Available online at www.sciencedirect.com
SCIENCE DIRECT®
LINEAR ALGEBRA AND ITS APPLICATIONS
ELSEVIER Linear Algebra and its Applications 389 (2004) 107–120
www.elsevier.com/locate/laa

Bases in max-algebra

R.A. Cuninghame-Green, P. Butkovič *

School of Mathematics and Statistics, The University of Birmingham, Edgbaston, Birmingham B15 2TT, UK

Received 12 May 2003; accepted 8 March 2004

Submitted by R.A. Brualdi

Abstract

For n-tuples over the algebraic system $(\mathbb{R}, \oplus, \otimes) = (\mathbb{R}, \max, +)$, concepts such as linear dependence, space and basis may be defined by analogy with classical linear algebra. Whenever a space is finitely generated, it possesses a basis and all its bases are trivially related and therefore have the same cardinality. However, for any given $n > 2$, spaces with bases of arbitrary cardinality may be constructed, as well as spaces with no basis.

AMS classification: 15A03

Keywords: Max-algebra; Independent set; Generating set; Basis

1. Introduction and preliminaries

1.1. Introduction

If we replace addition and multiplication of real numbers by the operations of taking the maximum of two numbers and of adding two numbers respectively, we obtain the so-called *max-algebra* which offers an attractive language to deal with certain problems in automata theory, scheduling theory, and discrete event systems, see e.g. the monographs of Baccelli et al. [1], Cuninghame-Green [3] and Zimmermann [10]. Among other papers in this area are Cuninghame-Green [4], Gaubert

* Corresponding author.
E-mail address: p.butkovic@bham.ac.uk (P. Butkovič).

0024-3795/$ - see front matter
doi:10.1016/j.laa.2004.03.022

[5] and Gondran and Minoux [7]. Specifically, significant effort has been devoted to building a theory similar to that of linear algebra, as for instance in [3], to systems of linear equations, eigenvalue problems, independence, rank and dimension.

The attractiveness of max-algebra is related to the fact that both its algebraic operations are commutative and associative, and that they satisfy the distributive law. Hence many of the basic tools from classical linear algebra are available in max-algebra as well.

The aim of this paper is to examine the possibility of defining *bases* in max algebra. The emphasis is on bases of finite or finitely generated sets. In the remainder of Section 1 we present introduction, definitions and preliminary results. Section 2 provides the main results: every finite set has a basis which can be found efficiently and all bases have the same cardinality. The question of upper bounds on the size of a basis is discussed. Building on the results of Section 2 we prove in Section 3 that every set which is finitely generated has a finite basis and that these two properties are essentially equivalent (Theorem 3.2). We then use *range seminorms* to prove non-existence of finite bases for a certain type of subspace. It follows from these results that in particular $\mathbb{R}^n$ has no finite basis. In Section 4 we then show that although $\mathbb{R}^n$ has a countable generating set, it does not have a basis of any cardinality.

We note that the results of this paper are strongly related to [8] in which similar questions were studied for pseudomodules, and even more general structures. If our ground set $\mathbb{R}$ was extended to $\mathbb{R} \cup \{-\infty\}$ then some of the results of this paper would immediately follow from [8], most importantly the existence of a unique basis up to scaling for any finitely generated set. To the authors' knowledge this immediate inference does not apply to finitely generated sets of vectors with finite entries, which is obviously a case of practical importance. The proof of this unique existence statement in the present paper relies on arguments completely different from those in [8]. Unlike that paper we also study quantitative and algorithmic aspects of dimension, bases of seminorm-bounded sets (Theorems 3.4–3.6), bases of $\mathbb{R}^n$ (Theorem 3.7) and infinite bases (Theorem 4.1).

1.2. Definitions

Let us denote $a \oplus b = \max(a, b)$ and $a \otimes b = a + b$ for $a, b \in \mathbb{R}$. The iterated product $a \otimes a \otimes \cdots \otimes a$ in which the element a is used k-times will be denoted by $a^{(k)}$. Consistently, we should write $a^{(-1)}$ for $-a$ but to avoid notational complexity we shall write simply a^{-1}.

Let us extend the pair of operations $(\oplus, \otimes)$ to matrices and vectors in the same way as in conventional linear algebra. That is, if $A = (a_{ij})$, $B = (b_{ij})$ are matrices or vectors over $\mathbb{R}$ of compatible sizes then we write $C = A \oplus B$ if $c_{ij} = a_{ij} \oplus b_{ij}$ for all i, j and $C = A \otimes B$ if $c_{ij} = \sum_k^{\oplus} a_{ik} \otimes b_{kj}$ for all i, j. We also define $\alpha \otimes A = (\alpha \otimes a_{ij})$ for $\alpha \in \mathbb{R}$.

For any set X and positive integers n, m the symbol $X^{n \times m}$ will denote the set of all $n \times m$ matrices over X.

