begin{aligned} & Bx \leq x \\ \therefore & (\forall i \in N) \max_{j \in N} (b_{ij} + x_j) \leq x_i \\ \therefore & (\forall i \in N) \max_{j \in N} (b_{ij} + \lfloor x_j \rfloor) \leq x_i \\ \therefore & (\forall i \in N) \max_{j \in N} (b_{ij} + \lfloor x_j \rfloor) \leq \lfloor x_i \rfloor \qquad (\because \text{LHS is integer}) \\ \therefore & B\lfloor x \rfloor \leq \lfloor x \rfloor. \end{aligned}$$

Hence $A\lfloor x \rfloor \leq \lambda \lfloor x \rfloor$. □

Let A be in FNF (see (9)) and $R = \{1, ..., r\}$. Lemma 8.4 and Corollary 8.5 can be deduced immediately from the proof of Theorem 4.5.14 in [1]. We state the proofs only for completeness.

Lemma 8.4 *[1] Suppose* $Ax \leq \lambda x$. *If* $\lambda < \lambda(A[N_k])$, *then* $x[N_k] = \varepsilon$.

Proof. If $x[N_k] \neq \varepsilon$ then $A[N_k]x[N_k] \leq \lambda x[N_k]$ which, since $A[N_k]$ is irreducible means that $x[N_k]$ is finite and $\lambda \geq \lambda(A[N_k])$. □

Corollary 8.5 *[1] Suppose* $Ax \leq \lambda x$. *If* $\lambda < \lambda(A[N_k])$ *and* $N_k \to N_i$, *then* $x[N_i] = \varepsilon$.

Proof. Assume $x[N_i] \neq \varepsilon$, so $x[N_i]$ is finite. First suppose that there exists an edge from some $s \in N_k$ to some $t \in N_i$. So $a_{st} > \varepsilon$. But then $\lambda x_s \geq a_{st} x_t > \varepsilon$ which implies that $x[N_k] \neq \varepsilon$, a contradiction.

If, instead, there is a path from some $s \in N_k$ to some $t \in N_i$ then we can apply the same argument, formally by induction, to again reach the contradiction that $x[N_k] \neq \varepsilon$. □

Define

$$I^*(\lambda) = \{i \in R : \lambda(N_i) \leq \lambda\}.$$

Note that although $(\lambda^{-1}A)^+$, and hence also $(\lambda^{-1}A)^*$, may contain $+\infty$ values it holds that

$$(\forall i \in I^*(\lambda)) \ \lambda(\lambda^{-1} A_{ii}) = \lambda^{-1} \lambda(A_{ii}) \leq 0$$

and hence $(\lambda^{-1}A_{ii})^*$ is finite (since A_{ii} is irreducible) for all $i \in I^*(\lambda)$.

Let $N^*(A,\lambda) = \{j \in N_i : i \in I^*(\lambda)\}$.

Proposition 8.6 *Let $A \in \overline{\mathbb{R}}^{n\times n}$ and $\lambda \in \mathbb{R}$ with $\lambda \geq \min \Lambda(A)$. Then*

$$\overline{IV}^*(A,\lambda) = \left\{ \bigoplus_{j\in N^*(A,\lambda)} (\lambda^{-1}A)^*_j z_j : z_j \in \mathbb{Z} \right\}.$$

Proof. Let $M = N^*(A,\lambda)$. By Lemma 2.6 we may assume without loss of generality that A has the form

$$\begin{pmatrix} * & \varepsilon \\ * & A[M] \end{pmatrix}$$

where $A[M]$ is in FNF.

Now $\lambda \geq \lambda(A[M]) = \max_{i\in I^*(\lambda)} \lambda(A_{ii})$ so we have that $x \in \overline{IV}^*(A[M],\lambda)$ if and only if $x = (\lambda^{-1}A[M])^*u$ for some u. It remains to observe the following two points:

(1) If $\tilde{x} \in \overline{IV}^*(A[M],\lambda)$, then $(\varepsilon,\tilde{x})^T \in \overline{IV}^*(A,\lambda)$.

(2) If $(x,\tilde{x})^T \in \overline{IV}^*(A,\lambda)$, then $x = \varepsilon$ by Lemma 8.4 and $\tilde{x} \in \overline{IV}^*(A[M],\lambda)$.

$\square$

9 Extended Integer Image

Here, given $A \in \overline{\mathbb{R}}^{m\times n}$ doubly $\mathbb{R}$-astic, we aim to decide whether there exist $z \in \overline{\mathbb{Z}}^m$ such that $A \otimes x = z$. We call this the $\overline{\mathbb{Z}}$-image problem and define,

$$\overline{IIm}(A) := \{z \in \overline{\mathbb{Z}}^m - \{\varepsilon\} : \exists x \in \overline{\mathbb{R}}^n, Ax = z\}.$$

We call $z \in \overline{IIm}(A)$ a $\overline{\mathbb{Z}}$-image of A.

Observation 9.1 *Let $A \in \overline{\mathbb{R}}^{m\times n}$ be doubly $\mathbb{R}$-astic.*

(i) If A is finite, then $\overline{IIm}(A) = IIm(A)$.

(ii) If A is not finite and $m = 2$, then $\overline{IIm}(A) \neq \emptyset$. This is because there exists a column of the form

$$\begin{pmatrix} a_{1j} \\ \varepsilon \end{pmatrix} \text{ or } \begin{pmatrix} \varepsilon \\ a_{2j} \end{pmatrix},$$

both of which have a $\mathbb{Z}$-image.

Since the interesting case of the $\overline{\mathbb{Z}}$-image problem is when the matrix is not finite, the known algorithms for finding integer solutions cannot obviously be adapted. In this section we instead show that the $\overline{\mathbb{Z}}$-image problem can be solved in strongly polynomial time in a special case.

We will use the following definitions.

A matrix Q is called a *generalised permutation matrix* if it can be obtained from a diagonal matrix by permuting the rows and/or columns. We use P_n to denote the set of permutations on N. For $A \in \overline{\mathbb{R}}^{n \times n}$ the *max-algebraic permanent* is given by

$$maper(A) = \bigoplus_{\pi \in P_n} \bigotimes_{i \in N} a_{i,\pi(i)}.$$

For a given $\pi \in P_n$ its *weight* with respect to A is

$$w(\pi, A) = \bigotimes_{i \in N} a_{i,\pi(i)}$$

and the set of permutations whose weight is maximum is

$$ap(A) = \{\pi \in P_n : w(\pi, A) = maper(A)\}.$$

9.1 NNI matrices and $\overline{\mathbb{Z}}$-solutions

We assume that $maper(A) > \varepsilon$. Then, from [1], there exists a generalised permutation matrix Q such that $A \otimes Q$ is strongly definite. As in [8] we say that a strongly definite matrix A is called *nearly non integer* (NNI) if the only integer entries are the zeros on the diagonal.

The result below extends [8, Theorem 3.21] to extended integer images.

Theorem 9.2 *Let $A \in \overline{\mathbb{R}}^{n \times n}$ be a strongly definite NNI matrix. Then it holds that*

$$\overline{IIm}(A) = \overline{IV}^*(A, 0) = \overline{IV}(A, 0) - \{\varepsilon\}.$$

Claim 9.3 *Let $A \in \overline{\mathbb{R}}^{n \times n}$, $n \geq 3$, be strongly definite and NNI. Then for any $x \in \overline{\mathbb{R}}^n$ satisfying $A \otimes x = z \in \overline{\mathbb{Z}}^n$, $z \neq \varepsilon$ all a_{ii} are active.*

Proof. **[of Claim]** Let A be a strongly definite, NNI matrix. Suppose that there exists a vector x satisfying $A \otimes x = z \in \overline{IIm}(A)$ and let $I = \{i \in N : z_i > \varepsilon\}$.

