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On visualization scaling, subeigenvectors and Kleene stars in max algebra[☆]

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ABSTRACT

The purpose of this paper is to investigate the interplay arising between max algebra, convexity and scaling problems. The latter, which have been studied in nonnegative matrix theory, are strongly related to max algebra. One problem is that of strict visualization scaling, defined as, for a given nonnegative matrix A , a diagonal matrix X such that all elements of $X^{-1}AX$ are less than or equal to the maximum cycle geometric mean of A , with strict inequality for the entries which do not lie on critical cycles. In this paper such scalings are described by means of the max algebraic subeigenvectors and Kleene stars of nonnegative matrices as well as by some concepts of convex geometry.

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1. Introduction

The purpose of this paper is to investigate the interplay arising between max algebra, convexity and matrix scaling. A nonnegative matrix A is called *visualized* if all its elements are less than or equal to the maximum cycle geometric mean $\lambda(A)$ of A , and it is called *strictly visualized* if, further, there is strict inequality for the entries which do not lie on critical cycles. Given a nonnegative matrix A , the chief aim of this paper is to identify and characterize in several ways diagonal matrices X with a positive diagonal for which $X^{-1}AX$ is strictly visualized, see Theorems 3.3, 3.7, 4.2 and 4.4.

In Section 2, we revisit and appropriately summarize the theory of max algebraic eigenvectors and subeigenvectors, and some properties of Kleene stars.

Sections 3 and 4 contain our principal results. In Section 4 our chief tool is the Kleene star A^* of A (defined for a definite matrix), and the max algebraic cone $V^*(A)$. The latter consists of the subeigenvectors of A for the eigenvalue $\lambda(A)$ or, equivalently, of the eigenvectors of A^* . We call $V^*(A)$ the *subeigencone* of A . It is also a convex cone. Diagonal matrices X corresponding to vectors x in its relative interior of the subeigencone are precisely the matrices X that strictly visualize A , see Theorem 3.7. Among those vectors x are all linear combinations of the columns of A^* with positive coefficients, see Theorem 3.3.

While in Section 3 our approach is convex geometric, the main idea of Section 4 is to start with a strictly visualized matrix and to describe all strict visualizers in matrix theoretic terms, see Theorem 4.2. We also show that the dimension of the linear hull of the subeigencone $V^*(A)$ equals the number of components of the critical graph of the Kleene star A^* , see Theorem 4.4. At the end of the section we show by example that the max algebraic dimension of $V^*(A)$ may exceed its linear algebraic dimension.

The interplay between max algebra (essentially equivalent to tropical algebra) and convexity, here explored via visualization, is also important for tropical convexity, see the papers [15,31,32], among many others. We also note that visualization scalings can be important for max algebra, due to the connections with the theory of 0–1 matrices that they provide. See [16,17,39] for recent developments and applications of this idea.

2. Eigenvectors and subeigenvectors

By *max algebra* we understand the analogue of linear algebra developed over the max-times semiring $\mathbb{R}_{\max,\times}$ which is the set of nonnegative numbers \mathbb{R}_+ equipped with the operations of “addition” $a \oplus b := \max(a, b)$ and the ordinary multiplication $a \otimes b := a \times b$. The operations of the semiring are extended to the nonnegative matrices and vectors in the same way as in conventional linear algebra. That is if $A = (a_{ij}), B = (b_{ij})$ and $C = (c_{ij})$ are matrices of compatible sizes with entries from \mathbb{R}_+ , we write $C = A \oplus B$ if $c_{ij} = a_{ij} \oplus b_{ij}$ for all i, j and $C = A \otimes B$ if $c_{ij} = \sum_k^\oplus a_{ik} b_{kj} = \max_k(a_{ik} b_{kj})$ for all i, j . If $\alpha \in \mathbb{R}_+$ then $\alpha A = (\alpha a_{ij})$. We assume everywhere in this paper that $n \geq 1$ is an integer. P_n will stand for the set of permutations of the set $\{1, \dots, n\}$, and the sets like $\{1, \dots, m\}$ or $\{1, \dots, n\}$ will be denoted by $[m]$ or $[n]$, respectively. If A is an $n \times n$ matrix then the iterated product $A \otimes A \otimes \dots \otimes A$ in which the symbol A appears k times will be denoted by A^k .

