Comparison of methods for solving two-sided systems in max-algebra

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Abstract

We compare the computational performance of two methods for solving two-sided systems of the form $A \otimes x = B \otimes x$ in max-algebra.

1 Introduction

Given $A = (a_{ij}), B = (b_{ij}) \in \mathbb{R}^{m\times n}$, a system of the form

$$A \otimes x = B \otimes x \tag{1}$$

where $x \in \mathbb{R}^n$, is called homogeneous max-linear system. Homogeneous max-linear systems (1) have been investigated in several articles e.g [2], [4], [6], [8], [3]. Several solution methods for system (1) have also been developed. The aim of this paper is to compare the performance of two of these methods: The Stepping Stone Method (SSM) [3] and the Alternating Method (AM) [4] .

2 The Stepping Stone Method

The Stepping Stone Method (SSM) for solving two-sided linear systems was presented in [3]. The method finds a solution to system (1) or decides that no solution exists. The algorithm is as follows: Assuming that m and n are integers, define $x = (x_1, \dots, x_n) \in \mathbb{Q}^n$, $M = \{1, \dots, m\}$ and $N = \{1, \dots, n\}$. It is also assumed that a_{ij}, b_{ij} $(i \in M, j \in N)$ are given rational numbers. Denote for all $i \in M$, $a_i(x) = max_{j\in N}(a_{ij} + x_j)$, $b_i(x) = max_{j\in N}(b_{ij} + x_j)$. Therefore, the algorithm finds a solution to the system:

$$a_i(x) = b_i(x),\ i \in M \tag{2}$$

or decides that no solution exists.
Denote by T the set of all rational solutions of (2). Furthermore define for any $x^{\#} \in \mathbb{Q}^n$

$$T(x^{\#}) \equiv \{x \mid x \in M \ \&\ x \leq x^{\#}\}. \tag{3}$$

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Lemma 2.1. [3]
Let $x^{\#} \in \mathbb{Q}^n$ be given. Then $T \neq \emptyset$ if and only if $T(x^{\#}) \neq \emptyset$.

Let $K \subseteq \mathbb{Q}^n, \tilde{x} \in K$. Then $\tilde{x}$ is called the maximum element of K if $x \leq \tilde{x}$ for every $x \in K$.
The algorithm starts with arbitrary chosen and fixed $x^{\#} \in \mathbb{R}^n$ and either finds the maximum element of $T(x^{\#})$ for a given upper bound $x^{\#}$ or find out that $T(x^{\#})$ is empty. Thus it is assumed that $x^{\#} \notin T(x^{\#})$ else $x^{\#}$ is the maximum element of $T(x^{\#})$. It is assumed that:

$$a_i(x^{\#}) \geq b_i(x^{\#}),\ i \in M \tag{4}$$

otherwise swap the inequality sides appropriately.
Denote for any $x \in \mathbb{R}^n$ and for all $i \in M$:

$$F_i(x) = \{j \in N \mid a_{ij} + x_j = a_i(x)\}.$$
$$G_i(x) = \{j \in N \mid b_{ij} + x_j = b_i(x)\}.$$

Let $H(x^{\#}) \equiv \{j \in N \mid a_i(x^{\#}) > b_i(x^{\#})\}$. To find an element of $T(x^{\#})$, the algorithm decreases $a_i(x^{\#})$ for all $i \in H(x^{\#})$. That is all variables $x_j^{\#},\ j \in A(x^{\#})$ where:

$$A(x^{\#}) \equiv \bigcup_{i \in H(x^{\#})} F_i(x^{\#}). \tag{5}$$

The following subroutine is designed to find the set $P(x^{\#})$ of indices of all variables, which have to be decreased while the equalities $a_i(x^{\#}) = b_i(x^{\#})$ for $i \in M \setminus H(x^{\#})$ be preserved.

Subroutine 1.

1. $P(x^{\#}) := A(x^{\#})$
2. $E_1 := \{i \in M \setminus H(x^{\#}) \mid F_i(x^{\#}) \subseteq P(x^{\#})\ \&\ G_i(x^{\#}) \not\subseteq P(x^{\#})\}$,
   $E_2 := \{i \in M \setminus H(x^{\#}) \mid F_i(x^{\#}) \not\subseteq P(x^{\#}) \& G_i(x^{\#}) \subseteq P(x^{\#})\}$.
3. If $E_1 \cup E_2 = \emptyset$ then $P(x^{\#})$ is the set of indices of variables to be decreased, STOP.
4. $P(x^{\#}) := P(x^{\#}) \cup \bigcup_{i \in E_1} (G_i(x^{\#}) \setminus P(x^{\#})) \cup \bigcup_{i \in E_2} (F_i(x^{\#}) \setminus P(x^{\#}))$.
5. Go to 2.

