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Generalised eigenproblem in max-algebra

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***Abstract*— In this paper we consider the generalized eigenproblem in max-algebra, i.e. given matrices A, B, find x and λ such that $A \otimes x = \lambda \otimes B \otimes x$. We present several conditions that are necessary or sufficient for the existence of a solution to this problem.**

CONTENTS

I. INTRODUCTION

A. Definitions

Let $a \oplus b = \max(a, b)$ and $a \otimes b = a + b$ for $a, b \in \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\}$. Obviously, $-\infty$ plays the role of a neutral element for $\oplus$ and a null for $\otimes$. Throughout the paper we denote $-\infty$ by ε and for convenience we also denote by the same symbol any vector or matrix whose every component is $-\infty$. If $a \in \overline{\mathbb{R}}$ then the symbol a^{-1} stands for $-a$. If $a_1, ..., a_n \in \overline{\mathbb{R}}$ then the expression $a_1 \oplus ... \oplus a_n$ will be denoted by $\sum_{i=1,...,n}^{\oplus} a_i$. The iterated expression $a \otimes a \otimes ... \otimes a$ where the symbol a appears k-times ($k \geq 1$) will be denoted $a^{(k)}$ and $a^{(0)} = 0$ by definition.

By *max-algebra* we understand the analogue of linear algebra developed for the pair of operations $(\oplus, \otimes)$, extended to matrices and vectors. That is if $A = (a_{ij}),\ B = (b_{ij})$ and $C = (c_{ij})$ are matrices of compatible sizes (this is also assumed in all matrix expressions below) with entries from $\overline{\mathbb{R}}$, we write $C = A \oplus B$ if $c_{ij} = a_{ij} \oplus b_{ij}$ for all i, j and $C = A \otimes B$ if $c_{ij} = \sum_k^{\oplus} a_{ik} \otimes b_{kj} = \max_k(a_{ik} + b_{kj})$ for all i, j. If $\alpha \in \overline{\mathbb{R}}$ then $\alpha \otimes A = (\alpha \otimes a_{ij})$. We assume everywhere in this paper that $n \geq 1$ is an integer. P_n will stand for the set of permutations of the set $\{1, ..., n\}$.

A square matrix D is called *diagonal*, notation $D = diag(d_1, ..., d_n)$, if its diagonal entries are $d_1, ..., d_n \in \mathbb{R}$ and off-diagonal entries are ε. We also denote $I = diag(0, ..., 0)$. Obviously, $A \otimes I = A$ for every $m \times n$ matrix A. Any matrix arising from I and D by permuting its rows and/or columns is called a *permutation matrix* and *generalised permutation matrix*, respectively.

If A is an $n \times n$ matrix then the iterated product $A \otimes A \otimes ... \otimes A$ in which the symbol A appears k-times ($k \geq 1$) will be denoted by $A^{(k)}$; and $\Gamma(A) = A \oplus A^{(2)} \oplus ... \oplus A^{(n)}$. We set $A^{(0)} = I$ by definition. The *conjugate* of an $m \times n$ matrix $A = (a_{ij})$ is the $n \times m$ matrix $A^* = \left(a_{ji}^{-1}\right)$.

Of the elementary properties of matrices in max-algebra we mention at least the isotonicity of $\otimes$:

$$A \otimes C \leq B \otimes C \text{ and } C \otimes A \leq C \otimes B$$

whenever $A \leq B$ and A, B, C are matrices including possibly vectors and scalars.

Max algebra has been studied by various authors [1], [3], [8], [9], [11].

B. Problem formulation

The aim of this paper is to study the problem: *Given $A, B \in \overline{\mathbb{R}}^{m \times n}$, find all $\lambda \in \overline{\mathbb{R}}$* and *$x \in \overline{\mathbb{R}}^n, x \neq \varepsilon$ such that*

$$A \otimes x = \lambda \otimes B \otimes x. \quad (1)$$

A motivation for this research is given by the following practical interpretation: Consider the *multi-machine interactive production process* (MMIPP) where products $P_1, ..., P_m$ are prepared using n machines (or processors), every machine contributing to the completion of each product by producing a partial product. It is assumed that every machine can work for all products simultaneously and that all these actions on a machine start as soon as the machine starts to work. Let a_{ij} be the duration of the work of the j^{th} machine needed to complete the partial product for P_i $(i = 1, ..., m; j = 1, ..., n)$. Let us denote by x_j the starting time of the j^{th} machine $(j = 1, ..., n)$. Then all partial products for P_i $(i = 1, ..., m)$ will be ready at time $\max(x_1 + a_{i1}, ..., x_n + a_{in})$. Now suppose that independently, n other machines prepare partial products for products $Q_1, ..., Q_m$ and the duration and starting times are b_{ij} and y_j, respectively. Then the *synchronisation problem* is to find starting times of all $2n$ machines so that each pair (P_i, Q_i) $(i = 1, ..., m)$ is completed at the same time. This task is equivalent to finding $x_1, ..., x_n, y_1, ..., y_n \in \mathbb{R}$ satisfying the system