Throughout the paper W will be a given (finite or infinite) set of n -tuples (called *vectors*) from $\mathbb{R}^n$. If U is a nonempty finite subset $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_t\}$ of W and $\mathbf{w} \in W$, we write $\mathbf{w} \sim U$ to denote the existence of a *linear dependence*

$$\mathbf{w} = \sum_{\substack{j=1,\ldots,t \\ \mathbf{u}_j \neq \mathbf{w}}}^{\oplus} \lambda_j \otimes \mathbf{u}_j. \tag{1.1}$$

If V is also a subset of W, we write $V \sim U$ to mean that $\mathbf{v} \sim U$ for all $\mathbf{v} \in V$. Now 1.1 states that $\mathbf{w}$ is expressible as a linear combination of all elements of U, with the exception of $\mathbf{w}$ if $\mathbf{w} \in U$. As the following lemma suggests, apart from this exception, there is no loss of generality in including all elements of U in the above linear combination.

Lemma 1.1. *For any* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ *there is a* $\lambda \in \mathbb{R}$ *such that* $\mathbf{x} \oplus \lambda \otimes \mathbf{y} = \mathbf{x}$.

Proof. If $\mathbf{x} = (x_1, \ldots, x_n)^{\mathrm{T}}, \mathbf{y} = (y_1, \ldots, y_n)^{\mathrm{T}}$ then for the λ we may take any value not exceeding $\min(x_1 \otimes y_1^{-1}, \ldots, x_n \otimes y_n^{-1})$. $\square$

A nonempty, finite subset U of the set W is called

- *generating* (W) if $\mathbf{w} \sim U$ for every $\mathbf{w} \in W \setminus U$,
- *independent* if $\mathbf{w} \sim U$ does not hold for any $\mathbf{w} \in U$ and
- a *basis* of W if it is both a generating and independent subset of W.

It immediately follows from these definitions that the empty set and all one-element sets are independent. Also, U is a basis of W if and only if for each $\mathbf{w} \in W$, either $\mathbf{w} \in U$ or $\mathbf{w} \sim U$ but not both. Clearly, if U, V are bases of W and $U \subseteq V$ then $U = V$. In fact a somewhat stronger statement holds: If U is generating and V is independent then $U \subseteq V$ implies $U = V$.

For convenience we shall use the word "space" to denote any set of vectors closed with respect to $\oplus$ and to $\otimes$-multiplication by a scalar. Notice that this does not presume the existence of a neutral element.

1.3. Checking linear dependence

If $A = (a_{ij}) \in \mathbb{R}^{n \times m}$ and $\mathbf{b} = (b_1, \ldots, b_n)^{\mathrm{T}} \in \mathbb{R}^n$ then a *max-algebraic linear equation system* (briefly a *linear system* or *system*) is

$$\sum_{j=1,\ldots,m}^{\oplus} a_{ij} \otimes x_j = b_i \quad (i = 1, \ldots, n) \tag{1.2}$$

or, in a more compact form

$$A \otimes \mathbf{x} = \mathbf{b}. \tag{1.3}$$

If we $\otimes$-multiply the ith equation in (1.2) by b_i^{-1} $(i = 1, \ldots, n)$ then all right-hand sides will become zero. We shall call such a system *normalized*. Let us denote the set of row indices $\{1, \ldots, n\}$ by N, the set of column indices $\{1, \ldots, m\}$ by M and let $M_j = \{k \in N; a_{kj} = \max_{i \in N} a_{ij}\}$ for all $j \in M$. The following is a standard solubility criterion for normalized systems [3,9].

Theorem 1.1. *The system $A \otimes \mathbf{x} = \mathbf{0}$ is soluble in $\mathbb{R}^m$ if and only if $\bigcup_{j=1}^{m} M_j = N$ (or, equivalently, every row of A contains a column maximum).*

We call $\mathbf{w} \in U$ *dependent in U* if (1.1) holds for some λ_j's and *free in U* otherwise. Theorem 1.1 offers a simple method for deciding which of the columns of a given matrix are dependent in the set of columns.

Algorithm FREECOLUMNS
Input: Matrix $A = (a_{ij}) \in \mathbb{R}^{n \times m}$
Output: Decision about each column of A whether it is dependent in the set of all columns of A.
<1> For each $l = 1, \ldots, m$ do steps <2 >–<4>:
<2> For all $i = 1, \ldots, n,\ j = 1, \ldots, m,\ j \neq l$ find $z_{ij} := a_{ij} \otimes a_{il}^{-1}$.
<3> For all $j = 1, \ldots, m,\ j \neq l$ find $M_j := \{k \in N; z_{kj} = \max_i z_{ij}\}$.
<4> If $\bigcup_{j=1, j \neq l}^{m} M_j = N$ then column l is dependent, else it is free, in the set of all columns.

It is easily verified that the computational complexity of **FREECOLUMNS** is $O(m^2 n)$.

2. Bases of finite sets

2.1. Free sibling classes

Throughout this and next section, $W = \{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ $(m \geqslant 2)$ will be a given finite set of n-tuples over $\mathbb{R}$.

By construing these n-tuples as columns of a matrix, we may use the algorithm **FREECOLUMNS** to determine the n-tuples free in W, and those dependent in W.