Note that for any $i \in N - I$ it holds that a_{ii} is active since $\max_j(a_{ij} + x_j) = \varepsilon$ means that all a_{ij} are active. So we must prove that a_{ii} are active when $i \in I$. Suppose for a contradiction that there exists $i \in I$ such that a_{ii} is inactive. Then there exists $k_1 \in N$, $k_1 \neq i$, such that a_{ik_1} is active and hence $x_{k_1} \in \mathbb{R} - \mathbb{Z}$.

Observe that $k_1 \in I$ since $z_{k_1} \geq a_{k_1 k_1} + x_{k_1} = x_{k_1} > \varepsilon$. Further $a_{k_1 k_1}$ is inactive because $x_{k_1} \notin \mathbb{Z}$.

So $\exists k_2 \in N,\ k_2 \neq k_1$ such that a_{k_1,k_2} is active and $k_2 \in I$. Then it holds that

$$a_{k_1,k_2} + x_{k_2} \geq a_{k_1,k_1} + x_{k_1} = x_{k_1}. \tag{11}$$

We know that there is an active element in every row so consider row k_2. Then a_{k_2,k_2} is inactive because $x_{k_2} \notin \mathbb{Z}$ and $z_{k_2} > \varepsilon$. Further a_{k_2,k_1} is inactive since, if not, $a_{k_2,k_1} + x_{k_1} > a_{k_2,k_2} + x_{k_2} = x_{k_2}$ which together with (11) would imply that the permutation in P_n containing the cycle (k_1, k_2) and sending all other indices to themselves has strictly positive weight, which is a contradiction.

Thus $\exists k_3 \in N,\ k_3 \neq k_1, k_2$ such that a_{k_2,k_3} is active, $k_3 \in I$ and we have

$$a_{k_2,k_3} + x_{k_3} > a_{k_2,k_2} + x_{k_2} = x_{k_2}. \tag{12}$$

Consider row k_3. Again it can be seen that both a_{k_3,k_3} and a_{k_3,k_2} are inactive. Further we show that a_{k_3,k_1} is inactive. If it was active then we would have $a_{k_3,k_1} + x_{k_3} > x_{k_1}$ which together with (11) and (12) would imply that the permutation containing the cycle (k_1, k_2, k_3) and sending all other indices to themselves would have strictly positive weight, a contradiction.

Thus $\exists k_4 \in N,\ k_4 \neq k_1, k_2, k_3$ such that a_{k_3,k_4} is active and $k_4 \in I$.

Continuing in this way we reach $k_{|I|}$ such that $k_{|I|} \in I$ and $a_{k_{|I|}k_{|I|}}$ is inactive, meaning that there exists $k_j \notin \{k_1, ..., k_{|I|}\}$ such that $k_j \in I$ which is impossible. Hence all a_{ii} are active. $\square$

Proof. **[of Theorem 9.2]** We have shown that, if A is strongly definite, NNI and $n \geq 3$, then all the diagonal elements are active. Clearly this is also true if $n = 1$. If $n = 2$, then the matrix is either finite (for which the result holds from [8]) or has an ε entry on the off diagonal, which implies that integer entries are active with respect to a $\overline{\mathbb{Z}}$-image.

Thus, if $z \in \overline{IIm}(A)$, then $A \otimes x = z$ for some x with a_{ii} active for all $i \in N$. But then $(\forall i)\ a_{ii} + x_i = z_i$ and hence $x = z$. Thus $z \in \overline{IV}(A, 0)$. We conclude that $\overline{IIm}(A) = \overline{IV}(A, 0)$ for matrices of this type. $\square$

Corollary 9.4 *If A is strongly definite, NNI and $y \in \overline{\mathbb{Z}}^n$, then*

$$Ax = y \Rightarrow x = y.$$

We finish this section with some observations about powers of NNI matrices.

Corollary 9.5 *If A is strongly definite and NNI, then $\forall t \in \mathbb{N}$,*

$$\overline{IIm}(A^t) = \overline{IV}(A^t, 0).$$

Proof. Assume $x \in \overline{IIm}(A^t)$. Then there exists $y \in \overline{\mathbb{R}}^n$ such that $x = A^t y = A(A^{t-1}y)$ and so $x \in \overline{IIm}(A)$. Then by Theorem 9.2 $x \in \overline{IV}(A, 0) \subseteq \overline{IV}(A^t, 0)$. Hence $\overline{IV}(A^t, 0) \supseteq \overline{IIm}(A^t)$. The other inclusion is clear. $\square$

Corollary 9.6 *If A is NNI then*
(i) $(\forall r \in \mathbb{N})\overline{IIm}(A^r) = \overline{IIm}(A)$.
(ii) $(\forall r \in \mathbb{N})\overline{IV}(A^r, 0) = \overline{IV}(A, 0)$.
(iii) $(\forall r, s \in \mathbb{N})\overline{IIm}(A^s) = \overline{IV}(A^r, 0)$.

Proof. Fix $t \in \mathbb{N}$. Then

$$\overline{IV}(A, 0) = \overline{IIm}(A) \supseteq \overline{IIm}(A^t) = \overline{IV}(A^t, 0) \supseteq \overline{IV}(A, 0)$$

so all these sets are equal. Hence for all $r, s \in \mathbb{N}$,

$$\overline{IIm}(A^r) = ... = \overline{IIm}(A^2) = \overline{IIm}(A) = \overline{IV}(A, 0) = \overline{IV}(A^2, 0) = ... = \overline{IV}(A^s, 0)$$

$\square$

Example 9.7 *It is possible for matrices B that are not NNI to satisfy $IIm(B) = IV(B, 0)$.*

Let

$$A = \begin{pmatrix} 0 & -0.2 & -0.2 \\ 0.2 & 0 & -0.3 \\ -0.1 & -0.2 & 0 \end{pmatrix}.$$

Then

$$B := A^2 = \begin{pmatrix} 0 & -0.2 & -0.2 \\ 0.2 & 0 & 0 \\ 0 & -0.2 & 0 \end{pmatrix}$$

is not NNI but is the square of a NNI matrix so satisfies $IIm(B) = IV(B, 0)$ by Corollary 9.5.

10 Extended Integer Solutions to Two-sided Systems: a Special Case

For $A \in \overline{\mathbb{R}}^{m\times n}, B \in \overline{\mathbb{R}}^{m\times k}$ doubly $\mathbb{R}$-astic we study the problem of determining whether there exist $\overline{\mathbb{Z}}$-solutions to $Ax = By$ or $Ax = Bx$. The following statement is obvious.

Proposition 10.1 *Let $A, B \in \overline{\mathbb{R}}^{m\times n}$. The problem of finding $x \in \overline{\mathbb{Z}}^n$ satisfying $Ax = Bx$ is equivalent to finding $x \in \overline{\mathbb{Z}}^n, y \in \overline{\mathbb{R}}^m$ such that*

$$\begin{pmatrix} A \\ B \end{pmatrix} x = \begin{pmatrix} I \\ I \end{pmatrix} y$$

where $I \in \overline{\mathbb{R}}^{m\times m}$.

Hence it is sufficient to only consider systems with separated variables.

First we describe a case where, for the question of existence, it is sufficient to consider finite integer solutions only.

Proposition 10.2 *If either of A, B is finite then there exist non trivial $x \in \overline{\mathbb{Z}}^n, y \in \overline{\mathbb{Z}}^k$ such that $Ax = By$ if and only if there exist $x \in \mathbb{Z}^n, y \in \mathbb{Z}^k$ satisfying $Ax = By$.*

Proof. Suppose that (x, y) is a $\overline{\mathbb{Z}}$-solution to $Ax = By$ and, without loss of generality, that A is finite. Since $x \neq \varepsilon$ by assumption we have $A \otimes x = \gamma \in \mathbb{R}^m$.