$$\max(x_1 + a_{i1}, ..., x_n + a_{in}) = \max(y_1 + b_{i1}, ..., y_n + b_{in})$$

for $i = 1, ..., m$. If the machines are linked it may also be required that the starting times (x_j, y_j) of each pair of machines $(j = 1, ..., n)$ differ by the same value. If we denote this value by λ then the equations read

$$\max(x_1+a_{i1}, ..., x_n+a_{in}) = \max(\lambda+x_1+b_{i1}, ..., \lambda+x_n+b_{in}) \quad (2)$$

for $i = 1, ..., m$. If we denote $a \oplus b = \max(a, b)$ and $a \otimes b = a + b$ for $a, b \in \mathbb{R}$ then this system gets the form

$$\sum_{j=1,...,n}^{\oplus} a_{ij} \otimes x_j = \lambda \otimes \sum_{j=1,...,n}^{\oplus} b_{ij} \otimes x_j \quad (i = 1, ..., m) \quad (3)$$

which is essentially (1).

A related problem, important in max-algebra, is:

EIGENPROBLEM (EP): Given $A \in \overline{\mathbb{R}}^{n\times n}$, find all $x \in \overline{\mathbb{R}}^{n}, x \neq \varepsilon$ (*eigenvectors*) such that $A \otimes x = \lambda \otimes x$ for some $\lambda \in \overline{\mathbb{R}}$ (*eigenvalue*).

The set of all eigenvalues and eigenvectors of A will be denoted by $\Lambda(A)$ and $V(A)$, respectively. If $\lambda \in \Lambda(A)$ then $V(A, \lambda)$ will stand for the set $\left\{x \in \overline{\mathbb{R}}^{n}; A \otimes x = \lambda \otimes x, x \neq \varepsilon\right\}$. EV has been studied since the 1960's and can now be efficiently solved [5], [7], [9], [1], [3], [6], [12]. Obviously, when $B = I$ then (1) coincides with EV. To mark this connection we will call (1) the GENERALISED EIGENPROBLEM (GEP).

For the GEP we denote

$$\Lambda(A, B) = \left\{\lambda \in \overline{\mathbb{R}}; \left(\exists x \in \overline{\mathbb{R}}^{n} - \{\varepsilon\}\right) A \otimes x = \lambda \otimes B \otimes x\right\}$$

and

$$V(A, B) = \left\{x \in \overline{\mathbb{R}}^{n} - \{\varepsilon\}; A \otimes x = \lambda \otimes B \otimes x, \lambda \in \overline{\mathbb{R}}\right\}.$$

When the GEP (1) is solvable then we write (A, B) *is solvable*. To our knowledge [2] is the only paper dealing with this generalisation of EP. That paper solves the problem completely when $m = 2$ and some special cases for general m and n. No solution method seems to exist either for finding a λ or an x satisfying (1) for general matrices.

In this paper we will present a number of solvability conditions for general matrices and show how to solve GEP in some special cases.

C. Previous results

We now give a brief overview of earlier results relevant for the present paper, especially on EP. The reader is referred to [1], [10], [5], [9], [12] for comprehensive information.

We start with terminology and notation. An ordered pair $D = (N, F)$ is called a *digraph* if N is a non-empty set (of *nodes*) and $F \subseteq N \times N$ (the set of *arcs*). A sequence $\pi = (v_1, ..., v_p)$ of nodes is called a *path* (in D) if $p = 1$, or $p > 1$ and $(v_i, v_{i+1}) \in F$ for all $i = 1, ..., p-1$. The node v_1 is called the *starting node* and v_p the *endnote* of π, respectively. If there is a path in D with starting node u and endnote v then we say that v is *reachable* from u, notation $u \to v$. Thus $u \to u$ for any $u \in N$. As usual a digraph D is called *strongly connected* if $u \to v$ for all nodes u, v in D. A path $(v_1, ..., v_p)$ is called a *cycle* if $v_1 = v_p$ and $p > 1$ and it is called an *elementary cycle* if, moreover, $v_i \neq v_j$ for $i, j = 1, ..., p-1, i \neq j$. The arcs $(v_i, v_{i+1}) \in F$ for $i = 1, ..., p-1$ are called the *arcs of the cycle*.

