Max-algebra and combinatorial optimisation: connections and open problems

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Let $a \oplus b = \max(a, b)$ and $a \otimes b = a + b$ for $a, b \in \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\}$. By *max-algebra* we understand the analogue of linear algebra developed for the pair of operations $(\oplus, \otimes)$, extended to matrices and vectors in the same way as in linear algebra, that is if $A = (a_{ij}),\ B = (b_{ij})$ and $C = (c_{ij})$ are matrices with entries from $\overline{\mathbb{R}}$ of compatible sizes, we write $C = A \oplus B$ if $c_{ij} = a_{ij} \oplus b_{ij}$ for all i, j and $C = A \otimes B$ if $c_{ij} = \sum_k^{\oplus} a_{ik} \otimes b_{kj} = \max_k(a_{ik} + b_{kj})$ for all i, j. The iterated product $A \otimes A \otimes ... \otimes A$ in which the symbol A appears k-times will be denoted by $A^{(k)}$.

Let $A = (a_{ij}) \in \overline{\mathbb{R}}^{n\times n}$. We denote $N = \{1, ..., n\}$. The complete arc-weighted digraph associated with A is $D_A = (N, N \times N, a_{ij})$, the finiteness digraph is $F_A = (N, \{(i, j); a_{ij} > -\infty\})$, the zero digraph is $Z_A = (N, \{(i, j); a_{ij} = 0\})$.

1. Links between max-algebraic problems and combinatorial or combinatorial optimisation problems: The set covering - solvability of max-algebraic linear systems, the minimal set covering - unique solvability of a linear system, existence of a directed cycle - strong regularity of a matrix, sign-nonsingularity or existence of an even directed cycle - regularity of a matrix, maximum cycle mean - eigenvalue, longest-distances vectors - eigenvectors, best principal submatrices - coefficients of a characteristic polynomial, linear assignment problem - permanent of a matrix.

2. Examples of combinatorial optimisation results obtained as a consequence of the max-algebraic theory.

2.a) The (assignment problem) normal form of a matrix and the longest-distances vectors. Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ and denote by P_n the set of all permutations of the set N. Then the (*max-algebraic*) *permanent* of A is $maper(A) = \sum_{\pi\in P_n}^{\oplus} \prod_{i\in N}^{\otimes} a_{i,\pi(i)}$. In the conventional notation this reads $maper(A) = \max_{\pi\in P_n} \sum_{i\in N} a_{i,\pi(i)}$ and thus $maper(A)$ is the optimal assignment problem value for A. For $\pi \in P_n$ we denote $w(A, \pi) = \prod_{i\in N}^{\otimes} a_{i,\pi(i)} = \sum_{i\in N} a_{i,\pi(i)}$. The set of all optimal permutations will be denoted by $ap(A)$, that is $ap(A) = \{\pi \in P_n; maper(A) = w(A, \pi)\}$.

A matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is called *normal* [*strictly normal*] if $a_{ij} \le a_{ii} = 0$ for all $i, j \in N$ [if $a_{ij} < a_{ii} = 0$ for all $i, j \in N, i \ne j$]. A normal matrix can be obtained from any $A \in \mathbb{R}^{n\times n}$ by adding suitable constants to the rows and columns and by permuting the columns (or rows), e.g. using the Hungarian method for solving the assignment problem for A. We say that a matrix A is *equivalent to* a matrix B if A can be obtained from B by adding constants to the rows or columns and by permuting the rows or columns. Thus every matrix is equivalent to a normal matrix. Not every matrix is equivalent to a strictly normal matrix. Note that if A is normal then the set of all optimal permutations of the assignment problem for A can conveniently be described: $ap(A) = \{\pi \in P_n; a_{i,\pi(i)} = 0 \text{ for all } i \in N\}$. We say that A is *max-algebraically* *definite* (or, shortly, *definite*) if $a_{ii} = 0$ for all $i \in N$ and D_A contains no positive cycles. Clearly $id \in ap(A)$ if A is definite.

Theorem 1 *[2] Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be definite. If $x = (x_1, ..., x_n)^T \in \mathbb{R}^n$ is a vector of longest distances from all nodes to any fixed node in D_A then the matrix $(a_{ij} + x_j - x_i)$ is normal.*

Theorem 2 *[2] $A \in \mathbb{R}^{n\times n}$ is equivalent to a strictly normal matrix if and only if $|ap(A)| = 1$.*

By V we denote the max-hull of the vectors of longest distances from all nodes to a fixed node in D_A.