Theorem 2.1. *The elements, if any, free in W lie in every basis, if any, of W.*

Proof. Let U be a basis of W. If $\mathbf{w}$ is free in W, we cannot have $\mathbf{w} \sim U$ and so we cannot have $\mathbf{w} \in W \setminus U$ since U is generating. $\square$

However, this takes us only part of the way towards establishing a basis, because the status of the elements dependent in W is unclear if there is more than one basis, since elements in one basis may be a linear combination of those in another. There may in consequence even be no elements free in W. And it is not clear that different bases must have the same cardinality.

These difficulties all flow from the relation of *siblinghood*: Elements $\mathbf{w}_i, \mathbf{w}_j$ of W are called *siblings* if $\mathbf{w}_i = \lambda \otimes \mathbf{w}_j$ for some $\lambda \in \mathbb{R}$. Since this definition allows the possibility that $i = j$, it implies the decomposition of W into equivalence classes of siblings, or *sibling classes*. It is clear that the elements, if any, free in W all have singleton sibling classes.

A subset of W which contains exactly one element from each sibling class will be called a *section* of W.

Theorem 2.2. *If in a certain section of W there are elements free in this section, suppose they are exactly those lying in sibling classes $C_1, \ldots, C_h$. Then there are elements free in any section of W, namely those lying in sibling classes $C_1, \ldots, C_h$.*

Proof. Since any linear dependence, say of $\mathbf{w}_i$ on $\mathbf{w}_{j_1}, \ldots, \mathbf{w}_{j_t}$ trivially implies a linear dependence of any sibling of $\mathbf{w}_i$ on any siblings of $\mathbf{w}_{j_1}, \ldots, \mathbf{w}_{j_t}$, no element of C_1 in any other section can be linearly dependent on elements of other sibling classes in that other section. Similarly for $C_2, \ldots, C_h$. Since the situation is symmetrical between the two sections, the result follows. □

In the affirmative case of the foregoing theorem, we shall call $C_1, \ldots, C_h$ the *free sibling classes*. The algorithm **FREECOLUMNS** for finding the columns of a matrix free in the set of all columns may be adapted to finding free sibling classes of W. Specifically, at each pass through the elements of W to determine the dependence or freedom of a particular $\mathbf{w} \in W$, we may recognize the siblings of $\mathbf{w}$ and remove them from consideration, simultaneously building the sibling class of $\mathbf{w}$. Siblings of $\mathbf{w}$ are recognized as constant columns after the normalization using $\mathbf{w}$ in step <2> (or, equivalently as columns whose every entry is a column maximum, i.e. $M_j = N$). The following algorithm results:

Algorithm SIBLINGCLASS

Input: Set $W = \{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ $(m \geqslant 2)$

Output: The sibling class $S(\mathbf{w}_l)$ of each vector $\mathbf{w}_l \in W$ and the decision whether this class is a free sibling class.

<1> $S := \emptyset$

<2> For each $l = 1, \ldots, m, l \notin S$ do steps <3>–<6>:

<3> For all $i = 1, \ldots, n$, $j \in M \setminus S$ find $z_{ij} := w_{ij} \otimes w_{il}^{-1}$

<4> Set $M_l := N$ and for all $j \in M \setminus S$, $j \neq l$
find $M_j := \{k \in N; z_{kj} = \max_i z_{ij}\}$

<5> $S(\mathbf{w}_l) := \{j \in M; M_j = N\}$, $S := S \cup (S(\mathbf{w}_l) \setminus \{l\})$

<6> If $\bigcup_{\substack{j \in M \setminus S \\ j \neq l}} M_j \neq N$ then $S(\mathbf{w}_l)$ is a free sibling class.

Example 2.1. Let W be the set $\left\{\begin{pmatrix}3\\1\end{pmatrix}, \begin{pmatrix}0\\-6\end{pmatrix}, \begin{pmatrix}1\\7\end{pmatrix}, \begin{pmatrix}5\\3\end{pmatrix}, \begin{pmatrix}-1\\5\end{pmatrix}\right\}$. Then the algorithm finds successively

$l = 1 : (z_{ij}) = \begin{pmatrix}0 & -3 & -2 & 2 & -4\\0 & -7 & 6 & 2 & 4\end{pmatrix}$,
$M_1 = \{1, 2\}$, $M_2 = \{1\}$, $M_3 = \{2\}$, $M_4 = \{1, 2\}$, $M_5 = \{2\}$,
$S(\mathbf{w}_1) = \{1, 4\}$, $S = \{4\}$.
$l = 2 : (z_{ij}) = \begin{pmatrix}3 & 0 & 1 & \cdot & -1\\7 & 0 & 13 & \cdot & 11\end{pmatrix}$,
$M_1 = \{2\}$, $M_2 = \{1, 2\}$, $M_3 = \{2\}$, $M_5 = \{2\}$,
$S(\mathbf{w}_2) = \{2\}$, $S = \{4\}$, $S(\mathbf{w}_2)$ is free.
$l = 3 : (z_{ij}) = \begin{pmatrix}2 & -1 & 0 & \cdot & -2\\-6 & -13 & 0 & \cdot & -2\end{pmatrix}$,
$M_1 = \{1\}$, $M_2 = \{1\}$, $M_3 = \{1, 2\}$, $M_5 = \{1, 2\}$,
$S(\mathbf{w}_3) = \{3, 5\}$, $S = \{4, 5\}$, $S(\mathbf{w}_3)$ is free.