In the rest of the paper $N = \{1, ..., n\}$. The digraph *associated* with $A = (a_{ij}) \in \overline{\mathbb{R}}^{n\times n}$ is

$$D_A = (N, \{(i, j); a_{ij} > \varepsilon\}).$$

The matrix A is called *irreducible* if D_A is strongly connected, *reducible* otherwise.

If $\pi = (i_1, ..., i_p)$ is a path in D_A then the *weight* of π is $w(\pi, A) = a_{i_1 i_2} + a_{i_2 i_3} + ... + a_{i_{p-1} i_p}$ if $p > 1$, and ε if $p = 1$. The symbol $\lambda(A)$ stands for the *maximum cycle mean* of A, that is if D_A has at least one cycle then

$$\lambda(A) = \max_{\sigma} \mu(\sigma, A), \quad (4)$$

where the maximisation is taken over all cycles in D_A and

$$\mu(\sigma, A) = \frac{w(\sigma, A)}{k} \quad (5)$$

denotes the *mean* of the cycle $\sigma = (i_1, ..., i_k, i_1)$. If D_A is acyclic we set $\lambda(A) = \varepsilon$. Various algorithms for finding $\lambda(A)$ exist. One of them is Karp's [13] of computational complexity $O(n^3)$.

In this subsection we we will concentrate on known properties of irreducible matrices. However the following fundamental results for general matrices will also be useful:

Theorem 1.1: [1], [10], [5] The maximum cycle mean $\lambda(A)$ is the greatest eigenvalue for every $A \in \overline{\mathbb{R}}^{n\times n}$ and $|\Lambda(A)| \leq n$.

We now present some results on EP for irreducible matrices.

Theorem 1.2: [8] Let $A = (a_{ij}) \in \overline{\mathbb{R}}^{n\times n}$ be irreducible, $n > 1$, and k be the number of columns of $\Gamma((\lambda(A))^{-1} \otimes A)$ having zero diagonal entries. Then

1) $\lambda(A) > \varepsilon$ and it is the unique eigenvalue of A,
2) $k > 0$ and $V(A) = \{\Gamma_0 \otimes x; x \in \mathbb{R}^k\} \subseteq \mathbb{R}^n$ where Γ_0 is the $n \times k$ matrix consisting of the columns of $\Gamma((\lambda(A))^{-1} \otimes A)$ having zero diagonal entries.

In this paper we follow terminology introduced in [8]. Being motivated by Theorem 1.2, the columns of $\Gamma((\lambda(A))^{-1} \otimes A)$ having zero diagonal entries are called the *fundamental eigenvectors* of A (FEV). Two vectors $x, y \in \mathbb{R}^n$ are called *equivalent* ($x \sim y$) if $x = \alpha \otimes y$ for some $\alpha \in \mathbb{R}$. It is easily seen that if $\lambda \in \Lambda(A)$ and $x, y \in V(A, \lambda)$ then $\alpha \otimes x \oplus \beta \otimes y \in V(A, \lambda)$ for all $\alpha, \beta \in \mathbb{R}$. Therefore when $|\Lambda(A)| = 1$ (in particular when A is irreducible) we will call $V(A)$ the *eigenspace* of A.

Corollary 1.3: $V(A) = \{\Gamma_0' \otimes x; x \in \mathbb{R}^d\}$ where d is the maximal number of non-equivalent fundamental eigenvectors of A and Γ_0' is any matrix consisting of d non-equivalent fundamental eigenvectors of A.

It is known that none of the FEVs can be expressed as a linear combination of other (non-equivalent) FEVs [1], [8]. The number d in Corollary 1.3 is called the *dimension* of the eigenspace for A and will be denoted by $d(A)$.

We denote $E(A) = \{i \in N; \exists\sigma = (i = i_1, ..., i_k, i_1) : \mu(\sigma, A) = \lambda(A)\}$. The elements of $E(A)$ are called *critical nodes (of D_A)*. A cycle σ is called *critical* if $\mu(\sigma, A) = \lambda(A)$. *The critical digraph* of A is the digraph $C(A)$ with the set of nodes N; the set of arcs is the union of the sets of arcs of all critical cycles. All cycles in a critical digraph are critical [1].