Hence for each i we have $\max_j(a_{ij} + x_j) = \gamma_i = \max_t(b_{it} + y_t)$ where the maximum values are attained for finite a_{ij}, x_j, b_{it}, y_t. Let $x' \in \mathbb{Z}^n$ be obtained from x by replacing each ε component with a small enough integer. Define y' similarly. Then $Ax' = \gamma = By'$ since we have not changed the active finite entries of x and y and the new integer components are small enough not to influence the maximums. The other direction is trivial. $\square$

In what follows, we develop a method to find an extended integer solution to $Ax = By$ in strongly polynomial time in a special case. This uses the special case for integer solutions from [3, 8] which we recap below.

Let $A \in \overline{\mathbb{R}}^{m\times n}, B \in \overline{\mathbb{R}}^{m\times k}$. We say (A, B) satisfies *Property OneFP* if for each $i \in M$ there is exactly one pair $(r(i), r'(i))$ such that

$$a_{ir(i)}, b_{ir'(i)} \in \mathbb{Z}, \text{ and}$$

for all $i \in M$, if $j \neq r(i)$ and $t \neq r'(i)$, then

$$a_{ij}, b_{it} > \varepsilon \Rightarrow fr(a_{ij}) \neq fr(b_{it}).$$

Theorem 10.3 *[3, 8] Let $A \in \overline{\mathbb{R}}^{m\times n}$, $B \in \overline{\mathbb{R}}^{m\times k}$ satisfy Property OneFP. For all $i, j \in M$ let*

$$w_{ij} = \max(\lceil a_{j,r(i)} \rceil - a_{i,r(i)},\ \lceil b_{j,r'(i)} \rceil - b_{i,r'(i)}).$$

Then an integer solution to $Ax = By$ exists if and only if $\lambda(W) \leq 0$. If this is the case then $Ax = By = z^{-1}$ where $z \in IV^*(W, 0)$ and x and y can be found using Proposition 2.2.

10.1 A strongly polynomial method to find $\overline{\mathbb{Z}}$ solutions to two sided systems weakly satisfying Property OneFP

We say that the pair (A, B) is said to *weakly satisfy Property OneFP* if for each i there exists at most one pair $(a_{ir(i)}, b_{ir'(i)})$ sharing the same fractional part.

Throughout this section we assume that $A \in \overline{\mathbb{R}}^{m\times n}, B \in \overline{\mathbb{R}}^{m\times k}$ are doubly $\mathbb{R}$-astic matrices such that (A, B) weakly satisfy Property OneFP. Further we assume without loss of generality that, for each row containing two entries with the same fractional part, the entries $(a_{ir(i)}, b_{ir'(i)})$ are integer.

Claim 10.4 *We can assume without loss of generality that the matrices satisfy Property OneFP.*

Proof. Assume that no entry in row i of A shares its fractional part with an entry in row i of B. Then we have

$$\max_j(a_{ij} + x_j) = \varepsilon = \max_t(b_{it} + y_t).$$

Let $T_x(i) = \{j \in N : a_{ij} > \varepsilon\}$ and $T_y(i) = \{t \in K : b_{it} > \varepsilon\}$. Note that $|T_x(i)|, |T_y(i)| > 0$. Then $x_j = \varepsilon$ for all $j \in T_x(i)$ and $y_t = \varepsilon$ for all $t \in T_y(i)$.

Therefore we can remove row i from the system as well as columns $A_j, j \in T_x(i)$ and $B_t, t \in T_y(i)$. This can be repeated until either $x = \varepsilon = y$, in which case we conclude that no non trivial solution exists, or until we are left with A', B' which satisfy Property $OneFP$. $\square$

In light of this we develop a subroutine for an algorithm which will take a pair (A, B) weakly satisfying Property OneFP and convert to a pair (A', B') satisfying Property OneFP such that $Ax = By$ has a $\overline{\mathbb{Z}}$-solution if and only if $A'x' = B'y'$ has a $\overline{\mathbb{Z}}$-solution.

Subroutine 1: Convert to Property OneFP

Input: $A \in \overline{\mathbb{R}}^{m\times n}, B \in \overline{\mathbb{R}}^{m\times k}$ doubly $\mathbb{R}$-astic weakly satisfying Property OneFP.

Output: Matrices A', B' satisfying Property OneFP and sets T_x, T_y such that $x_j = \varepsilon, \forall j \in T_x$ and $\forall y_j = \varepsilon, j \in T_y$ or indication that no $\overline{\mathbb{Z}}$-solution exists.

(1) $A' := A, B' := B, T_x := \emptyset, T_y := \emptyset$.

(2) $R := \{i : (\forall j \in N, t \in K) fr(a'_{ij}) \neq fr(b'_{it})\}$.
If $R = \emptyset$ *then STOP: output* A', B', T_x, T_y.

(3) Take some $i \in R$. *Set*
$T_x(i) = \{j \in N : a'_{ij} > \varepsilon\}$ *and* $T_y(i) = \{t \in K : b'_{it} > \varepsilon\}$.

(4) $T_x := T_x \cup T_x(i), T_y := T_y \cup T_y(i)$
Remove row i from both A' and B'.
Remove columns $A'_j, j \in T_x$ from A' and $B'_t, t \in T_y$ from B'.

(5) If there exists an ε row in either A' or B' remove it.
If $A'_j = \varepsilon$ then remove the column from A' and add j to T_x.
If $B'_j = \varepsilon$ then remove the column from B' and add j to T_y.

(6) If $T_x = N$ or $T_y = K$ STOP: no $\overline{\mathbb{Z}}$-solution to $Ax = By$. Else go to (2).

It is important to note that we keep the original numbering of columns throughout this subroutine and all other steps described in this section.

Proposition 10.5 *Subroutine 1 is correct and runs in $\mathcal{O}(m^2nk)$ time*

Proof. Correctness follows from the proof of Claim 10.4.

Further, each iteration requires $\mathcal{O}(mnk)$ operations and there are at most m iterations (each time we do not halt a row is removed from the system).

□

We will first show how to find a $\overline{\mathbb{Z}}$-solution to $Ax = By$ in a special case.

Suppose that $A, B \in \overline{\mathbb{R}}^{n\times n}$ and that the active entries $a_{ir(i)}$ and $b_{ir'(i)}$ are spread over all columns in A and B respectively. Then, by swapping columns if necessary, we can assume without loss of generality that $r(i) = i = r'(i)$.

Recall that Theorem 10.3 says, if we construct the matrix $W = (w_{ij})$, $i, j \in N$ where

$$w_{ij} = \max(\lceil a_{j,r(i)} \rceil - a_{i,r(i)},\ \lceil b_{j,r'(i)} \rceil - b_{i,r'(i)}),$$

then an integer solution to $Ax = By$ exists if and only if $\lambda(W) \leq 0$. In particular $\lambda(W) = 0$ is a sufficient condition for $Ax = By$ to have a $\overline{\mathbb{Z}}$ solution since $w_{ii} = 0$.

To deal with the case $\lambda(W) > 0$ we will need the following result. We will use $W(A, B)$ to denote the matrix W calculated from A and B according to Theorem 10.3.

Proposition 10.6 *Suppose $A, B \in \overline{\mathbb{R}}^{n\times n}$ and $r(i) = i = r'(i)$. Let $S \subseteq N$. If $W = W(A, B)$ then $W[S] = W(A[S], B[S])$.*

Proof. By definition the i^{th} row of W is equal to H^T where

$$H = a_{ir(i)}^{-1}\lceil A_{r(i)}\rceil \oplus b_{ir'(i)}^{-1}\lceil B_{r'(i)}\rceil = a_{ii}^{-1}\lceil A_i\rceil \oplus b_{ii}^{-1}\lceil B_i\rceil.$$

So row i of W is linked only to columns A_i and B_i.