2.2. Gaussian analogue

Let $U = \{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ and $V = \{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ be subsets of W. We assume V is nonempty, but initially that U could be empty.

Lemma 2.1. *Suppose $r > 1$ and that for some k $(1 \leqslant k < r)$ there holds*

$$\{\mathbf{v}_k, \ldots, \mathbf{v}_r\} \sim U \cup \{\mathbf{v}_k, \ldots, \mathbf{v}_r\}.$$

Then, provided that none of $\mathbf{v}_{k+1}, \ldots, \mathbf{v}_r$ is a sibling of $\mathbf{v}_k$ there also holds

$$\{\mathbf{v}_{k+1}, \ldots, \mathbf{v}_r\} \sim U \cup \{\mathbf{v}_{k+1}, \ldots, \mathbf{v}_r\}.$$

Proof. We present the argument by analogy with Gaussian elimination. The given dependencies may be written as

$$\mathbf{v}_j = \sum_{h\in\{k,\ldots,r\}\backslash\{j\}}^{\oplus} a_{jh} \otimes \mathbf{v}_h \oplus \sum_{i\in\{1,\ldots,p\}}^{\oplus} \beta_{ji} \otimes \mathbf{u}_i \quad (j = k, \ldots, r).$$

Using the expression for $\mathbf{v}_k$ to substitute in the others, we obtain for $j = k + 1, \ldots, r$:

$$\mathbf{v}_j = a_{jk} \otimes a_{kj} \otimes \mathbf{v}_j \oplus \sum_{h\in\{k+1,\ldots,r\}\backslash\{j\}}^{\oplus} (a_{jk} \otimes a_{kh} \oplus a_{jh}) \otimes \mathbf{v}_h$$
$$\oplus \sum_{i\in\{1,\ldots,p\}}^{\oplus} (a_{jk} \otimes \beta_{ki} \oplus \beta_{ji}) \otimes \mathbf{u}_i.$$

Now if equality could hold between $\mathbf{v}_j$ and $a_{jk} \otimes a_{kj} \otimes \mathbf{v}_j$ on any row, it would imply

$$a_{jk} \otimes a_{kj} = 0.$$

But, from the given dependencies, $\mathbf{v}_j \geqslant a_{jk} \otimes \mathbf{v}_k$ and $\mathbf{v}_k \geqslant a_{kj} \otimes \mathbf{v}_j$, whence

$$\mathbf{v}_j \geqslant a_{jk} \otimes \mathbf{v}_k \geqslant a_{jk} \otimes a_{kj} \otimes \mathbf{v}_j = \mathbf{v}_j,$$

implying that $\mathbf{v}_j = a_{jk} \otimes \mathbf{v}_k$, i.e. that $\mathbf{v}_k$ and $\mathbf{v}_j$ are siblings.

Hence, the term $a_{jk} \otimes a_{kj} \otimes \mathbf{v}_j$ is dominated on every row in the expression for $\mathbf{v}_j$ and may be deleted, giving for $j = k+1, \ldots, r$ a set of dependencies $\mathbf{v}_j \sim U \cup \{\mathbf{v}_{k+1,\ldots}, \mathbf{v}_r\}$. □

Theorem 2.3. *If $V \sim V \cup U$, and no two elements of V are siblings, then U is nonempty and $V \sim U$.*

Proof. If $r \geqslant 2$, apply Lemma 2.1 (if necessary) for $k = 1, 2, \ldots, r-2$ to arrive finally at $\{\mathbf{v}_{r-1}, \mathbf{v}_r\} \sim U \cup \{\mathbf{v}_{r-1}, \mathbf{v}_r\}$. We infer that U must be nonempty, since $\{\mathbf{v}_{r-1}, \mathbf{v}_r\} \sim \{\mathbf{v}_{r-1}, \mathbf{v}_r\}$ would state that $\mathbf{v}_{r-1}, \mathbf{v}_r$ are siblings. Applying Lemma 2.1 now gives $\{\mathbf{v}_r\} \sim U \cup \{\mathbf{v}_r\}$. Hence we must have U nonempty, otherwise (from the definition of the symbol "$\sim$") the notation would be vacuous. So in all cases, $\mathbf{v}_r \sim U$ (from the definition of the symbol "$\sim$"). But since the numbering of the elements of V is arbitrary, the result follows. □

We remark that given dependencies may be written in matrix form as

$$V = V \otimes M \oplus U \otimes P.$$

The iterative solution of equations of this kind, with V regarded as unknown, has been systematically studied in e.g. Zimmermann [10], following work by Carré [2], Gondran and Minoux [6,7], and others, drawing attention to the analogue with classical iterative schemes by Jacobi, Gauss and Jordan.