Theorem 1.4: [8] Suppose that $A \in \overline{\mathbb{R}}^{n\times n}$ is irreducible, $n > 1$, $\Gamma((\lambda(A))^{-1} \otimes A) = (g_{ij})$ and let $g_1, ..., g_n$ be the columns of $\Gamma((\lambda(A))^{-1} \otimes A)$. Then

- $i \in E(A)$ if and only if $g_{ii} = 0$.
- If $i, j \in E(A)$ then $g_i \sim g_j$ if and only if i and j belong to the same critical cycle of A.

Corollary 1.5: $V(A) = \{\sum^{\oplus}_{i \in E^*(A)} x_i \otimes g_i; x_i \in \mathbb{R}\}$

where $E^*(A)$ is any maximal set of indices of non-equivalent FEVs of A and $d(A) = |E^*(A)|$ is the number of non-trivial strongly connected components of $C(A)$.

We also define LEFT EIGENPROBLEM (LEP) [8]: Given $A \in \overline{\mathbb{R}}^{n\times n}$ find all $x \in \overline{\mathbb{R}}^n, x \neq \varepsilon$ (*left eigenvectors*) such that $x^T \otimes A = \lambda \otimes x^T$ for some $\lambda \in \overline{\mathbb{R}}$ (*left eigenvalue*). The expression "left eigenspace" will have the meaning similar to that of "eigenspace". Note that we will sometimes refer to EP as to the "right" eigenproblem (and correspondingly we say "right eigenvalue" and "right eigenvector").

It easily seen that LEP is equivalent to EP for A^T. Since $\lambda(A) = \lambda(A^T)$ and $\Gamma(A^T) = \Gamma(A)^T$ it follows:

Theorem 1.6: Irreducible matrices have a unique left eigenvalue that coincides with the right eigenvalue, critical nodes for EP and LEP are identical and the left eigenspace is generated by the rows of $\Gamma((\lambda(A))^{-1} \otimes A)$ with indices corresponding to the critical nodes.

Similarly we define GENERALISED LEFT EIGENPROBLEM (GLEP) as the task of finding $\lambda \in \overline{\mathbb{R}}$ and $x = (x_1, ..., x_n)^T \in \overline{\mathbb{R}}^n, x \neq \varepsilon$ such that

$$x^T \otimes A = \lambda \otimes x^T \otimes B.$$

For similar reasons as in the case of LEP this task is equivalent to GEP for the pair of matrices (A^T, B^T).

Note that we refer to GEP as to the "right" generalised eigenproblem.

Min-algebra can be defined in a similar way as max-algebra: We denote $a \oplus' b = \min(a, b), a \otimes' b = a \otimes b$ for $a, b \in \mathbb{R}$ and extend the pair of operations $(\oplus', \otimes')$ to matrices and vectors in the same way as for $(\oplus, \otimes)$. All statements hold after the replacement of maximisation by minimisation, $-\infty$ by $+\infty$ and converting the inequality signs. In some cases we need to work with both max-algebra and min-algebra at the same time. For this we extend the four operations to $\overline{\overline{\mathbb{R}}} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$. The operations $\oplus$ and $\oplus'$ are extended in the natural way. Further we define

$$-\infty \otimes +\infty = -\infty = +\infty \otimes -\infty$$

and

$$-\infty \otimes' +\infty = +\infty = +\infty \otimes' -\infty.$$

The *minimum cycle mean* of a matrix A will be denoted by $\lambda'(A)$. It is easily seen that

$$\lambda'(A) \leq a_{ii} \leq \lambda(A) \tag{6}$$

holds for any $i \in N$ and $A = (a_{ij}) \in \mathbb{R}^{n\times n}$.

Min-algebra is useful when solving *max-algebraic linear systems* (shortly *max-linear systems*).

Theorem 1.7: [8], [9] If $A \in \overline{\mathbb{R}}^{m\times n}, b \in \overline{\mathbb{R}}^m$ and $x \in \overline{\mathbb{R}}^n$ then

$$A \otimes x \leq b \text{ if and only if } x \leq A^* \otimes' b.$$

We will denote $A^* \otimes' b$ everywhere by $\hat{x}(A, b)$ or just $\hat{x}$ and call it the *principal solution* to the system $A \otimes x \leq b$.

Corollary 1.8: If $A \in \overline{\mathbb{R}}^{m\times n}$ and $b \in \overline{\mathbb{R}}^m$ then

(a) $\hat{x}$ is the greatest solution to $A \otimes x \leq b$ and
(b) $A \otimes x = b$ has a solution if and only if $\hat{x}$ is a solution.