After the collection of all indices of variables to be decreased, the process of decreasing variables $x_j,\ j \in P(x^{\#})$ is described as follows:
Define $x(t) = (x_1(t), \ldots, x_n(t))$ for $t \geq 0$ as follows:

$$x_j(t) \equiv \begin{cases} x_j^{\#} - t & \text{if } j \in P(x^{\#}), \\ x_j^{\#} & \text{otherwise.} \end{cases} \tag{6}$$

So that if the parameter t is increased the variable $x_j,\ j \in P(x^{\#})$ will be decreased. That is $x_j(0) = x_j^{\#},\ x_j(t) < x_j^{\#}$ for any $t > 0$ and $j \in P(x^{\#})$.

It can be assumed without loss of generality that $P(x^{\#}) \neq N$ (it follows from theorem (2.1) below that $T(x^{\#}) = \emptyset$, if $H(x^{\#}) \neq \emptyset$ and $P(x^{\#}) = N$).
Define the following:

$$L_1 \equiv \{i \in M \mid F_i(x^{\#}) \subseteq P(x^{\#})\},$$
$$L_2 \equiv \{i \in M \mid G_i(x^{\#}) \subseteq P(x^{\#})\},$$
$$L_3 \equiv \{i \in H(x^{\#}) \mid G_i(x^{\#}) \nsubseteq P(x^{\#})\}.$$

The parameter t will be increased until at least one of the following occurs for the first time:

$$(i) F_i(x(t)) \neq F_i(x^{\#}), \text{i.e } a_i(x(t)) = \alpha_i(x^{\#}) \equiv \max_{j \in N \setminus P(x^{\#})} (a_{ij} + x_j^{\#}) \text{ for some } i \in L_1;$$

$$(ii) G_i(x(t)) \neq G_i(x^{\#}), \text{i.e } b_i(x(t)) = \beta_i(x^{\#}) \equiv \max_{j \in N \setminus P(x^{\#})} (b_{ij} + x_j^{\#}), \text{for some } i \in L_2;$$

$$(iii) H(x(t)) \neq H(x^{\#}), \text{i.e } a_i(x(t)) = \beta_i(x^{\#}) \text{ for some } i \in L_3.$$

If $P(x^{\#}) = N$, set $\alpha_i(x^{\#}) = \beta_i(x^{\#}) = -\infty$ for all $i \in M$.
Also determine the values $t_1,\ t_2,\ t_3$ at which the cases $(i),\ (ii),\ (iii)$ occur.
The values $\alpha_i(x^{\#}),\ \beta_i(x^{\#})$ are always finite if $P(x^{\#}) \neq N$.
The formulae for determining $t_1,\ t_2$ are determined from case (i) and (ii) as follows:

$$t_1 = \min_{i \in L_1} (a_i(x^{\#}) - \alpha_i(x^{\#})). \tag{7}$$

$$t_2 = \min_{i \in L_2} (b_i(x^{\#}) - \beta_i(x^{\#})). \tag{8}$$

In the case (iii) t_3 was obtained as follows:

$$t_3 = \min_{i \in L_3} (a_i(\check{x}) - \beta_i(x^{\#})). \tag{9}$$

Set $t_i = \infty$ if $L_i = \emptyset$ $(1 \leq i \leq 3)$. Also set $\tau = min(t_1, t_2, t_3)$ and use on the next iteration $x(\tau)$ as the new upper bound. The next iteration will begin with the new upper bound if $H(x(\tau)) \neq \emptyset$.
If $P(x^{\#}) = N$, set by definition $t_i = \infty$ for $i = 1, 2, 3$ and thus $\tau = \infty$.
The following algorithm summarizes all the previous considerations, calls Subroutine 1 in its step 3 and finds a solution to a given two-sided system if it exists and returns with 'no solution' otherwise.

Algorithm 1. (The Stepping Stone Method)
Input: $A, B \in \mathbb{Q}^{m \times n}$, $x^{\#}$.
Output: $\bar{y}$, a solution to $A \otimes x = B \otimes x$ or 'no solution'.