Max algebra is often presented in settings which seem to be different from $\mathbb{R}_{\max,\times}$, namely, over the max-plus semiring $\mathbb{R}_{\max,+} = (\mathbb{R} \cup \{-\infty\}, \oplus = \max, \otimes = +)$ and the min-plus (or tropical) semiring $\mathbb{R}_{\min,+} = (\mathbb{R} \cup \{+\infty\}, \oplus = \min, \otimes = +)$. The semirings are isomorphic to each other and to $\mathbb{R}_{\max,\times}$. In particular, $x \mapsto \exp(x)$ yields an isomorphism between $\mathbb{R}_{\max,+}$ and $\mathbb{R}_{\max,\times}$.

Let $A = (a_{ij}) \in \mathbb{R}_+^{n \times n}$. The max algebraic eigenproblem consists in finding $\lambda \in \mathbb{R}_+$ and $x \in \mathbb{R}_+^n \setminus \{0\}$ such that $A \otimes x = \lambda x$. If this equation is satisfied, then λ is called a *max algebraic eigenvalue* of A and x is called a *max algebraic eigenvector* of A associated with the eigenvalue λ .

We will also be interested in the *max algebraic subeigenvectors* associated with λ , that is, nonzero $x \in \mathbb{R}_+^n$ such that $A \otimes x \leq \lambda x$. Their first appearance in max algebra seems to be [22, Chapter IV] and [23]. For a more recent reference, see generalization of the max-plus spectral theory [1], where they are called super-eigenvectors.

Next we explain two notions important for both the eigenproblem and the subeigenproblem: that of the maximum cycle mean and that of the Kleene star.

Let $A = (a_{ij}) \in \mathbb{R}_+^{n \times n}$. The weighted digraph $D_A = (N(A), E(A))$, with the set of nodes $N(A) = [n]$ and the set of edges $E(A) = N(A) \times N(A)$ with weights $w(i, j) = a_{ij}$, is called the *digraph associated with A*. Suppose that $\pi = (i_1, \dots, i_p)$ is a path in D_A , then the *weight* of π is defined to be $w(\pi, A) = a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{p-1} i_p}$ if $p > 1$, and 1 if $p = 1$. If $i_1 = i_p$ then π is called a cycle. A path π is called *positive* if $w(\pi, A) > 0$. A path which begins at i and ends at j will be called an $i \rightarrow j$ path. The *maximum cycle geometric mean* of A , further denoted by $\lambda(A)$, is defined by the formula

$$\lambda(A) = \max_{\sigma} \mu(\sigma, A),$$

where the maximization is taken over all cycles in the digraph and

$$\mu(\sigma, A) = w(\sigma, A)^{1/k}$$

denotes the *geometric mean* of the cycle $\sigma = (i_1, \dots, i_k, i_1)$.

If the series $I \oplus A \oplus A^2 \oplus \dots$ converges to a finite matrix, then this matrix is called the *Kleene star* of A and denoted by $A^* = (a_{ij}^*)$. The next proposition gives a necessary and sufficient condition for a matrix to be a Kleene star.

Proposition 2.1 [4]. *Let $A = (a_{ij}) \in \mathbb{R}_+^{n \times n}$. The following are equivalent:*

1. A is a Kleene star;
2. $A^* = A$;
3. $A^2 = A$ and $a_{ii} = 1$ for all $i = 1, \dots, n$.

The next theorem explains some of the interplay between the maximum cycle geometric mean $\lambda(A)$, the Kleene star A^* , and the max algebraic eigenproblem.

Theorem 2.2 [4,5,12,13,40]. *Let $A \in \mathbb{R}_+^{n \times n}$. Then*

1. *the series $I \oplus A \oplus A^2 \oplus \dots$ converges to a finite matrix A^* if and only if $\lambda(A) \leq 1$, and then $A^* = I \oplus A \oplus A^2 \oplus \dots \oplus A^{n-1}$ and $\lambda(A^*) = 1$;*
2. *$\lambda(A)$ is the greatest max algebraic eigenvalue of A .*

This theorem shows great similarity between max algebra and nonnegative linear algebra. However, it also reveals a crucial difference: the series $I \oplus A \oplus A^2 \oplus A^3 \oplus \dots$ converges also if $\lambda(A) = 1$.