Further the i^{th} column of W is equal to

$$\begin{pmatrix} \lceil a_{i1}\rceil - a_{11} \\ \lceil a_{i2}\rceil - a_{22} \\ \vdots \\ \lceil a_{in}\rceil - a_{nn} \end{pmatrix} \oplus \begin{pmatrix} \lceil b_{i1}\rceil - b_{11} \\ \lceil b_{i2}\rceil - b_{22} \\ \vdots \\ \lceil b_{in}\rceil - b_{nn} \end{pmatrix}$$

so the i^{th} column of W is linked only to the i^{th} rows of A and B

Therefore, for any $i \in N$, $W[N-i] = W(A[N-i], B[N-i])$. This argument can be repeated to obtain $W[S] = W(A[S], B[S])$. $\square$

Now we show that, when $\lambda(W) > 0$, we can identify a set T such that, for any $\overline{\mathbb{Z}}$-solution of $Ax = By$ it holds that $x_j = \varepsilon = y_j$ for all $j \in T$. Recall that, for a matrix $G \in \overline{\mathbb{R}}^{n\times n}$, D_G is the digraph associated with D.

Proposition 10.7 *Suppose $A, B \in \overline{\mathbb{R}}^{n\times n}$, $r(i) = i = r'(i)$ and $\lambda(W) > 0$. Let σ be a critical cycle with $S = \{i \in N : i$ is a vertex on the cycle σ in $D_W\}$.*

Define

$$T = \{j \in N : \exists i \in S \text{ such that } a_{ij} > \varepsilon \text{ or } b_{ij} > \varepsilon\}.$$

If $Ax = By$, $x, y \in \overline{\mathbb{Z}}^n$, then $x_i = y_i = z_i = \varepsilon$ for all $i \in T$.

Proof. Since $\sum_{i\in S} l_{i\sigma(i)} = |S|\lambda(W) > 0$ it holds that,

$$(\forall i \in S) \max(\lceil a_{\sigma(i)r(i)}\rceil - a_{ir(i)}, \lceil b_{\sigma(i)r'(i)}\rceil - b_{ir'(i)}) = w_{i\sigma(i)} > \varepsilon. \quad (13)$$

Hence, for all $i \in S$, at least one of $a_{\sigma(i)r(i)}$ or $b_{\sigma(i)r'(i)}$ is finite.

Suppose that a non trivial $\overline{\mathbb{Z}}$-solution to $Ax = z = By$ exists and some $i \in S$ satisfies $z_i > \varepsilon$. Then, since $a_{ii}x_i = z_i = b_{ii}y_i$, we conclude that $x_i > \varepsilon$ and $y_i > \varepsilon$.

In fact,

$$(\forall i \in N) x_i = \varepsilon \Leftrightarrow y_i = \varepsilon \Leftrightarrow z_i = \varepsilon. \quad (14)$$

Now, since $i = r(i) = r'(i)$ by (13), we have $a_{\sigma(i)i} > \varepsilon$ or $b_{\sigma(i)i} > \varepsilon$ which implies that $z_{\sigma(i)} > \varepsilon$ and hence $x_{\sigma(i)} > \varepsilon$ and $y_{\sigma(i)} > \varepsilon$. Continuing this argument, and using the fact that $\sigma(i) \neq i$ for all $i \in S$, we conclude that for all $i \in S$, $z_i, x_i, y_i > \varepsilon$.

This implies $A[S]x[S] = z[S] = B[S]y[S]$ has an integer solution. Therefore $\lambda(W(A[S], B[S])) = 0$. This is a contradiction since, using Proposition 10.6 and the fact that σ is critical in W we get,

$$0 = \lambda(W[S]) = \lambda(W) > 0.$$

Hence we have shown $z_i = \varepsilon$ for all $i \in S$.

Fix $i \in S$. Then $\max_j(a_{ij} + x_j) = \varepsilon = \max_j(b_{ij} + y_j)$ which implies $x_j = \varepsilon$ for all $j \in \{j \in N : \exists i \in S \text{ such that } a_{ij} > \varepsilon\} := T'$. Similarly $y_j = \varepsilon$ for all $j \in \{j \in N : \exists i \in S \text{ such that } b_{ij} > \varepsilon\} := T''$. Observe that $T' \cup T'' = T$ and hence by (14) $x_i = y_i = z_i = \varepsilon$ for all $i \in T$. $\square$

Remark 10.8 *In fact, this proof shows that if $w(\sigma, W) > 0$ then any node i on σ in D_W satisfies $x_i = y_i = z_i = \varepsilon$, as well as any node accessible from S.*

So we propose a subroutine that, when given a pair of square matrices (A, B) satisfying Property OneFP with $r(i) = i = r'(i)$, determines whether a $\overline{\mathbb{Z}}$-solution to $Ax = By$ exists.

Subroutine 2: Solve a special square case

Input: $A \in \overline{\mathbb{R}}^{n\times n}, B \in \overline{\mathbb{R}}^{n\times n}$ doubly $\mathbb{R}$-astic satisfying Property OneFP with $r(i) = i = r'(i)$.

Output: Indication whether or not a $\overline{\mathbb{Z}}$-solution to $Ax = By$ exists.

(1) Calculate $W = W(A, B)$.

If $\lambda(W) = 0$ STOP: solution exists, output A, B, W.

(2) $S := \{i \in N : i$ is critical in $W\}$.

$T(0) := \{j \in N : \exists i \in S$ such that $a_{ij} > \varepsilon$ or $b_{ij} > \varepsilon\} \supseteq S$.

(3) Set $l = 1$.

(3.1) $T(l) = \{j \in N : \exists i \in T(l-1)$ such that $a_{ij} > \varepsilon$ or $b_{ij} > \varepsilon\}$

(3.2) If $T(l) = N$ STOP: no solution.

(3.3) If $T(l) = T(l-1)$ go to (4).

(3.4) $l := l + 1$, go to (3.1).

(4) $T := T(l)$

(5) $A := A[N - T], B := B[N - T]$, $x_i = y_i = z_i = \varepsilon$ for all $i \in T$.

(6) Go to (1).

Proposition 10.9 *Subroutine 2 is correct and terminates after $\mathcal{O}(n^4)$ operations.*

Proof. Correctness follows from Proposition 10.7 and noting that if the pair (A, B) satisfies Property OneFP with $r(i) = i = r'(i)$ then so does $(A[N-T], B[N-T])$. Observe that the loop in step (3) uses the fact that if $z_i = \varepsilon$ for $i \in T(l-1)$ then there may be finite entries a_{it} or b_{it} where $t \notin T(l-1)$. In this case also $z_t = x_t = y_t = \varepsilon$ by (14) so we add these indices to the list $T(l)$. Also this loop will terminate since $|T(l)| \leq n$ and the algorithm stops if no new index is added to $T(l-1)$.

In each iteration where a solution is not found at least two rows and columns are removed from each matrix, this is since σ is never a loop in D_L so $|\bar{T}| \geq |S| \geq 2$. Therefore there are at most $\lceil \frac{n}{2} \rceil$ iterations. Each iteration is completed in $\mathcal{O}(n^3)$ time. □

Now we show how to use Subroutine 2 to determine whether a general TSS satisfying Property OneFP has a $\overline{\mathbb{Z}}$-solution. To do this we transform matrices $A \in \overline{\mathbb{R}}^{m \times n}, B \in \overline{\mathbb{R}}^{m \times k}$ satisfying Property OneFP to square matrices A^{sc}, B^{sc} also satisfying Property OneFP with the extra condition that the entries $a_{ir(i)}, b_{ir'(i)}$ are spread over all columns. We do this in such a way that $Ax = By$ has a $\overline{\mathbb{Z}}$-solution if any only if $A^{sc}x' = B^{sc}y'$ has a $\overline{\mathbb{Z}}$-solution, and further x, y are easily described using x', y'.