Corollary 1.9: If $A \in \overline{\mathbb{R}}^{m\times n}, B \in \overline{\mathbb{R}}^{m\times k}$ and $\hat{X} = A^* \otimes' B$ then

(a) $\hat{X}$ is the greatest solution to $A \otimes X \leq B$ and
(b) $A \otimes X = B$ has a solution if and only if $\hat{X}$ is a solution.

If $A \otimes X = B$ has a solution then we say "B is a right multiple of A" and $\hat{X}$ will be called the *principal solution to the matrix inequality* $A \otimes X \leq B$.

Given $A \in \overline{\mathbb{R}}^{n\times n}$ and $x \in \overline{\mathbb{R}}^{n}$, the set

$$\left\{A^{(k)} \otimes x; k = 0, 1, ...\right\}$$

is called the *orbit of* A *starting at* x and will be denoted by $T(A, x)$. Obviously, if $A^{(k_0)} \otimes x \in V(A)$ for some k_0 then $A^{(k)} \otimes x \in V(A)$ for all $k \geq k_0$. In general it depends on x whether $T(A, x) \cap V(A) \neq \emptyset$. If $T(A, x) \cap V(A) \neq \emptyset$ for every $x \in \overline{\mathbb{R}}^{n}, x \neq \varepsilon$ then the orbit of A reaches an eigenvector of A with any non-trivial starting vector and A is therefore called *robust*. Robust matrices have been fully characterised [4], [5].

Following this review of relevant earlier results, we proceed to a consideration of GEP. The results following are believed to be new, except where the contrary is explicitly stated.

II. Solvability conditions

In this section we present some solvability conditions for GEP provided that A and B are finite matrices. We therefore assume that $A = (a_{ij}), B = (b_{ij}) \in \mathbb{R}^{m\times n}$ are given matrices and we denote $M = \{1, ..., m\}$ and, as before, $N = \{1, ..., n\}$. We will also denote

$$C = (c_{ij}) = (a_{ij} \otimes b_{ij}^{-1})$$

and

$$D = (d_{ij}) = (b_{ij} \otimes a_{ij}^{-1}).$$

A. Necessary conditions

Theorem 2.1: If (A, B) is solvable and $\lambda \in \Lambda(A, B)$ then C satisfies

$$\max_{i\in M} \min_{j\in N} c_{ij} \leq \lambda \leq \min_{i\in M} \max_{j\in N} c_{ij}. \tag{7}$$

Proof: No row of $\lambda \otimes B$ strictly dominates the corresponding row of A, so for every i there is a j such that $a_{ij} \geq \lambda \otimes b_{ij}$, i.e. $\lambda \leq c_{ij}$. Hence for all i we have $\lambda \leq \max_j c_{ij}$, thus $\lambda \leq \min_i \max_j c_{ij}$. Similarly, no row of A strictly dominates the corresponding row of $\lambda \otimes B$, yielding for all $i : \lambda \geq \min_j c_{ij}$, thus $\lambda \geq \max_i \min_j c_{ij}$. ■

Corollary 2.2: If (A, B) is solvable then C satisfies

$$\max_{i\in M} \min_{j\in N} c_{ij} \leq \min_{i\in M} \max_{j\in N} c_{ij}. \tag{8}$$

Example 2.3: If $A = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ then (A, B) is not solvable because $C = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$ does not satisfy (8).

Corollary 2.4: If $m = n, (A, B)$ is solvable and $\lambda \in \Lambda(A, B)$ then C satisfies

$$\lambda'(C) \leq \lambda \leq \lambda(C).$$

Proof: A cycle in D_C whose every arc has the weight equal to a row maximum in C exists. The arc weights on this cycle are all at least the smallest row maximum, thus $\lambda(C) \geq \min_{i\in M} \max_{j\in N} c_{ij}$. The second inequality now follows from Theorem 2.1 and the other inequality by swapping max and min. ■

Recall that the conjugate of B is $B^* = (b^*_{ij}) = (b_{ji}^{-1})$. Then the i^{th} element of the diagonal of $A \otimes B^*$ equals

$$\max_j (a_{ij} + b^*_{ji}) = \max_j (a_{ij} \otimes b_{ij}^{-1}) = \max_j c_{ij}.$$

Similarly, the i^{th} element of the diagonal of $A \otimes' B^*$ equals $\min_j c_{ij}$. Hence by Theorem 2.1 we have:

Corollary 2.5: If (A, B) is solvable then the greatest element of the diagonal of $A \otimes' B^*$ does not exceed the least element of the diagonal of $A \otimes B^*$.