$A \in \mathbb{R}_+^{n \times n}$ is called *irreducible* if for any nodes i and j in D_A a positive $i \rightarrow j$ path exists.

Proposition 2.3 [4,13]. *If A is irreducible and $\lambda(A) \leq 1$, then A^* has all entries positive.*

More generally, it is important that Kleene stars accumulate the paths with greatest weights. Namely, if $i \neq j$ then $a_{ij}^* = \max w(\pi, A)$ where π ranges over paths from i to j .

Matrices with $\lambda(A) = 1$ are called *definite*.

Results involving a Kleene star A^* will be stated for definite matrices. There is no real loss of generality here in the case of matrices A with $\lambda(A) > 0$. Indeed, for any such A we have that $\lambda(\alpha A) = \alpha \lambda(A)$, and if $\alpha > 0$, then any eigenvector of A associated with $\lambda(A)$ is also an eigenvector of αA associated with $\lambda(\alpha A)$ and conversely. Hence if $\lambda(A) > 0$, then the eigenproblems for A and $A/\lambda(A)$, which is definite, are equivalent.

Note that $\lambda(A) = 0$ implies that A contains a zero column, and then eigenvectors and subeigenvectors are just vectors x satisfying $x_i = 0$ whenever the corresponding column $A_{\cdot i} \neq 0$. In what follows, we will not treat this trivial case and we will always assume that $\lambda(A) > 0$.

The spaces that we consider in max algebra are subsets of \mathbb{R}_+^n closed under componentwise maximization \oplus , and scalar multiplication. They are called *max cones*, due to the apparent analogy and important connections with conventionally convex cones in \mathbb{R}_+^n .

The set consisting of the zero vector and all subeigenvectors of A associated with $\lambda(A)$ will be denoted by $V^*(A)$. The set consisting of the zero vector and all eigenvectors associated with $\lambda(A)$ will be denoted by $V(A)$. Both sets are max cones, and hence $V(A)$ will be called the *eigencone* of A , and $V^*(A)$ will be called the *subeigencone* of A . Next we study some simple relations between $V(A)$ and $V^*(A)$. The first one is immediate.

Proposition 2.4. $V(A) \subseteq V^*(A)$.

Further we denote by $\text{span}_{\oplus}(A)$ the *max algebraic column span* of A , which is the set of *max combinations* $\left\{ \sum_i^{\oplus} \lambda_i a_i, \lambda_i \in \mathbb{R}_+ \right\}$ of the columns of A . Note that $V(A) \subseteq \text{span}_{\oplus}(A)$ for any matrix A .

Proposition 2.5. *If A is definite, then $V^*(A) = V(A^*) = V^*(A^*) = \text{span}_{\oplus}(A^*)$.*

Proof. First note that by Theorem 2.2, if $\lambda(A) = 1$ then A^* exists and $\lambda(A^*) = 1$. Now we show that $V^*(A) = V(A^*)$. Suppose that $A^* \otimes x = x$, then $A \otimes x \leq x$, because $A \leq A^*$. If $A \otimes x \leq x$, then $(I \oplus A) \otimes x = x$ and also $A^* \otimes x = x$, since $A^m \otimes x \leq x$ for any m (due to the monotonicity of matrix multiplication). As $(A^*)^* = A^*$ by Proposition 2.1, we also have that $V^*(A^*) = V(A^*)$.

We show that $V^*(A) = \text{span}_{\oplus}(A^*)$. As $A \otimes A^* \leq A^*$, each column of A^* is a subeigenvector of A , hence $\text{span}_{\oplus}(A^*) \subseteq V^*(A)$. The converse inclusion follows from $V^*(A) = V(A^*)$ and the inclusion $V(A^*) \subseteq \text{span}_{\oplus}(A^*)$. \square

A matrix A will be called *strongly definite*, if it is definite and if all its diagonal entries equal 1. Note that any Kleene star is strongly definite by Proposition 2.1.