1. $\bar{y} := x^{\#}$.
2. If $H(\bar{y}) = \emptyset$ then $\bar{y}$ is the maximum element of $T(x^{\#})$, STOP.
3. Find $A(\bar{y})$ and $P(\bar{y})$ using the Subroutine 1 (replacing $x^{\#}$ with $\bar{y}$). If $P(\bar{y}) = N$ then $T(x^{\#}) = \emptyset$, STOP ('no solution').
4. Define $x(t)$ using (6) and t_1, t_2, t_3 using (7),(8),(9) (with $x^{\#}$ replaced by $\bar{y}$ everywhere).

5. $\tau := min(t_1, t_2, t_3)$,$\bar{y} := x(\tau)$.

6. Go to 2.

Theorem 2.1. [3]
If on some current iteration of Algorithm 1 $\bar{y} \notin T(\bar{y})$ and $P(\bar{y}) = N$, then $T(\bar{y}) = \emptyset$.

3 The Alternating Method

The Alternating Method (AM) is an iterative procedure designed for solving systems of the form $A \otimes x = B \otimes y$ where $A \in \mathbb{R}^{m\times n},\ B \in \overline{\mathbb{R}}^{m\times k}, \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$. This method was presented in [4] and converges to a finite solution from any finite starting point whenever a finite solution exists.
A matrix A is called *doubly G-astic* [7, 6], if it has at least one finite element on each row and on each column.

Define by $a \oplus^{'} b = min(a, b)$ and $a \otimes^{'} b = a + b$ for $a, b \in \mathbb{R}$ and extend this pair operations to matrices and vectors in the same way as in linear algebra.

Let $x,\ y$ be vectors such that $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$. $x < y$ if and only if for all i, $x_i < y_i$. Systems of one-sided linear inequalities $A \otimes x \leq b$ always possesses a solution [5] and the greatest is :

$$\bar{x} = A^* \otimes^{'} b \tag{10}$$

where $A^* = -A^T$ (that is negation and transposition of matrix A). Also A^* is finite for A doubly G-astic and b finite. The system of equation $A \otimes x = b$ has solution if and ony if $\bar{x}$ is a solution [5, 1]. Based on these the following method for solving two-sided linear systems $A \otimes x = B \otimes y$ was developed:

Algorithm 2. (The Alternating Method)
Input: $A \in \mathbb{R}^{m\times n}$, $B \in \overline{\mathbb{R}}^{m\times k}$, $x \in \mathbb{R}^n$.
Output: $x,\ y$. A solution to $A \otimes x = B \otimes y$ or an indication that there is no solution.

1. $r = 0$, $x(0) = x$.

2. $y = B^* \otimes^{'} (A \otimes x)$, $y(r) = y$.

3. If $r \geq 1$, $y(r) < y(r-1)$, STOP ('no solution').

4. $x = A^* \otimes^{'} (B \otimes y)$, $x(r+1) = x$.

5. If $x(r+1) < x(r)$, STOP ('no solution').

6. $r = r + 1$.

7. If $A \otimes x = B \otimes y$, STOP.

8. Go to 2.

Theorem 3.1. [4]
Let $A = (a_{ij}) \in \mathbb{Z}^{m\times n}$, $B = (b_{ij}) \in \overline{\mathbb{Z}}^{m\times k}$, $\rho = \max\{|a_{ij}|; i \in M,\ j \in N\}$. If B is doubly-G-astic then the Alternating Method terminates in $O(mn(m+k)\rho)$ steps and thus is pseudopolynomial.

4 Conversion of $A \otimes x = B \otimes x$ to $A \otimes x = B \otimes y$

Given $A, B \in \mathbb{R}^{m \times n}$ finding a solution to the problem:

$$A \otimes x = B \otimes x \tag{11}$$

is equivalent to finding a vector $y = (y_1, \dots, y_m)^T$, such that:

$$\begin{aligned} A \otimes x &= y \\ B \otimes x &= y \end{aligned} \tag{12}$$

Max-algebraic *identity matrix* is a matrix with all diagonal entries zero and $-\infty$ else where. We denote by I an identity matrix. Therefore (12) can be written as:

$$\begin{pmatrix} A \\ B \end{pmatrix} \otimes x = \begin{pmatrix} I \\ I \end{pmatrix} \otimes y, \tag{13}$$

where I is the $m \times m$ identity matrix.
System (13) is equivalent to:

$$C \otimes x = D \otimes y, \tag{14}$$

where,

$$C = \begin{pmatrix} A \\ B \end{pmatrix} \in \mathbb{R}^{2m \times n}, \quad D = \begin{pmatrix} I \\ I \end{pmatrix} \in \overline{\mathbb{R}}^{2m \times m}.$$