Given $A \in \overline{\mathbb{R}}^{m \times n}, B \in \overline{\mathbb{R}}^{m \times k}$ satisfying Property OneFP we calculate the *square counterparts*, $A^{sc}, B^{sc} \in \overline{\mathbb{R}}^{m \times m}$, as follows:

Subroutine 3: Calculate square counterparts

Input: $A \in \overline{\mathbb{R}}^{m \times n}, B \in \overline{\mathbb{R}}^{m \times k}$ *doubly $\mathbb{R}$-astic satisfying Property OneFP*
.

Output: Square A^{sc}, B^{sc} satisfying Property OneFP with $r(i) = i = r'(i)$.

(1) $F := \{1 - fr(a_{ij}) : i \in M, j \in N\} \cup \{1 - fr(b_{ij}) : i \in M, j \in K\}$.

(2) $C(A_j) := \{i \in M : r(i) = j\} = \{i \in M : a_{ij} \in \mathbb{Z}\}$,
$C(B_l) := \{i \in M : r'(i) = l\} = \{i \in M : b_{il} \in \mathbb{Z}\}$.

(3) For each $j \in N$,
If $|C(A_j)| = 1$, *then column* A_j *remains unchanged.*
If $|C(A_j)| = 0$, *then column* A_j *is removed.*
If $|C(A_j)| > 1$, *choose* $0 < \delta_A < 1$ *with* $\delta_A \notin F$. *Column* A_j *is replaced by* $|C(A_j)|$ *columns* $A^{sc}_{j_p}, p \in C(A_j)$ *where*

$$a^{sc}_{ij_p} = \begin{cases} a_{ij} & \text{if } i \in (M - C(A_j)) \cup \{p\} \\ a_{ij} - \delta_A & \text{if } i \in C(A_j) - \{p\}. \end{cases}$$

(4) For each $l \in K$,
If $|C(B_l)| = 1$, *then column* B_l *is unchanged.*

If $|C(B_l)| = 0$, then column B_l is removed.

If $|C(B_l)| > 1$, choose $0 < \delta_B < 1$ with $\delta_B \notin F \cup \{\delta_A\}$. Column B_l is replaced by $|C(B_l)|$ columns $B^{sc}_{l_q}, q \in C(B_l)$ where

$$b^{sc}_{il_q} = \begin{cases} b_{il} & \text{if } i \in (M - C(B_l)) \cup \{q\} \\ b_{il} - \delta_B & \text{if } i \in C(B_l) - \{q\}. \end{cases}$$

Remark 10.10 *Calculating A^{sc} from $A \in \overline{\mathbb{R}}^{m\times n}$ requires $\mathcal{O}(m^2 n)$ operations.*

It should be observed that choosing δ_A and δ_B can be done efficiently since there are at most $mn + mk + 1$ values that they cannot take. Further, this choice of δ_A, δ_B preserves Property OneFP since we only subtract δ_A from integer entries of A, meaning that the only fractional part we add to the left hand side of the system is $1 - \delta_A$. Similarly, the only fractional part added to the right hand side of the system is $1 - \delta_B$.

If $A^{sc}_{j_p}$ corresponds to column A_j in A, then we say that $A^{sc}_{j_p}$ is the counterpart, of A_j. Similarly for B

Proposition 10.11 *Let $A \in \overline{\mathbb{R}}^{m\times n}, B \in \overline{\mathbb{R}}^{m\times k}$ satisfy Property OneFP and A^{sc}, B^{sc} be the square counterparts of A and B. A $\overline{\mathbb{Z}}$-solution to $Ax = By$ exists if and only if a $\overline{\mathbb{Z}}$-solution to $A^{sc}\bar{x} = B^{sc}\bar{y}$ exists.*

Proof. We show that

$$(\exists x \in \overline{\mathbb{Z}}^n) Ax = z \Leftrightarrow (\exists \bar{x} \in \overline{\mathbb{Z}}^m) A^{sc}\bar{x} = z$$

where, in both equations, only integer entries are active in rows for which $z_i > \varepsilon$. Note that, if $z_i = \varepsilon$, then integer entries are still active because every entry in row i is active. An almost identical argument can be used to show the same holds for B and B^{sc}.

If $|C(A_j)| = 0$, then column A_j is inactive and so a solution exists if and only if a $\overline{\mathbb{Z}}$-solution exists with $x_j = \varepsilon$. This justifies the deletion of such columns from A^{sc}.

If $|C(A_j)| = 1$, then let A^{sc}_j be the counterpart of A_j in A^{sc}.

If $|C(A_j)| > 1$, then let $A^{sc}_{j_p}, p \in C(A_j)$ be the counterparts of A_j in A^{sc}, where the integer entry in $A^{sc}_{j_p}$ is $a^{sc}_{pj_p} = a_{pj}$.

For all $j \in N$ let $|C(A_j)| = \alpha_j$.

First assume that $Ax = z$ for some $x \in \overline{\mathbb{Z}}$ and only integer entries in A are active.

Let

$$\bar{x} = (x_1, ..., x_1, x_2, ..., x_2, ..., x_n, ..., x_n)^T \in \overline{\mathbb{Z}}^m$$

where each x_j is repeated $|C(A_j)|$ times.

Then $A^{sc}\bar{x} = z$ where integer entries are active.

For the other direction assume that $A^{sc}\bar{x} = z$ where integer entries are active.

Observe first that if $A^{sc}_{j_p}$ and $A^{sc}_{j_q}$ correspond to the same column A_j, then $\bar{x}_{j_p} = \bar{x}_{j_q}$ To see this note that $a^{sc}_{pj_p} + \bar{x}_{j_p} = z_p$ and $a^{sc}_{qj_q} + \bar{x}_{j_q} = z_q$.

Therefore

$$a_{pj} + \bar{x}_{j_p} = a^{sc}_{pj_p} + \bar{x}_{j_p} = z_p > a^{sc}_{pj_q} + \bar{x}_{j_q} = a_{pj} - \delta_A + \bar{x}_{j_q} \text{ and}$$
$$a_{qj} + \bar{x}_{j_q} = a^{sc}_{qj_q} + \bar{x}_{j_q} = z_q > a^{sc}_{qj_p} + \bar{x}_{j_p} = a_{qj} - \delta_A + \bar{x}_{j_p}.$$

Hence $\bar{x}_{j_q} + \delta_A > \bar{x}_{j_p} > \bar{x}_{j_q} - \delta_A$ which, since $\bar{x} \in \overline{\mathbb{Z}}^m$ means that $\bar{x}_{j_p} = \bar{x}_{j_q}$.

Therefore we can set $x_j = \bar{x}_j$ if $|C(A_j)| \geq 1$ and $x_j = \varepsilon$ otherwise to obtain x such that $Ax = z$. □

We propose the following algorithm to determine whether or not $Ax = By$ has a $\overline{\mathbb{Z}}$-solution in the case when (A, B) weakly satisfy Property OneFP.

Algorithm: $\overline{\mathbb{Z}}$-SEP-TSS-P1

Input: $A \in \overline{\mathbb{R}}^{m\times n}, B \in \overline{\mathbb{R}}^{m\times k}$ weakly satisfying Property OneFP.

Output: Indication of whether or not $Ax = By$ has a $\overline{\mathbb{Z}}$-solution.

(1) Input A, B into Subroutine 1.
If output is no solution then STOP.
Otherwise output is A', B', T_x, T_y.

(2) Input A', B' into Subroutine 3 to obtain the square counterparts A^{sc}, B^{sc}.

(3) Rearrange columns of A^{sc}, B^{sc} so that $r(i) = i = r'(i)$.