Proposition 2.6. *For a strongly definite matrix, $V(A) = V^*(A)$.*

Proof. To establish $V(A) = V^*(A)$, it is enough to show $V^*(A) \subseteq V(A)$, as the converse inclusion is trivially true. Take $y \in V^*(A)$. We have that $\sum_{j \neq i}^{\oplus} a_{ij} y_j \oplus y_i \leq y_i$ which is equivalent to $\sum_{j \neq i}^{\oplus} a_{ij} y_j \oplus y_i = y_i$, so $y \in V(A)$. \square

By the above propositions, the subeigenvectors of A , and in the strongly definite case also the eigenvectors of A , are described as the vectors from the max algebraic column span of A^* , which we call *Kleene cone*.

More generally, a set S is called a *generating set* for a max cone K , written $K = \text{span}_{\oplus}(S)$, if every vector $y \in K$ can be expressed as a max combination $y = \sum_{i=1}^m \oplus \lambda_i x^i$ of some elements $x^1, \dots, x^m \in S$, with $\lambda_i \geq 0$ for $i \in [m]$. A set S is called a (*weak*) *basis* for K if $\text{span}_{\oplus}(S) = K$ and none of the vectors in S can be expressed as a max combination of the other vectors in S . A vector $y \in K$ is called a *max extremal* of K , if $y = u \oplus w$, $u, w \in K$ implies that $y = u$ or $y = w$. The set of max extremals u of K scaled with respect to the max norm, which means that $\|u\| = \max_i u_i = 1$, will be denoted by $\text{ext}_{\oplus}(K)$. We have the following general result describing max extremals of closed max cones.

Theorem 2.7 [11,24]. *If $K \subseteq \mathbb{R}_+^n$ is a closed max cone, then the set $\text{ext}_{\oplus}(K)$ is non-empty and it is the unique scaled basis for K .*

If $K = \text{span}_{\oplus}(A)$ for some matrix A , then K is closed, so the set $\text{ext}_{\oplus}(\text{span}_{\oplus}(A))$ denoted by $\text{ext}_{\oplus}(A)$ for brevity, is non-empty and constitutes the unique scaled basis for $\text{span}_{\oplus}(A)$. In this case the vectors of $\text{ext}_{\oplus}(A)$ are some of the columns of A scaled with respect to the max norm.

Next we describe the eigencone and the subeigencone of $A \in \mathbb{R}_+^{n \times n}$, and the sets of their scaled max extremals, in the case $\lambda(A) > 0$. For this we will need the following notions and notation. The cycles with the cycle geometric mean equal to $\lambda(A)$ are called *critical*, and the nodes and the edges of D_A that belong to critical cycles are called *critical*. The set of critical nodes is denoted by $N_c(A)$, the set of critical

edges is denoted by $E_c(A)$, and the *critical digraph* of A , further denoted by $C(A) = (N_c(A), E_c(A))$, is the digraph which consists of all critical nodes and critical edges of D_A . All cycles of $C(A)$ are critical [4]. The set of nodes that are not critical is denoted by $\overline{N_c(A)}$. By $C^*(A)$ we denote the digraph with the set of nodes $[n]$ and the set of edges $E_c^*(A)$ containing all the loops (i, i) for $i \in [n]$ and such that $(i, j) \in E_c^*(A)$, for $i \neq j$, if and only if there exists an $i \rightarrow j$ path in $C(A)$. The following theorem describes both subeigencone and eigencone in the case when A is definite. For two vectors x and y , we write $x \sim y$ if $x = \lambda y$ for $\lambda > 0$.

Theorem 2.8. *Let $A \in \mathbb{R}_+^{n \times n}$ be a definite matrix, and let $M(A)$ denote a fixed set of indices such that for each strongly connected component of $C(A)$ there is a unique index of that component in $M(A)$. Then A^* is strongly definite, and*

1. *the following are equivalent: $(i, j) \in E_c(A)$, $a_{ij}a_{jk}^* = a_{ik}^*$ for all $k \in [n]$, $a_{kj}^* = a_{ki}^*a_{ij}$ for all $k \in [n]$.*
2. *the following are equivalent: $(i, j) \in E_c^*(A)$, $A_{\cdot i}^* \sim A_{\cdot j}^*$, $A_{i \cdot}^* \sim A_{j \cdot}^*$;*
3. *any column of A^* is a max extremal of $\text{span}_{\oplus}(A^*)$;*
4. *$V(A)$ is described by*

$$V(A) = \left\{ \sum_{i \in M(A)} \oplus \lambda_i A_{\cdot i}^*; \lambda_i \in \mathbb{R}_+ \right\},$$

and $\text{ext}_{\oplus}(V(A))$ is the set of scaled columns of A^ whose indices belong to $M(A)$;*