Since A, B are finite and I is doubly G-astic then C is finite and D doubly G-astic.
Therefore, finding solution to (11) is equivalent to solving (14). Hence if (11) has a solution then (12) has a solution and conversely. Thus, we have obtained:

Theorem 4.1.
$x \in \mathbb{R}^n$ is a solution to (11) if and only if (x, y) is also a solution to (13) for some $y \in \mathbb{R}^m$.

Proof. Straightforward. □

Due to Theorem 4.1 Alternating Method (AM) can therefore be used to find a solution to (1) or decide that it does not exist.

5 Performance of the two methods

Performance of the two methods for solving $A \otimes x = B \otimes x$ is presented in this section, we also compare performance of these methods on the same problem. Matlab programming language was used for designing programs for this task. Specification of the computer used is 1.66.Hz of CPU and 0.99 GB of RAM. These programs can be downloaded from http://web.mat.bham.ac.uk/P.Butkovic/software/. In what follows *range* k means inputs entries for matrices are generated from the interval $[-k, k]$. *Dimension* n means a square matrix of order n.

5.1 The Stepping Stone Method

Performace of the Stepping Stone Method (SSM) was measured within five minutes on matrices of 6 different dimensions. The following table gives the summary of the results.

S/NO	DIMENSION	RANGE	TIME	NO. OF ITERATIONS
1	200	200	1min 40sec	7
	,,	,,	1min 38sec	7
	,,	,,	1min 23sec	6
	,,	,,	1min 25sec	6
	,,	,,	1min 23sec	6
2	200	300	2min	8
	,,	,,	3min 4sec	12
	,,	,,	1 min 42sec	7
	,,	,,	1min 42sec	7
	,,	,,	2min 15sec	9
3	200	1000	6min 41sec	25
	,,	,,	6min 24sec	24
	,,	,,	5 min 32sec	21
	,,	,,	3min 11sec	22
	,,	,,	3min	21
4	200	2000	10min 20sec	38
	,,	,,	10min 31sec	38
	,,	,,	9 min 37sec	35
5	300	300	3min 18sec	6
	,,	,,	4min 30sec	8
	,,	,,	3min 15sec	6
6	400	400	4min 28sec	8
	,,	,,	3min 11sec	6

Table 1 : Test results for the Stepping Stone Method.

5.2 The Alternating Method

Performance of the Alternating Method for solving $A \otimes x = B \otimes x$ is tested on six different problems. The table below gives the summary of the results.

S/NO	DIMENSION	RANGE	TIME	NO. OF ITERATIONS
1	200×200	200	3sec	25
2	,,	10 000	3sec	28
3	,,	500 000	2sec	13
4	500×500	3 000 000	4sec	14
5	1000×1000	10 000 000	17sec	29
6	2000×2000	40 000 000	5min 41sec	202

Table 2: Test results for the Alternating Method.

5.3 Stepping Stone and Alternating Methods on the same problem

Performance of the Stepping Stone and the Alternating Methods are compared using six different examples. Below is the summary of the results.

S/NO	DIMENSION	METHOD	RANGE	ITERATIONS	TIME
1	50	SSM	3000	37	00:22
	,,	AM	3000	4	00:00:6
2	50	SSM	7000	28	00:16
	,,	AM	,,	3	00:00:5
3	200	SSM	5000	68	09:38 :3
	,,	AM	,,	3	00:00:7
4	200	SSM	10000	80	10:06 :02
	,,	AM	,,	3	00:01:2
5	300	SSM	9000	94	30:55 :5
	,,	AM	,,	5	00:01:6
6	300	SSM	3000	43	14:00:9
	,,	AM	,,	4	00:01:4

Table 3 : Stepping Stone and Alternating Methods on the same problem.

Conclusions: From the results obtained in tables 1, 2 and 3, we can conclude that the Alternating Method (AM) is much faster compared to the Stepping Stone Method (SSM).

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