It is easy to derive by iteration the necessary condition

$$V = V \otimes M^{(s)} \oplus U \otimes P \otimes \left(I \oplus M \oplus M^{(2)} \oplus \cdots \oplus M^{(s-1)}\right), \forall s \geqslant 1.$$

We may then adapt the proof of Lemma 2.1 to show that to avoid siblings, not only all products $a_{jk} \otimes a_{kj}$ must be negative, but all cycle-products of the form $a_{j_1 j_2} \otimes a_{j_2 j_3} \otimes \cdots \otimes a_{j_r j_1}$ must be negative. This condition is sufficient (e.g. [4,10]) to ensure that the sequence

$$I \oplus M \oplus M^{(2)} \oplus \cdots \oplus M^{(s-1)} \oplus \cdots$$

converges at a finite value of s to the *transitive closure matrix* $\Gamma(M)$; but also that for sufficiently large values of s we have $V > V \otimes M^{(s)}$ whence $V = U \otimes P \otimes \Gamma(M)$, giving an alternative proof that $V \sim U$.

2.3. Characterising bases

Theorem 2.4. *If W' is a section of W, then there are elements free in W' and they form a basis of W.*

Proof. Let U be the set of elements free in W', let $V = W' \backslash U$ be the set of elements dependent in W' and let $S = W \backslash W'$. Then $V \sim V \cup U$, so from Theorem 2.3, U is nonempty and $V \sim U$ since V is sibling-free. Obviously, U is an independent set, its elements being free in W'. And clearly $S \sim V \cup U$ since every element of S is a sibling of some element of W', so $S \sim U$. Hence $W \backslash U = V \cup S \sim U$, so U is a generating set in W. □

Theorem 2.5. *If U is any basis of W then there exists a section W' of W such that U is the set of elements free in W'.*

Proof. As U is independent, each of its elements must come from a different sibling class of W. From each remaining sibling class, choose one element, to form a set V. Evidently, $W' = U \cup V$ is a section of W. Let T be the set of elements free in W'. Evidently $T \subseteq U$ since U generates W. But T is a basis of W by Theorem 2.4, so $T = U$. □

Theorem 2.6. *All bases of W have the same cardinality.*

Proof. From Theorems 2.4 and 2.5, the bases of W are exactly the sets produced by choosing one element from each free sibling class. □

Taking the earlier example, the algorithm **SIBLINGCLASSES** establishes the following sibling classes:

$$\left\{\begin{pmatrix}3\\1\end{pmatrix}, \begin{pmatrix}5\\3\end{pmatrix}\right\}, \left\{\begin{pmatrix}0\\-6\end{pmatrix}\right\}, \left\{\begin{pmatrix}1\\7\end{pmatrix}, \begin{pmatrix}-1\\5\end{pmatrix}\right\}$$

and simultaneously extracts the following section

$$\left\{\begin{pmatrix}3\\1\end{pmatrix}, \begin{pmatrix}0\\-6\end{pmatrix}, \begin{pmatrix}1\\7\end{pmatrix}\right\},$$

in which the second and third elements are found to be free. Thus the second and third sibling classes are free, and there are two bases, both of cardinality 2:

$$\left\{\begin{pmatrix}0\\-6\end{pmatrix}, \begin{pmatrix}1\\7\end{pmatrix}\right\}, \left\{\begin{pmatrix}0\\-6\end{pmatrix}, \begin{pmatrix}-1\\5\end{pmatrix}\right\}.$$

2.4. Cardinality of bases

Since elements of W are drawn from $\mathbb{R}^n$, one might intuitively suppose that the possible cardinalities of bases of finite sets would be bounded as a function of n. Indeed, for $n = 1$, it is clear that every basis of every finite set has cardinality 1. Moreover:

Proposition 2.1. *Let $n = 2$. Then the cardinality of a basis of any finite $W \subseteq \mathbb{R}^n$ is* 1 *or* 2.

Proof. Let $W = \left\{ \begin{pmatrix} x_{1l} \\ x_{2l} \end{pmatrix}; l = 1, \ldots, m \right\}$, with $m \geqslant 3$, where without loss of generality we assume $x_{11} \otimes x_{21}^{-1} \geqslant x_{12} \otimes x_{22}^{-1} \geqslant \cdots \geqslant x_{1m} \otimes x_{2m}^{-1}$. Then the algorithm **FREECOLUMNS** finds that the elements of $\left\{ \begin{pmatrix} x_{1l} \\ x_{2l} \end{pmatrix}; 1 < l < m \right\}$ are not free in W. For, at stage l, $(1 < l < m)$, the algorithm seeks column-maxima in a set containing $\begin{pmatrix} x_{11} \otimes x_{1l}^{-1} \\ x_{21} \otimes x_{2l}^{-1} \end{pmatrix}$ and $\begin{pmatrix} x_{1m} \otimes x_{1l}^{-1} \\ x_{2m} \otimes x_{2l}^{-1} \end{pmatrix}$. Clearly, it finds $1 \in M_1, 2 \in M_m$ and so $\begin{pmatrix} x_{1l} \\ x_{2l} \end{pmatrix}$ is a dependent element in W. $\square$

However, for greater values of n, the intuitive result does not apply.