5. *for any $y \in V^*(A)$ and any $(i, j) \in E_c(A)$ we have $a_{ij}y_j = y_i$;*
6. *$V^*(A)$ is described by*

$$V^*(A) = V(A^*) = \left\{ \sum_{i \in M(A)} \oplus \lambda_i A_{\cdot i}^* \oplus \sum_{j \in \overline{N_c(A)}} \oplus \lambda_j A_{\cdot j}^*; \lambda_i, \lambda_j \in \mathbb{R}_+ \right\},$$

and $\text{ext}_{\oplus}(V^(A)) = \text{ext}_{\oplus}(A^*)$ is the set of scaled columns of A^* whose indices belong to $M(A) \cup \overline{N_c(A)}$.*

Proof. Statements 1–4 are well-known [4,13,14,22,26].

We show 5: By Proposition 2.5, any $y \in V^*(A)$ is a max combination of the columns of A^* . Let $(i, j) \in E_c(A)$, then part 1 implies that $a_{ij}z_j = z_i$ for any $z = A_{\cdot k}^*$, $k \in [n]$. As y is a max combination of all these, it follows that $a_{ij}y_j = y_i$.

We show 6: By Proposition 2.5 we have $V^*(A) = \text{span}_{\oplus}(A^*)$ and any column of A^* is a max extremal of $\text{span}_{\oplus}(A^*)$ by part 3. By 2 we have that $A_{\cdot i}^* \sim A_{\cdot j}^*$ if and only if $(i, j) \in E_c^*(A)$, hence all the columns in $M(A)$ are independent max extremals and any other columns with indices in $N_c(A)$ are proportional to them. Also note that there are no edges $(i, j) \in E_c^*(A)$ such that $i \notin N_c(A)$ or $j \notin N_c(A)$ except for the loops, and therefore all columns in $\overline{N_c(A)}$ are also independent max extremals. \square

The number of connected components of $C(A)$ will be denoted by $n(C(A))$. For a finitely generated max cone K the cardinality of its unique scaled basis will be called the *max algebraic dimension* of K . Parts 4 and 6 of Theorem 2.8 yield the following corollary.

Proposition 2.9. *For any matrix $A \in \mathbb{R}_+^{n \times n}$ with $\lambda(A) > 0$ we have that the max algebraic dimension of $V(A)$ is equal to $n(C(A))$, and the max algebraic dimension of $V^*(A)$ is equal to $n(C(A)) + |\overline{N_c(A)}|$.*

For $x \in \mathbb{R}_+^n$ denote by $\text{diag}(x)$ the diagonal matrix with entries $\delta_{ij}x_i$, for $i, j \in [n]$, where δ_{ij} is the Kronecker symbol (that is, $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$). Note that the max algebraic multiplication by a diagonal matrix is not different from the conventional multiplication, and therefore the notation \otimes will be omitted in this case. If x is positive, then $X = \text{diag}(x)$ is invertible both in max algebra and in the ordinary linear algebra, and the inverse X^{-1} has entries $\delta_{ij}x_i^{-1}$, for $i, j \in [n]$. The

spectral properties of a matrix A do not change significantly if we apply a diagonal similarity scaling $A \mapsto X^{-1}AX$, where $X = \text{diag}(x)$, with a positive $x \in \mathbb{R}_+^n$.

The following proposition follows very easily from results in the diagonal scaling literature, see, e.g. Remark 2.9 of [18].

Proposition 2.10. *Let $A \in \mathbb{R}_+^{n \times n}$ and let $B = X^{-1}AX$, where $X = \text{diag}(x)$, with positive $x \in \mathbb{R}_+^n$. Then*

1. $w(\sigma, A) = w(\sigma, B)$ for every cycle σ , hence $\lambda(A) = \lambda(B)$ and $C(A) = C(B)$;
2. $V(A) = \{Xy | y \in V(B)\}$ and $V^*(A) = \{Xy | y \in V^*(B)\}$;
3. A is definite if and only if B is definite, and in this case $B^* = X^{-1}A^*X$.