By Corollary 2.4 we also have:

Corollary 2.6: If (A, B) is solvable and $\lambda \in \Lambda(A, B)$ then

$$\lambda'(A \otimes' B^*) \leq \lambda \leq \lambda(A \otimes B^*).$$

B. A necessary and sufficient condition

Let

$$D_i = \begin{pmatrix} a_{i1} & ... & a_{in} \\ b_{i1} & ... & b_{in} \end{pmatrix} \quad (i = 1, ..., m).$$

Theorem 2.7: A necessary and sufficient condition for (A, B) to be solvable is that the column spaces of the D_i have a common element such that the same multipliers express the dependences for all i, that is there exists a $\xi = (\xi_1, \xi_2)^T$ and $x = (x_1, ..., x_n)^T$ such that $D_i \otimes x = y_i \otimes \xi$ for some $y_i \in \mathbb{R}$ and for all $i = 1, ..., m$. If it is the case then $\xi_1 \otimes \xi_2^{-1} \in \Lambda(A, B)$.

Proof: Let $x \in \mathbb{R}^n$ and $B \otimes x = y$. Then $A \otimes x = \lambda \otimes B \otimes x$ if and only if $D_i \otimes x = \begin{pmatrix} \lambda \otimes y_i \\ y_i \end{pmatrix}$ for all $i = 1, ..., m$. We see that (A, B) is solvable if and only if there exist x and λ such that $D_i \otimes x$ is a multiple of $\begin{pmatrix} \lambda \\ 0 \end{pmatrix}$ for all $i = 1, ..., m$, that is all $D_i \otimes x$ have the same differences between the first and second component. ■

C. On the uniqueness of λ

Theorem 2.8: [2] If both GEP and LGEP for (A, B) are solvable then both have a unique and identical eigenvalue, that is

$$\Lambda(A, B) = \Lambda(A^T, B^T) = \{\lambda\}$$

provided that $\Lambda(A, B) \neq \emptyset, \Lambda(A^T, B^T) \neq \emptyset$.

Proof: Suppose

$$\begin{aligned} A \otimes x &= \lambda \otimes B \otimes x \\ A^T \otimes y &= \mu \otimes B^T \otimes y \end{aligned}$$

for some λ, μ, x, y. Then

$$\begin{aligned} \lambda \otimes y^T \otimes B \otimes x &= y^T \otimes A \otimes x \\ &= x^T \otimes A^T \otimes y \\ &= \mu \otimes x^T \otimes B^T \otimes y \\ &= \mu \otimes y^T \otimes B \otimes x. \end{aligned}$$

Since x, y, B are finite it follows that $\lambda = \mu$. ■

Corollary 2.9: If $A, B \in \mathbb{R}^{n\times n}$ are symmetric then $|\Lambda(A, B)| \leq 1$.

If the GEP for matrices A, B has at most one eigenvalue then it is extremely unlikely that 0 is such an eigenvalue, thus this corollary also implies that two-sided systems $A \otimes x = B \otimes x$ with $A, B \in \mathbb{R}^{n\times n}$ symmetric are "almost never" solvable.

III. Solvable special cases

A. Essentially EP

If either A or B is a generalised permutation matrix then (1) is easily solvable. If (say) B is a generalised permutation matrix then B has the inverse B^{-1} and after multiplying (1) by B^{-1} the GEP is transformed to EP. Since in max-algebra matrices other than generalised permutation matrices do not have an inverse [8], this approach cannot be extended to other GEPs.

B. When A and B have a common eigenvector

Proposition 3.1: $V(A) \cap V(B) \subseteq V(A, B)$ for any $A, B \in \overline{\mathbb{R}}^{n\times n}$ and if $A \otimes x = \lambda \otimes x$ and $B \otimes x = \mu \otimes x$ for some $x \in \overline{\mathbb{R}}^n, x \neq \varepsilon, \mu > \varepsilon$ then $\lambda \otimes \mu^{-1} \in \Lambda(A, B)$.

Proof: $A \otimes x = \lambda \otimes x = \lambda \otimes \mu^{-1} \otimes B \otimes x.$ ■

C. When one of A, B is a right-multiple of the other

Theorem 3.2: If one of $A, B \in \overline{\mathbb{R}}^{m\times n}$ is a right-multiple of the other then (A, B) is solvable.