Theorem 3 *[2] Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be definite. A is equivalent to a strictly normal matrix if and only if $int(V) \neq \emptyset$. If $x = (x_1, ..., x_n)^T \in int(V)$ then the matrix $(a_{ij} + x_j - x_i)$ is strictly normal.*

2.b) Another link between the assignment problem and the longest-distances problem. It is easily seen that if $id \in ap(A)$ for some matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ then $B = (a_{ij} - a_{ii})$ is definite. We will call B the *definite form* of A. If A is a definite matrix then it can be considered as the direct-distances matrix between all pairs of nodes in D_A. Note that the longest-distances matrix can max-algebraically be expressed as $A^{(n-1)}$.

Theorem 4 *[6] Let $A \in \mathbb{R}^{n\times n}$ and B and C be the definite forms of any two matrices B' and C' arising from A by permuting the columns so that $id \in ap(B') \cap ap(C')$. Then the longest-distances matrices of B and C coincide.*

2.c) The maximum cycle mean. Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$. If $\sigma = (i_1, ..., i_k, i_1)$ is a cycle in D_A then its *mean* is

$$\mu(\sigma, A) = \frac{a_{i_1 i_2} + a_{i_2 i_3} + ... + a_{i_k i_1}}{k}.$$

The value $\lambda(A) = \max_\sigma \mu(\sigma, A)$ is called the *maximum cycle mean* of A. Next statement is an immediate corollary of the max-algebraic spectral theory.

Theorem 5 *$\lambda(A^{(k)}) = (\lambda(A))^{(k)}$, for all natural k and any $A \in \overline{\mathbb{R}}^{n\times n}$.*

3. Some open problems.

3.a) Even [odd] parity assignment problem (EPAP [OPAP]): Let P_n^+ [P_n^-] be the set of all even [odd] permutations of the set N. Given $A \in \mathbb{R}^{n\times n}$, find a permutation $\pi^* \in P_n^+$ [$\pi^* \in P_n^-$] such that $w(A, \pi^*) = \max_{\pi\in P_n^+} w(A, \pi)$ $\left[= \max_{\pi\in P_n^-} w(A, \pi)\right]$. Obviously one of these two problems is always solved by solving the assignment problem. No polynomial method is known in general for solving both problems at the same time. The question whether $\max_{\pi\in P_n^+} w(A, \pi)$ $= \max_{\pi\in P_n^-} w(A, \pi)$ is equivalent to the even cycle problem [2] and if this equality holds then optimal solutions to both problems can be found in $O(n^3)$ time. Some polynomially solvable special cases are studied in [3].

3.b) Strong linear independence. This concept is equivalent to the question: Given $A \in \mathbb{R}^{m\times n}$, $m > n$, is there an $n \times n$ submatrix B of A such that $|ap(B)| = 1$? For $m = n$ it reduces to checking $|ap(A)| = 1$, which can be done in $O(n^2)$ time after solving the assignment problem. Also, it is polynomially solvable for $m \times n$ $0-1$ matrices.

3.c) (Max-algebraic) rank. This is a generalisation of the previous problem important for applications in algebraic geometry [7]: Given $A \in \mathbb{R}^{m\times n}$, find the biggest natural number k for which there is a $k \times k$ submatrix B of A such that $|ap(B)| = 1$.

3.d) Coefficients of a (max-algebraic) characteristic polynomial. Given $A \in \overline{\mathbb{R}}^{n\times n}$ and $k < n$, find a $k \times k$ principal submatrix of A whose optimal assignment problem value is maximal (notation δ_k). No polynomial algorithm is known in general. A polynomial randomised algorithm exists [1]. Biggest k for which δ_k is finite can be found in $O(n^3)$ time [1]. An $O(n(m + n \log n))$ algorithm for finding all δ_k corresponding to so called essential terms exists [5] (here m is the number of finite entries of A). The problem arising after removing "principal" is easily solvable in $O(n^3)$ time [1].

3.e) Special case of the previous problem for matrices over $\{0, -\infty\}$. Given a digraph D with n nodes and $k < n$, are there pairwise node-disjoint cycles in D with exactly k nodes in total? No polynomial algorithm is known in general. Polynomially solvable for a number of special cases, including symmetric matrices and k even [4].

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