3. Subeigenvectors, visualization and convexity

We call $x \in \mathbb{R}_+^n$ a *nonnegative linear combination* (resp. a *log-convex combination*) of $y^1, \dots, y^m \in \mathbb{R}_+^n$, if $x = \sum_{i=1}^m \lambda_i y^i$ with $\lambda_i \geq 0$ (resp. $x = \prod_{i=1}^m (y^i)^{\lambda_i}$ with $\lambda_i \geq 0$ and $\sum_{i=1}^m \lambda_i = 1$, and both power and multiplication taken componentwise). The combinations are called *positive* if $\lambda_i > 0$ for all i . A set $K \subseteq \mathbb{R}_+^n$ is called a *convex cone* (resp. a *log-convex set*), if it is stable under linear combinations (resp. under log-convex combinations).

In max arithmetics, $a \oplus b \leq c$ is equivalent to $a \leq c$ and $b \leq c$. Using this, one can write out a system of very special homogeneous linear inequalities which define the subeigenscone of A , and hence this cone is also a convex cone and a log-convex set.

Proposition 3.1. *Let $A \in \mathbb{R}_+^{n \times n}$ and $\lambda(A) > 0$. Then $V^*(A)$ is a max cone, a convex cone and a log-convex set.*

Proof. We have that

$$V^*(A) = \{y | A \otimes y \leq \lambda(A)y\} = \left\{ y \mid \sum_j^\oplus a_{ij}y_j \leq \lambda(A)y_i \ \forall i \right\}$$

$$= \{y | a_{ij}y_j \leq \lambda(A)y_i \ \forall i, j\}.$$

Each set $\{y | a_{ij}y_j \leq \lambda(A)y_i\}$ is a max cone, a convex cone and a log-convex set, hence the same is true about $V^*(A)$, which is the intersection of these sets. \square

The log-convexity in $(\mathbb{R}_+ \setminus \{0\})^n$ (i.e. in the max-times setting) corresponds to the conventional convexity in \mathbb{R}^n (i.e., the max-plus setting or the min-plus setting). We also note that $\{y | a_{ij}y_j \leq \lambda(A)y_i\}$ and hence $V^*(A)$ are closed under some other operations. In particular, $V^*(A)$ is closed under componentwise p -norms \oplus_p defined by $(y \oplus_p z)_i = (y_i^p + z_i^p)^{1/p}$ for $p > 0$.

Proposition 3.1 raises a question whether or not there exist max cones containing positive vectors, which are finitely generated and convex, other than Kleene cones. The results of [32] suggest that the answer is negative.

Let K be a convex cone, then $y \in K$ is called an *extremal* of K if and only if $y = \lambda u + \mu v$, where $u, v \in K$, implies $y \sim u$ (and hence also $y \sim v$). The set of scaled extremals of K will be denoted by $\text{ext}(K)$.

Proposition 3.2. *Let $A \in \mathbb{R}_+^{n \times n}$ and $\lambda(A) > 0$, then $\text{ext}_{\oplus}(V^*(A)) \subseteq \text{ext}(V^*(A))$.*

Proof. Without loss of generality we assume that A is definite. By Theorem 2.8 part 6, $\text{ext}_{\oplus}(V^*(A))$ is the set of scaled columns of A^* , after eliminating the repetitions. As $a_{ik}^* a_{kk}^* = a_{ik}^*$, for all $i, k \in [n]$, we have that the $x := A_{\cdot k}^*$ satisfies $a_{ik}^* x_k = x_i$ for all $i \in [n]$. As $V^*(A) = V^*(A^*)$ by Proposition 2.5, we have

that $a_{ik}^* z_k \leq z_i$ for any $z \in V^*(A)$ and all $i \in [n]$, implying that if $x = \lambda z^1 + \mu z^2$ with $z^1, z^2 \in V^*(A)$, then $a_{ik}^* z_k^s = z_i^s$ for all $i \in [n]$ and $s = 1, 2$. Hence $z^1 \sim x$ and $z^2 \sim x$ meaning that $x \in \text{ext}(V^*(A))$. \square

We note that the convex extremals $\text{ext}(V^*(A))$ correspond to the pseudoverties of tropical polytopes [32] (Kleene cones in the min-plus setting), and it is known that the number of these may be up to $(2(n - 1))! / (n - 1)!$ [15,32], unlike the number of max extremals $\text{ext}_{\oplus}(V^*(A))$ which is not more than n .