Proof: Suppose e.g. $A = B \otimes P$, where $P \in \overline{\mathbb{R}}^{n\times n}$. Let $\lambda \in \Lambda(P)$ and $x \in V(P, \lambda)$. Then

$$A \otimes x = B \otimes P \otimes x = B \otimes (\lambda \otimes x) = \lambda \otimes B \otimes x.$$

■

Example 3.3: Suppose $A = \begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 3 & 1 \end{pmatrix}, P = \begin{pmatrix} 4 & 6 \\ -2 & 0 \end{pmatrix}$. Then $\lambda(P) = 4$, $\Gamma(\lambda^{-1} \otimes P) = \begin{pmatrix} 0 & 2 \\ -6 & -4 \end{pmatrix}$ and $x = \begin{pmatrix} 0 \\ -6 \end{pmatrix}, A \otimes x = \begin{pmatrix} 4 \\ 7 \end{pmatrix}, B \otimes x = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$.

We can also prove a sufficient condition for the upper bound for λ in (7) to be attained when (say) A is a right-multiple of B and $A, B \in \mathbb{R}^{m\times n}$. Recall that $C = (c_{ij})$ is the matrix $(a_{ij} \otimes b_{ij}^{-1})$, $D = (d_{ij}) = (b_{ij} \otimes a_{ij}^{-1})$ and let us denote

$$\begin{aligned} L &= \max_i \min_j c_{ij} \\ U &= \min_i \max_j c_{ij}. \end{aligned}$$

It follows from the proof of Theorem 3.2 that $\lambda(P) \in [L, U]$ if $A = B \otimes P$. In this case we also know by Corollary 1.9 that

$$A = B \otimes (B^* \otimes' A) = B \otimes \hat{P}.$$

Let us denote $\hat{\lambda} = \lambda\left(\hat{P}\right)$; thus $L \leq \hat{\lambda} \leq U$.

Lemma 3.4: If $A, B \in \mathbb{R}^{m\times n}$ and $L' = \max_j \min_i c_{ij}$ then $L' \leq \hat{\lambda}$.

Proof:

$$\begin{aligned} \hat{\lambda} &= \lambda\left(\hat{P}\right) \\ &\geq \max_i \hat{p}_{ii} \\ &= \max_i \min_j (b_{ij}^* \otimes a_{ji}) \\ &= \max_i \min_j (a_{ji} \otimes b_{ji}^{-1}) \\ &= \max_i \min_j c_{ji} \\ &= \max_j \min_i c_{ij} = L'. \end{aligned}$$

■

Theorem 3.5: If $A, B \in \mathbb{R}^{m\times n}, D$ has a saddle point and there is a matrix P such that $A = B \otimes P$ then $\hat{\lambda} = U$ where $\hat{\lambda} = \lambda\left(\hat{P}\right) = \lambda(B^* \otimes' A)$.

Proof: D has a saddle point means

$$\max_i \min_j d_{ij} = \min_j \max_i d_{ij}.$$

Therefore the inverses of both sides are equal:

$$U = \min_i \max_j c_{ij} = \max_j \min_i c_{ij} = L'.$$

Hence by Lemma 3.4: $L' = \hat{\lambda} = U.$ ■

The following dual statement is proved in a dual way:

Theorem 3.6: Let $A, B \in \mathbb{R}^{m\times n}$. If there is a matrix P such that $A = B \otimes' P$ and C has a saddle point then $\hat{\lambda}' = L$ where $\hat{\lambda}' = \lambda'\left(\hat{P}\right) = \lambda'(B^* \otimes' A)$.

Even if one of A, B is a right-multiple of the other, the eigenvalue may not be unique as the following example shows.

Example 3.7: With A, B as in Example 3.3, we find for the principal solution matrix $\hat{P}$:

$\hat{P} = \begin{pmatrix} 4 & 6 \\ 3 & 5 \end{pmatrix}$, $\lambda\left(\hat{P}\right) = 5, \Gamma\left(\lambda^{-1} \otimes \hat{P}\right) = \begin{pmatrix} -1 & 1 \\ -2 & 0 \end{pmatrix}, A \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 9 \end{pmatrix}, B \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$.

Hence for the same A, B we find two solutions to relation (1), with different values of λ.

It may be useful to note that unlike in EP the set of all eigenvalues for a GEP may contain an interval. From Examples 3.3 and 3.7, since $A = B \otimes X$ when $X = P$ and when $X = \hat{P}$, it is clear by isotonicity that $A = B \otimes X$ whenever $P \leq X \leq \hat{P}$, and therefore that every $\lambda(X)$ with X in this range is an achievable value of λ in a solution of relation (1).

In particular, $P \leq X(a) \leq \hat{P}$ where

$$X(a) = \begin{pmatrix} 4 & 6 \\ a-2 & a \end{pmatrix}$$

and $a \in [0, 5]$. It is easily seen that $\lambda(X(a)) = a$ when $a \in [4, 5]$, giving a continuum of attainable values of λ.