Proposition 2.2. *Let $n \geqslant 3$. Then for each natural number $m = 1, 2, \ldots,$ there exists a finite $W \subseteq \mathbb{R}^n$ with a basis of cardinality m.*

Proof. First, suppose $n = 3$, and consider $W = \left\{ \begin{pmatrix} x_j \\ 0 \\ x_j^{-1} \end{pmatrix}; j = 1, \ldots, m \right\}$, for any distinct nonzero $x_1, \ldots, x_m \in \mathbb{R}$. Evidently the sibling classes of W are singletons. Moreover, the algorithm **FREECOLUMNS** finds that all elements of W are free in W. For example, at stage l, the algorithm seeks column-maxima in the set $\left\{ \begin{pmatrix} x_j \otimes x_l^{-1} \\ 0 \\ x_j^{-1} \otimes x_l \end{pmatrix}; j \neq l \right\}$. Since $x_j \neq x_l$ when $j \neq l$, it follows that $2 \notin M_j$ for any $j \neq l$, so no $\begin{pmatrix} x_l \\ 0 \\ x_l^{-1} \end{pmatrix}$ can be dependent in W. Hence W is a basis of itself. For $n > 3$ it suffices to extend all elements of W by $n - 3$ components of arbitrary value. $\square$

Propositions 2.1 and 2.2 are based on results previously presented in [3].

3. Bases of infinite sets

3.1. Finitely generated sets

Suppose now that the (possibly infinite) set W contains a generating subset $\widetilde{W} = \{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$, so W is a set of some expressions of the form $\sum^{\oplus}_{j=1,\ldots,m} \lambda_j \otimes \mathbf{w}_j$. In other words W is a (possibly proper) subset of the column-space of the matrix whose columns are $\mathbf{w}_1, \ldots, \mathbf{w}_m$. In such a case we say that $\widetilde{W}$ is *finitely generated*.

Theorem 3.1. *Let U be a basis of $\widetilde{W}$. Then U is also a basis of W. In fact, a given subset S of $\widetilde{W}$ is a basis of W if and only if: $|S| = |U|$ and the elements of S are, in some order, siblings of the corresponding elements of U.*

Proof. It is clear that U is also a basis of W, and then so is any set consisting of exactly one sibling of each element of U. Conversely, let S be any basis of W. If $U \subseteq S$ or $S \subseteq U$ then $U = S$ and the result follows. Else, consider the finite set $X = U \cup S$. Clearly both U and S are bases of X, so the result follows by (the proof of) Theorem 2.6. □

Theorem 3.2. *For any $W \subseteq \mathbb{R}^n$, the following conditions are equivalent:*

1. *W has a basis,*
2. *W is finitely generated.*

And then every basis of W has the same cardinality.

Proof. 1. implies 2. by definition, and Theorem 3.1 implies the converse. □

From this, the existence of a generating set gives a powerful criterion for the existence of a basis. This is exploited further in the following sections.

In the case when W is the full column-space of the matrix whose columns are $\mathbf{w}_1, \ldots, \mathbf{w}_m$, each of the free sibling classes $C_1, \ldots, C_h$ within $\widetilde{W}$ is of course a subset of a one-dimensional space or *principal ideal* $D_i = \{\mathbf{y}; \mathbf{y} = \lambda \otimes \mathbf{u_i}, \mathbf{u_i} \in C_i\}$ within W and we may speak of the *free ideals*. Summarising,

Theorem 3.3. *Every column-space W has a decomposition as a sum of disjunct ideals:*

$$W = \sum_i^{\oplus} D_i,$$

which is unique (apart from order).

3.2. The range seminorm

Define the *range seminorm* as the function $\tau : \mathbb{R}^n \to \mathbb{R}$ given by:

$$\tau : \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \left(\sum_j^{\oplus} x_j \right) \otimes \left(\sum_j^{\oplus} x_j^{-1} \right).$$

The RHS is called the *range* of $\mathbf{x}$ and arithmetically equals the excess of the greatest over the least of the components x_j.

Proposition 3.1. *The range seminorm satisfies, for all* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ *and* $\lambda \in \mathbb{R}$:

(i) $\tau(\lambda \otimes \mathbf{x}) = \tau(\mathbf{x})$,
(ii) $\tau(\mathbf{x} \oplus \mathbf{y}) \leqslant \tau(\mathbf{x}) \oplus \tau(\mathbf{y})$.

Proof. (i) is immediate.

For (ii), we have $x_j \leqslant x_j \oplus y_j$, whence $x_j^{-1} \geqslant (x_j \oplus y_j)^{-1}$. Thus $\sum_j^{\oplus} x_j^{-1} \geqslant \sum_j^{\oplus} (x_j \oplus y_j)^{-1}$ and similarly $\sum_j^{\oplus} y_j^{-1} \geqslant \sum_j^{\oplus} (x_j \oplus y_j)^{-1}$.