(4) Input A^{sc}, B^{sc} into Subroutine 2.
If output is no solution, then STOP.
Otherwise, output $\bar{A}, \bar{B}$ and $\bar{W}$ with $\lambda(\bar{W}) = 0$ and so a solution exists.

Remark 10.12 *If $\overline{\mathbb{Z}}$-SEP-INT-TSS outputs that a solution exists then we can find a solution easily. Since $\lambda(\bar{W}) = 0$ we can find an integer solution to $\bar{A}x = \bar{B}y$ in polynomial time. By extending the vectors x, y with ε entries if necessary we can obtain a $\overline{\mathbb{Z}}$ solution to $A^{sc}x = B^{sc}y$. Using the ideas in the proof of Proposition 10.11 we get a $\overline{\mathbb{Z}}$-solution to $A'x = B'y$ which can be adapted to obtain a $\overline{\mathbb{Z}}$-solution to the original problem.*

Theorem 10.13 *Algorithm $\overline{\mathbb{Z}}$-SEP-TSS-P1 is correct and terminates after*

$$\mathcal{O}(m^2(m^2+nk))$$

operations.

Proof. Correctness follows from Propositions 10.5, 10.9 and 10.11.

Running Subroutine 1 requires $\mathcal{O}(m^2nk)$ operations, calculating A^{sc} and B^{sc} and rearranging columns requires $\mathcal{O}(m^2(n+k))$ operations and running Subroutine 2 requires $\mathcal{O}(m^4)$ operations. So in total the algorithm requires

$$\mathcal{O}(m^2nk+m^2(n+k)+m^4) = \mathcal{O}(m^2(m^2+nk+n+k)) = \mathcal{O}(m^2(m^2+nk))$$

operations. □

Example 10.14 *Applying Subroutine 1. Let*

$$A = \begin{pmatrix} 0 & -0.5 & \varepsilon \\ 0.5 & 0 & \varepsilon \\ -1.5 & 0 & -0.5 \end{pmatrix} \text{ and } B = \begin{pmatrix} -0.6 & 1 & \varepsilon \\ -1.6 & -0.6 & \varepsilon \\ \varepsilon & \varepsilon & 0 \end{pmatrix}.$$

We calculate $R=\{2\}$, $T_x(2)=\{1,2\}$, $T_y(2)=\{1,2\}$. Then

$$A' = \begin{pmatrix} \varepsilon \\ -0.5 \end{pmatrix}, B' = \begin{pmatrix} \varepsilon \\ 0 \end{pmatrix}.$$

Now, since there exists an ε row we remove the first row from both A' and B'. So

$$A' = \begin{pmatrix} -0.5 \end{pmatrix}, B' = \begin{pmatrix} 0 \end{pmatrix}$$

and we begin a new iteration.

Now $R=\{3\}$ and $T_x(3)=\{3\}=T_y(3)$ so $T_x=N=T_y$. Hence there is no $\overline{\mathbb{Z}}$-solution to $Ax=By$.

Example 10.15 *Calculating square counterparts. Let*

$$A = \begin{pmatrix} 0 & -0.5 & \varepsilon \\ 0.5 & 0 & \varepsilon \\ -1.5 & 0 & -0.5 \end{pmatrix} \text{ and } B = \begin{pmatrix} -0.6 & 1 & \varepsilon \\ -1.6 & -1 & \varepsilon \\ \varepsilon & 0 & 0.6 \end{pmatrix}.$$

Let $\delta_A=0.01$ and $\delta_B=0.02$. In calculating A^{sc} note that A_1 will remain the same, A_3 will be removed and A_2 will be split into two columns

$$A^{sc}_{2_2} = \begin{pmatrix} -0.5 \\ 0 \\ -0.01 \end{pmatrix} \text{ and } A^{sc}_{2_3} = \begin{pmatrix} -0.5 \\ -0.01 \\ 0 \end{pmatrix}.$$

Hence

$$A^{sc} = \begin{pmatrix} 0 & -0.5 & -0.5 \\ 0.5 & 0 & -0.01 \\ -1.5 & -0.01 & 0 \end{pmatrix}, B^{sc} = \begin{pmatrix} 1 & 0.98 & 0.98 \\ -1.02 & -1 & -1.02 \\ -0.02 & -0.02 & 0 \end{pmatrix}.$$

So there exists a $\overline{\mathbb{Z}}$-solution to $Ax = By$ if and only if there exists a $\overline{\mathbb{Z}}$ solution to $A^{sc}x' = B^{sc}y'$. Further since A^{sc} is finite such a solution exists if and only if $\lambda(W) = 0$, i.e. there exists an integer solution to $A^{sc}x' = B^{sc}y'$.

Example 10.16 *Applying Subroutine 2. Let*

$$A = \begin{pmatrix} 0 & -0.5 & \varepsilon & -0.5 \\ 0.5 & 0 & \varepsilon & \varepsilon \\ -1.5 & \varepsilon & 0 & -0.5 \\ -1.5 & 0.5 & \varepsilon & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & -0.6 & \varepsilon & -0.6 \\ -0.6 & -1 & \varepsilon & \varepsilon \\ -1.6 & \varepsilon & 0 & \varepsilon \\ \varepsilon & -0.6 & -0.6 & 0 \end{pmatrix}.$$

Now

$$W = \begin{pmatrix} 0 & 1 & -1 & -1 \\ 1 & 0 & \varepsilon & 1 \\ \varepsilon & \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 & 0 \end{pmatrix}$$

for which $\lambda(W) = 1$ and $\sigma = (1, 2)$ is a critical cycle.

So $S = \{1, 2\}$ and $T(0) = \{1, 2, 4\}$ since $a_{14} > \varepsilon$. Now $T(1) = \{1, 2, 3, 4\}$ since $b_{43} > \varepsilon$. But then this means that there is no $\overline{\mathbb{Z}}$ solution.

Example 10.17 *Determine whether there exists a $\overline{\mathbb{Z}}$ solution to $Ax = By$ where*

$$A = \begin{pmatrix} 0 & -1.5 & \varepsilon & \varepsilon \\ 0.5 & 0 & \varepsilon & \varepsilon \\ \varepsilon & -0.5 & 0 & -0.5 \\ 0.5 & \varepsilon & -0.5 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & -0.6 & \varepsilon & -0.6 \\ \varepsilon & -1 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0 & \varepsilon \\ -0.6 & 0.6 & 0 & 0.6 \end{pmatrix}.$$

First note that (A, B) satisfies Property OneFP. So we calculate the square counterparts using $\delta_A = 0.01$ and $\delta_B = 0.02$:

$$A^{sc} = A, B^{sc} = \begin{pmatrix} 0 & -0.6 & \varepsilon & \varepsilon \\ \varepsilon & -1 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0 & -0.02 \\ -0.6 & 0.6 & -0.02 & 0 \end{pmatrix}.$$

So we know that y_4 is inactive and look for $x \in \overline{\mathbb{Z}}^4$ and $y = (y_1, y_2, y_3^{(1)}, y_3^{(2)})^T \in \overline{\mathbb{Z}}^4$ such that $A^{sc}x = B^{sc}y$. To do this we calculate that,

$$W = \begin{pmatrix} 0 & 1 & \varepsilon & 1 \\ 1 & 0 & 0 & 2 \\ \varepsilon & \varepsilon & 0 & 0 \\ \varepsilon & \varepsilon & 0 & 0 \end{pmatrix}$$

for which $\lambda(W) = 1$ and $\sigma = (1, 2)$.