D. The case of commuting A and B

In this subsection $A, B \in \overline{\mathbb{R}}^{n\times n}$ and $A \otimes B = B \otimes A$.

Lemma 3.8: Let $x \in V(B)$. Then $A \otimes x \in V(B)$.

Proof: $B \otimes x = \lambda \otimes x$ for some $\lambda \in \overline{\mathbb{R}}$ and thus

$$B \otimes (A \otimes x) = A \otimes (B \otimes x) = A \otimes \lambda \otimes x = \lambda \otimes (A \otimes x).$$

■

Corollary 3.9: If A, B commute then the two-sided equation system

$$A \otimes x = B \otimes y$$

is solvable, and a solution can be found by solving the EP for either of A, B.

Proof: Let e.g. $x \in V(B)$, then by Lemma 3.8 $A \otimes x \in V(B)$ and thus for the corresponding eigenvalue λ we have

$$B \otimes (A \otimes x) = \lambda \otimes (A \otimes x) = A \otimes (\lambda \otimes x).$$

■

Any function of the form

$$f(x) = a_0 \otimes x^{(n)} \oplus a_1 \otimes x^{(n-1)} \oplus ... \oplus a_n \otimes x^{(0)}$$

where $a_j \in \overline{\mathbb{R}}$ for all $j = 0, 1, ..., n$ is called a *max-algebraic polynomial*, shortly *maxpolynomial*.

Although most of the statements below hold for general matrices we will assume throughout what follows that A, B are finite. For any matrix U and column-index j, the j^{th} column of U is denoted U_j; and φ, ψ denote maxpolynomials.

Lemma 3.10: $\varphi(A) \otimes (\psi(B))_j = \psi(B) \otimes (\varphi(A))_j$.

Proof: Evidently,

$$\varphi(A) \otimes \psi(B) = \psi(B) \otimes \varphi(A) \tag{9}$$

Now, for any matrix multiplication $U \otimes V$, we have $(U \otimes V)_j = U \otimes V_j$, and the result follows. ■

Lemma 3.11: If $x \in V(B)$, then $\varphi(A) \otimes x \in V(B)$.

Proof: Let $\lambda = \lambda(B)$. Using (9) with $\psi(B) = B$, we have

$$B \otimes \varphi(A) \otimes x = \varphi(A) \otimes B \otimes x = \lambda(B) \otimes \varphi(A) \otimes x \tag{10}$$

■

Theorem 3.12: If A, B have a common critical index, say j, then they have both a common eigenvector and a common left eigenvector and accordingly both GEP and GLEP are solvable, with identical, unique eigenvalue.

Proof: Take $\varphi(A) = \Gamma((\lambda(A))^{-1} \otimes A)$ and $\psi(B) = \Gamma((\lambda(B))^{-1} \otimes B)$. Then $(\varphi(A))_j \in V(A)$ and $(\psi(B))_j \in V(B)$, so by Lemma 3.11,

$$\varphi(A) \otimes (\psi(B))_j \in V(B) \text{ and } \psi(B) \otimes (\varphi(A))_j \in V(A). \tag{11}$$

Hence by Lemma 3.10, $V(A), V(B)$ have a common element and the solvability of problem (A, B) follows immediately by Proposition 3.1.

Using properties of the left eigenproblem (Theorem 1.6), the solvability of the GLEP is proved similarly, with the consequent equality of all generalised eigenvalues by Theorem 2.8. ■

Theorem 3.13: Suppose one of A, B is robust. Then A, B have both a common eigenvector and a common left eigenvector and accordingly both GEP and GLEP are solvable, with identical, unique eigenvalue.

Proof: Suppose A is robust. Then for r sufficiently large, $A^{(r)} \otimes x \in V(A)$ for any finite vector x. But if we choose $x \in V(B)$, then also $A^{(r)} \otimes x \in V(B)$, by taking $\varphi(A) = A^{(r)}$ in Lemma 3.11. Hence, A, B have a common eigenvector and similarly a common left eigenvector. The rest now follows as in Theorem 3.12. ■

IV. Conclusions

We have presented a generalisation of the classical max-algebraic eigenproblem for which, to our knowledge, only paper [2] has been devoted before. No solution method seems to exist for this problem for general matrices over $\overline{\mathbb{R}}$. We have proved a number of necessary conditions for solvability of GEP (which in some cases are also sufficient) and cases of special matrices for which the problem is easily solvable. We have demonstrated that the set of generalised eigenvalues may be both discrete and continuum.

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