Hence

$$\begin{aligned}
\tau(\mathbf{x} \oplus \mathbf{y}) &= \left(\sum_j^{\oplus} x_j \oplus \sum_j^{\oplus} y_j \right) \otimes \left(\sum_j^{\oplus} (x_j \oplus y_j)^{-1} \right) \\
&= \left(\sum_j^{\oplus} x_j \right) \otimes \left(\sum_j^{\oplus} (x_j \oplus y_j)^{-1} \right) \oplus \left(\sum_j^{\oplus} y_j \right) \otimes \left(\sum_j^{\oplus} (x_j \oplus y_j)^{-1} \right) \\
&\leqslant \left(\sum_j^{\oplus} x_j \right) \otimes \left(\sum_j^{\oplus} x_j^{-1} \right) \oplus \left(\sum_j^{\oplus} y_j \right) \otimes \left(\sum_j^{\oplus} y_j^{-1} \right) \\
&= \tau(\mathbf{x}) \oplus \tau(\mathbf{y}). \qquad \square
\end{aligned}$$

For a given $k \geqslant 0$, we now define the set $S_k = \{\mathbf{x} \in \mathbb{R}^n; \tau(\mathbf{x}) \leqslant k\}$.

Proposition 3.2. *For* $n > 1$, *every* S_k *is a proper subspace of* $\mathbb{R}^n$, *and every* $W \subseteq \mathbb{R}^n$, *which is finitely generated, is a subset of* S_k *for some* k.

Proof. That S_k is a space follows immediately from Proposition 3.1, and since $\mathbb{R}^n$ has elements of arbitrarily large range, S_k is a proper subspace of $\mathbb{R}^n$. If U is a generating set in W, let $k = \max(\tau(\mathbf{u}); \mathbf{u} \in U)$. Then $U \subseteq S_k$, so $W \subseteq S_k$, using Proposition 3.1. $\square$

Propositions 3.1 and 3.2 are based on results previously presented in [3].

Theorem 3.4. *Every S_k has a basis.*

Proof. If $n = 1$, or $k = 0$, there is a generating set of cardinality 1, which is clearly a basis. For $n > 1$ and $k > 0$, define the $n \times n$ matrix I_k with diagonal elements zero and off-diagonal elements $-k$. For $\boldsymbol{\xi} \in S_k$, component i of $I_k \otimes \xi$ equals

$$\max(\xi_i, \max_{j \neq i}(-k + \xi_j)) = \xi_i + \max(0, \max_{j \neq i}(-k + \xi_j - \xi_i)) = \xi_i,$$

since $\xi_j - \xi_i \leqslant k$. Hence $I_k \otimes \boldsymbol{\xi} = \boldsymbol{\xi}$, showing that every $\boldsymbol{\xi} \in S_k$ is a linear combination of the columns of I_k. Clearly, the columns of I_k lie in S_k, so S_k is finitely generated and therefore has a basis by Theorem 3.2. $\square$

In fact, for $n > 1$ and $k > 0$, the columns of I_k form an independent set, as may be shown using the algorithm **FREECOLUMNS**. At stage j, the algorithm considers a set of columns all having $-k$ in component j, and k in one other component, so row j cannot provide a column-maximum. Hence:

Theorem 3.5. *If $k = 0$, S_k has a basis of cardinality 1; else, S_k has a basis of cardinality n.*

For given $n > 1$, $k > 0$, we now define the set $T_k = \{\mathbf{x} \in \mathbb{R}^n; \tau(\mathbf{x}) < k\}$.

Theorem 3.6. *Each T_k is a proper subspace of S_k, but with no generating set and hence no basis.*

Proof. That T_k is a space, and contained in S_k, is immediate. Every column of I_k is contained in $S_k \backslash T_k$, so T_k is a proper subspace. If U were a generating set in T_k, let $k' = \max(\tau(\mathbf{u}); \mathbf{u} \in U)$, so $k' < k$ and, as in Proposition 3.2, $T_k \subseteq S_{k'}$. But the n-tuple with first component $(k + k')/2$, and other components zero, belongs to $T_k \backslash S_{k'}$, a contradiction. $\square$

We conclude this section by noting the following for the whole space $\mathbb{R}^n$: Since $\mathbb{R}^n = \bigcup_{k=1}^{\infty} S_k$ we see that the union of all bases of S_k $(k = 1, 2, \ldots)$ is a countable generating set of $\mathbb{R}^n$. However, we also have:

Theorem 3.7. *For $n > 1$, $\mathbb{R}^n$ has no basis.*

Proof. If U were a generating set in $\mathbb{R}^n$, then by Proposition 3.2, $\mathbb{R}^n \subseteq S_k$, a proper subset of $\mathbb{R}^n$. $\square$

(Clearly, for $n = 1$, $\mathbb{R}^n = \mathbb{R} = S_k$, $\forall k \geqslant 0$, and $\mathbb{R}^n$ has a generating set of cardinality 1.)

4. Infinite bases

Up to this point "basis" meant by definition a finite basis. Previous sections have demonstrated that this concept "works well" while sets under consideration are finitely generated. The aim now is to show that if we allow generators to be chosen from an infinite set, the generating and independence properties may be inconsistent.