So $S = \{1, 2\} = T$ which gives us

$$\bar{A} = \begin{pmatrix} 0 & -0.5 \\ -0.5 & 0 \end{pmatrix}, \bar{B} = \begin{pmatrix} 0 & -0.02 \\ -0.02 & 0 \end{pmatrix}, \bar{W} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Now $\lambda(\bar{W}) = 0$ and so a $\overline{\mathbb{Z}}$-solution exists. We have, for any $\alpha \in \mathbb{Z}$,

$$\bar{A} \begin{pmatrix} \alpha \\ \alpha \end{pmatrix} = \bar{B} \begin{pmatrix} \alpha \\ \alpha \end{pmatrix}$$

which then means that

$$A^{sc} \begin{pmatrix} \varepsilon \\ \varepsilon \\ \alpha \\ \alpha \end{pmatrix} = B^{sc} \begin{pmatrix} \varepsilon \\ \varepsilon \\ \alpha \\ \alpha \end{pmatrix}.$$

We conclude that

$$A \begin{pmatrix} \varepsilon \\ \varepsilon \\ \alpha \\ \alpha \end{pmatrix} = B \begin{pmatrix} \varepsilon \\ \varepsilon \\ \alpha \\ y_4 \end{pmatrix}$$

for any small enough $y_4 \in \overline{\mathbb{Z}}$.

We end this section with a necessary and sufficient condition for the existence of a $\overline{\mathbb{Z}}$-solution to $Ax = By$ under the assumption that Property OneFP holds, this is Theorem 10.19 below. For this we will use the following result.

Proposition 10.18 *Suppose $A \in \overline{\mathbb{R}}^{m \times n}, B \in \overline{\mathbb{R}}^{m \times k}$ satisfy Property OneFP. Let $A^{sc}, B^{sc} \in \overline{\mathbb{R}}^{m \times m}$ be the square counterparts of A and B. Then $W(A, B) = W(A^{sc}, B^{sc})$.*

Proof. Let $W = W(A, B)$ and $W^{sc} = W(A^{sc}, B^{sc})$. Fix $i \in M$. Recall that row i of W is equal to H^T where

$$H = a_{ir(i)}^{-1} \lceil A_{r(i)} \rceil \oplus b_{ir'(i)} \lceil B_{r'(i)} \rceil.$$

Let $p \in M$ be such that A_p^{sc} corresponds to $A_{r(i)}$ and $a_{ip}^{sc} \in \mathbb{Z}$. Similarly let $q \in M$ be such that B_q^{sc} corresponds to $B_{r'(i)}$ and $b_{iq}^{sc} \in \mathbb{Z}$.

Then row i of W^{sc} is equal to G^T where

$$G = (a_{ip}^{sc})^{-1} \lceil A_p^{sc} \rceil \oplus (b_{iq}^{sc})^{-1} \lceil B_q^{sc} \rceil.$$

Finally observe that $a_{ir(i)} = a_{ip}^{sc}, b_{ir'(i)} = b_{iq}^{sc}$ and $\lceil A_{r(i)} \rceil = \lceil A_p^{sc} \rceil, \lceil B_{r'(i)} \rceil = \lceil B_q^{sc} \rceil$ so $H = G$. $\square$

Theorem 10.19 *Suppose $A \in \overline{\mathbb{R}}^{m \times n}, B \in \overline{\mathbb{R}}^{m \times k}$ satisfy Property OneFP. Let $W = W(A, B)$. Then there exists a $\overline{\mathbb{Z}}$-solution to $Ax = By$ if and only if there exists $T \subseteq N$ such that*

$$\lambda(W[T]) = 0 \textit{ and } (\forall i \in T)(\forall j \in N - T) w_{ij} = \varepsilon.$$

Proof. Let $A^{sc}, B^{sc} \in \overline{\mathbb{R}}^{m \times m}$ be the square counterparts of A and B. We know that there exists a $\overline{\mathbb{Z}}$-solution to $Ax = By$ if and only if there exists a $\overline{\mathbb{Z}}$-solution to $A^{sc}\bar{x} = B^{sc}\bar{y}$. Further $W(A, B) = W(A^{sc}, B^{sc})$ by Proposition 10.18. So we prove the result for A^{sc} and B^{sc}.

For the rest of the proof we assume $A := A^{sc}, B := B^{sc}$ and $r(i) = i = r'(i)$.

To prove the necessary direction assume that $Ax = z = By$ where $x, y, z \in \overline{\mathbb{Z}}^m$. Recall that (14) holds under our current assumptions and let

$$T = \{i \in M : x_i > \varepsilon\} = \{i \in M : y_i > \varepsilon\} = \{i \in M : z_i > \varepsilon\}.$$

Then $x[T], y[T] \in \mathbb{Z}^{|T|}$ and $A[T]x[T] = B[T]y[T]$ meaning that $\lambda(W[T]) = 0$. Now assume that there exists $i \in T, j \in N - T$ such that $w_{ij} > \varepsilon$. Then, by definition of W either $a_{ji} = a_{jr(i)} > \varepsilon$ or $b_{ji} = b_{jr'(i)} > \varepsilon$. Since $x_i, y_i > \varepsilon$ this implies that

$$z_j \geq \max(a_{ji} + x_i, b_{ji} + y_i) > \varepsilon$$

a contradiction. Therefore $w_{ij} = \varepsilon$ for all $i \in T, j \in N - T$.

For the sufficient direction assume

$$\lambda(W[T]) = 0 \text{ and } (\forall i \in T)(\forall j \in N - T) w_{ij} = \varepsilon.$$

Let $x, y, z \in \mathbb{Z}^{|T|}$ be such that $A[T]x = z = B[T]y$. Define $\bar{x}, \bar{y}, \bar{z} \in \overline{\mathbb{Z}}^m$ by

$$\bar{x}_j = \begin{cases} x_j & \text{if } j \in T \\ \varepsilon & \text{else,} \end{cases} \quad \bar{y}_j = \begin{cases} y_j & \text{if } j \in T \\ \varepsilon & \text{else} \end{cases} \quad \text{and } \bar{z}_j = \begin{cases} z_j & \text{if } j \in T \\ \varepsilon & \text{else.} \end{cases}$$

We claim that $A\bar{x} = \bar{z} = B\bar{y}$. Clearly $(A\bar{x})_i = \bar{z}_i = (B\bar{y})_i$ for all $i \in T$ so we need only show that $(A\bar{x})_i = \varepsilon = (B\bar{y})_i$ when $j \in N - T$.

Fix $j \in N - T$. Now

$$\begin{aligned} &(\forall i \in T) w_{ij} = \varepsilon \\ \therefore &(\forall i \in T) a_{ji} = \varepsilon = b_{ji} \\ \therefore &(\forall i \in T) a_{ji} x_i = \varepsilon = b_{ji} y_i \\ \therefore &(\forall i \in N) a_{ji} x_i = \varepsilon = b_{ji} y_i. \end{aligned}$$

This proves the claim and hence a $\overline{\mathbb{Z}}$-solution to $Ax = By$ exists. □

11 $\overline{\mathbb{Z}}$-solutions to IMLP under weak Property OneFP

In Subsection 10.1 we described a strongly polynomial method to determine whether $\overline{\mathbb{Z}}$-solutions to TSS weakly satisfying Property OneFP exist. Now we explore whether this extends to a method for solving the IMLP when we look for $\overline{\mathbb{Z}}$-solutions.

We will assume that $f \in \mathbb{R}^n$.

For a matrix A and vector c we use $(A|c)$ to denote the matrix obtained from A by adding c as an extra, final, column.

Note that if either if $(A|c)$ or $(B|d)$ is finite then by Proposition 10.2 it is enough to consider whether an integer solution exists and so our previous methods hold. We therefore assume that $A, B \in \overline{\mathbb{R}}^{m\times n}$ and $c, d \in \overline{\mathbb{R}}^m$. Further we assume throughout that the pair $((A|c), (B|d))$ weakly satisfies Property OneFP.