Theorem 4.1. *Let $S \subseteq \mathbb{R}^n$ be any set with the following property: For every $\mathbf{x} \in \mathbb{R}^n$ there exist a natural number m, $\mathbf{v}_1, \ldots, \mathbf{v}_m \in S$ and $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$ so that $\mathbf{x} = \sum^{\oplus}_{j=1,\ldots,m} \lambda_j \otimes \mathbf{v}_j$. Then for every $\mathbf{z} \in S$ there exist a natural number m, $\mathbf{v}_1, \ldots, \mathbf{v}_m \in S \setminus \{\mathbf{z}\}$ and $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$ so that $\mathbf{z} = \sum^{\oplus}_{j=1,\ldots,m} \lambda_j \otimes \mathbf{v}_j$.*

Proof. Let $\mathbf{z} = (z(1), \ldots, z(n))^{\mathrm{T}} \in S$ and suppose that $\mathbf{z} \neq \sum^{\oplus}_{j=1,\ldots,m} \lambda_j \otimes \mathbf{v}_j$ for any natural number m, $\mathbf{v}_1, \ldots, \mathbf{v}_m \in S \setminus \{\mathbf{z}\}$ and $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$. Set

$$S' = \{(z(1)^{-1} \otimes v(1), \ldots, z(n)^{-1} \otimes v(n))^{\mathrm{T}};\ (v(1), \ldots, v(n))^{\mathrm{T}} \in S\}.$$

Then $0 \neq \sum^{\oplus}_{j=1,\ldots,m} \lambda_j \otimes \mathbf{v}_j$ for any natural number m and for any $\mathbf{v}_1, \ldots, \mathbf{v}_m \in S'$ and $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$. If for every $k = 1, \ldots, n$ there was a $\mathbf{v}_k = (v_k(1), \ldots, v_k(n))^{\mathrm{T}} \in S'$ so that $v_k(k) = \max_{i=1,\ldots,n} v_k(i)$ then for $\lambda_k = (v_k(k))^{-1}$ $(k = 1, \ldots, n)$ we would have $\sum^{\oplus}_{j=1,\ldots,n} \lambda_j \otimes \mathbf{v}_j = 0$, a contradiction. Thus there is a $k \in N$ such that for every $\mathbf{v} = (v(1), \ldots, v(n))^{\mathrm{T}} \in S'$ the inequality

$$v(k) < \max_{i=1,\ldots,n} v(i) \tag{4.1}$$

holds. It follows then that the unit vector $\mathbf{e}_k \notin S'$ and so $\mathbf{y} = (z(1), \ldots, z(k) \otimes 1, \ldots, z(n))^{\mathrm{T}} \notin S$. Therefore, by the theorem hypothesis $\mathbf{y} = \sum^{\oplus}_{j=1,\ldots,m} \lambda_j \otimes \mathbf{v}_j$ for some natural number m, $\mathbf{v}_1, \ldots, \mathbf{v}_m \in S$ and $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$ and $\mathbf{e}_k$ is a linear combination of some vectors $\mathbf{u}_1, \ldots, \mathbf{u}_n \in S'$. Hence one of these vectors, say $\mathbf{u}_r = (u_r(1), \ldots, u_r(n))^{\mathrm{T}}$ satisfies the inequality

$$u_r(k) - 1 \geqslant \max_{i=1,\ldots,n, i \neq r} u_r(i).$$

But then $u_r(k) = \max_{i=1,\ldots,n} u_r(i)$ which contradicts (4.1). $\square$

References

[1] F.L. Baccelli, G. Cohen, G.-J. Olsder, J.-P. Quadrat, Synchronization and Linearity, John Wiley, Chichester, New York, 1992.

[2] B.A. Carré, An algebra for network routing problems, J. Inst. Math. Appl. 7 (1971) 273–294.

[3] R.A. Cuninghame-Green, Minimax Algebra, Lecture Notes in Economics and Math. Systems, vol. 166, Springer, Berlin, 1979.

[4] R.A. Cuninghame-Green, Minimax Algebra and Applications, in: Advances in Imaging and Electron Physics, vol. 90, Academic Press, New York, 1995, pp. 1–121.

[5] S. Gaubert, Théorie des systèmes linéaires dans les dioïdes, Thèse, Ecole des Mines de Paris, 1992.
[6] M. Gondran, M. Minoux, L'indépendance linéaire dans les dioïdes, Bulletin de la Direction Études et Recherches, EDF, Sé rie C, vol. 1, 1978, pp. 67–90.
[7] M. Gondran, M. Minoux, Linear algebra of dioïds: a survey of recent results, Ann. Discr. Math. 19 (1984) 147–164.
[8] E. Wagneur, Moduloids and pseudomodules 1. Dimension theory, Discrete Mathematics 98 (1991) 57–73.
[9] K. Zimmermann, Extremální algebra, Výzkumná publikace Ekonomicko - matematické laboratoře při Ekonomické m ústavě ČSAV, 46, Praha, 1976 (in Czech).
[10] U. Zimmermann, Linear and Combinatorial Optimization in Ordered Algebraic Structures. Ann. of Discrete Math., vol. 10, North Holland, Amsterdam, 1981.