11.1 Maximisation (finite f)

For $A, B \in \overline{\mathbb{R}}^{m\times n}, c, d \in \overline{\mathbb{R}}^m, f \in \mathbb{R}^n$ we want to find the optimal objective value, and an optimal solution of

$$\begin{aligned} &f^T \otimes x \to \max \\ &\text{s.t } Ax \oplus c = Bx \oplus d, \\ &x \in \overline{\mathbb{Z}}^n. \end{aligned}$$

The first constraint is equivalent to

$$(A|c)\begin{pmatrix}x\\0\end{pmatrix} = (B|d)\begin{pmatrix}x\\0\end{pmatrix}.$$

We conclude that feasible solutions $x \in \overline{\mathbb{Z}}^n$ satisfy

$$\begin{pmatrix}A|c\\I\end{pmatrix}\begin{pmatrix}x\\0\end{pmatrix} = \begin{pmatrix}B|d\\I\end{pmatrix} y \tag{15}$$

for some $y \in \overline{\mathbb{Z}}^{n+1}$.

We first test whether there exists a $\overline{\mathbb{Z}}$-solution to (15) using Algorithm $\overline{\mathbb{Z}}$-SEP-TSS-P1 with input

$$A' := \begin{pmatrix}A|c\\I\end{pmatrix}, B' := \begin{pmatrix}B|d\\I\end{pmatrix}.$$

If no solution exists then the problem is infeasible.

For any $x' \in \overline{\mathbb{Z}}^{n+1}$ satisfying $A'x' = B'y$ we require that $x'_{n+1} > \varepsilon$ since any feasible solution has $x'_{n+1} = 0$. Algorithm $\overline{\mathbb{Z}}$-SEP-TSS-P1 first removes columns j such that for any $\overline{\mathbb{Z}}$-solution $x'_j = \varepsilon$. If it concludes that $x'_{n+1} = \varepsilon$ then the corresponding maximisation problem is infeasible. When solving the equivalent problem with square counterparts the algorithm removes columns j in which no integer entry appears. Since each column of our input matrices contains a 0 entry from the identity column the algorithm will not remove any more columns from the problem.

Now we consider the case when Algorithm $\overline{\mathbb{Z}}$-SEP-TSS-P1 concludes that a $\overline{\mathbb{Z}}$-solution exists to $A'x' = B'y'$ for which $x'_{n+1} > \varepsilon$.

Algorithm $\overline{\mathbb{Z}}$-SEP-TSS-P1 will output square counterparts $\bar{A}, \bar{B}$ of A', B' and $\bar{W}$. Let $\bar{x}$ be the subvector of x' with duplicated components corresponding to the columns of $\bar{A}$, (recall that $\bar{A}$ is obtained from A' by removing inactive columns and duplicating columns with more than one integer entry, so the corresponding vector will have some duplicated components). We know that $\bar{A}\bar{x} = \bar{z}$ where $\bar{z} \in IV^*(\bar{W}, 0)$.

It should be observed here that $\bar{x}$ contains all entries of x' which can be finite. So these are the only possibilities for active entries of $f^{\max}$ (for $f^{\max}$ finite). Define $\bar{f}$ to be the vector obtained from f by removing entries corresponding to inactive columns and duplicating entries exactly as in $\bar{x}$.

Let $\bar{f}^{\max}$ be the optimal objective value of the IMLP given by

$$\begin{aligned}&\bar{f}^T \otimes \bar{x} \to \max\\ &\text{s.t } \bar{A}\bar{x} = \bar{B}\bar{x}\\ &\bar{x} \in \mathbb{Z}^k.\end{aligned}$$

We also denote the optimal solution of this IMLP by $\bar{x}^{opt}$. Observe that we can find the optimal solution to this system in strongly polynomial time using the methods in [5]. We therefore have the following result for IMLP$^{\max}$ under our current assumptions.

Theorem 11.1 *Assume that the system weakly satisfies Property OneFP. Then* $f^{\max} = \bar{f}^{\max}$ *and an optimal solution* x *can be obtained from* $\bar{x}^{opt}$ *by condensing duplicated entries and setting the remaining values to* ε.

11.2 Minimisation (finite f)

We will begin by proceeding as in the maximisation case, but the final result obtained in this way will not be as strong.

For $A, B \in \overline{\mathbb{R}}^{m\times n}, c, d \in \overline{\mathbb{R}}^{m}, f \in \mathbb{R}^n$ we want to find the optimal objective value, and an optimal solution of

$$\begin{aligned} &f^T \otimes x \rightarrow \min \\ &\text{s.t } Ax \oplus c = Bx \oplus d \\ &x \in \overline{\mathbb{Z}}^n. \end{aligned}$$

As before feasible solutions $x \in \overline{\mathbb{Z}}^n$ satisfy (15) for some $y \in \mathbb{Z}^{n+1}$ and we test whether there exists a $\overline{\mathbb{Z}}$-solution to (15) using Algorithm $\overline{\mathbb{Z}}$-SEP-TSS-P1 with input A', B' defined previously.

If $A'x' = B'y$ has no $\overline{\mathbb{Z}}$-solution, or if the algorithm concludes that for any solution $x'_{n+1} = \varepsilon$ then the corresponding minimisation problem is infeasible. Otherwise Algorithm $\overline{\mathbb{Z}}$-SEP-TSS-P1 will output square counterparts $\bar{A}, \bar{B}$ of A', B' and $\bar{W}$. Let $\bar{x}$ be the subvector of x' with duplicated components. We know that $\bar{A}\bar{x} = \bar{z}$ where $\bar{z} \in IV^*(\bar{W}, 0)$.

Let $\bar{f}^{\min}$ be the optimal objective value of the IMLP given by

$$\begin{aligned} &\bar{f}^T \otimes \bar{x} \rightarrow \min \\ &\text{s.t } \bar{A}\bar{x} = \bar{B}\bar{x} \\ &\bar{x} \in \mathbb{Z}^k. \end{aligned}$$

We also denote the optimal solution of this IMLP by $\bar{x}^{opt}$. Observe here that we cannot say that $f^{\min} = \bar{f}^{\min}$ as in the maximisation case. This is because the algorithm finds the set of all possible finite components of x, but for minimisation it could happen that a solution exists with less active entries, so more components of x could be set to ε.

Theorem 11.2 *Assume that the system weakly satisfies Property OneFP. Then $f^{\min} \le \bar{f}^{\min}$ and an upper bound on x^{opt} can be obtained from $\bar{x}^{opt}$ by condensing duplicated entries and setting the remaining values to ε.*

It remains to fully solve the minimisation problem and to consider when $f \in \overline{\mathbb{R}}^n$.

References

[1] P. Butkovič, *Max-linear Systems: Theory and Algorithms*, Springer-Verlag, London, 2010.

[2] P. Butkovič, H. Schneider, S. Sergeev, *Z-matrix equations in max-algebra, nonnegative linear algebra and other semirings*, Linear and Multilinear Algebra 60 (10), 1191-1210, 2012.

[3] P. Butkovič, M. MacCaig *On integer eigenvectors and subeigenvectors in the max-plus algebra*, Linear Algebra and its Applications 438 (2013), 3408 - 3424.

[4] P. Butkovič, M. MacCaig, *On the integer max-linear programming problem*, Discrete Applied Mathematics 162 (2014), 128-141.

[5] P. Butkovič, M. MacCaig, *A strongly polynomial method for solving integer max-linear programs in a generic case*, Journal of Optimization Theory and Applications (2014). DOI: 10.1007/s10957-014-0596-5.

[6] R. A. Cuninghame-Green, Process synchronisation in a steelworks-a problem of feasibility. In Banbury & Maitland (Eds.), *Proc. 2nd int. conf. on operational research* 323-328, London, English University Press, 1960.

[7] R. A. Cuninghame-Green, *Minimax algebra*, Lecture notes in economics and math systems, Vol. 166, Springer, Berlin, 1979.

[8] M. MacCaig, *Integrality in max-algebra*, PhD thesis, 2015.