Max algebraic subeigenvectors give rise to useful diagonal similarity scalings. A matrix A is called *visualized* (resp. *strictly visualized*), if $a_{ij} = \lambda(A)$ for all $(i, j) \in E_c(A)$, and $a_{ij} \leq \lambda(A)$ for all $(i, j) \notin E_c(A)$ (resp. $a_{ij} < \lambda(A)$ for all $(i, j) \notin E_c(A)$).

In the context of max algebra, visualizations have been used to obtain better bounds on the convergence of the power method [16,17]. Strong links between diagonal scaling and max algebra were established in [9].

Specifically, Corollary 2.9 of [9] shows that for a definite $A \in \mathbb{R}_+^{n \times n}$, $X^{-1}AX$ is visualized if and only if $X = \text{diag}(x)$ where x is nonnegative linear combination of the columns of A^* that is positive.

Strict visualization was treated in a special case [8], in connection with the strong regularity of max-plus matrices.

A preliminary version of the following theorem appeared in [10].

Theorem 3.3. *Let $A \in \mathbb{R}_+^{n \times n}$ be definite and $X = \text{diag}(x)$ with positive $x \in \mathbb{R}_+^n$. Then $X^{-1}AX$ is strictly visualized if any of the following conditions are true:*

1. x is a positive linear combination of all columns of A^* ;
2. A is irreducible and x is a positive log-convex combination of all columns of A^* .

Proof. The following argument goes for both cases. In both cases, x is positive: for positive linear combinations this is true since $a_{ii}^* = 1$ for all i , and for positive log-convex combinations, Proposition 2.3 assures that A^* is positive if A is irreducible. As $x \in V^*(A)$, we have that $a_{ij}x_j \leq x_i$ for all i, j . By Theorem 2.8 part 5, $a_{ij}x_j = x_i$ for all $(i, j) \in E_c(A)$. If $(i, j) \notin E_c(A)$, then, by Theorem 2.8 part 1, $a_{ij}z_j < z_i$ for $z = A_{\cdot i}^*$, while $a_{ij}z_j \leq z_i$ for all $z = A_{\cdot k}^*$ where $k \in [n]$. After summing these inequalities for all $z = A_{\cdot k}^*$ with positive coefficients, or after raising them in positive powers and multiplying, we obtain that $a_{ij}x_j < x_i$, taken into account the strict inequality for $z = A_{\cdot i}^*$. Thus x is positive, $x_i^{-1}a_{ij}x_j = 1$ for all $(i, j) \in E_c(A)$ and $x_i^{-1}a_{ij}x_j < 1$ for all $(i, j) \notin E_c(A)$. \square

Note that if A is definite, then every column of A^* can be used to obtain a visualization of A , which may not be strict. This result was known to Afriat [2,3] and Fiedler and Pták [20,21], and it has been a source of inspiration for many works on scaling problems, see [18,19,27,35–38].

Theorem 3.3 implies the following.

Proposition 3.4. *Let A have $\lambda(A) > 0$, then there exists $X = \text{diag}(x)$ with positive $x \in \mathbb{R}_+^n$ such that $X^{-1}AX$ is strictly visualized.*

If A is definite and irreducible then A^* is irreducible, and in this case A^* has an essentially unique positive linear algebraic eigenvector, called the *Perron eigenvector* [6]. As it is a positive linear combination of the columns of A^* , we have the following.

Proposition 3.5. *Let $A \in \mathbb{R}_+^{n \times n}$ be definite and irreducible and let x be the Perron eigenvector of A^* . Then $X^{-1}AX$, for $X = \text{diag}(x)$, is strictly visualized.*

We will now give a topological description of strict visualization scalings, using the linear hull and relative interior of $V^*(A)$.

By Theorem 2.8 part 5, for all $y \in V^*(A)$ and $(i, j) \in E_c(A)$ we have $a_{ij}y_j = y_i$. This can be formulated geometrically. For $A \subseteq \mathbb{R}_+^n$ consider the set

$$L(C(A)) = \{x \in \mathbb{R}^n \mid a_{ij}x_j = \lambda(A)x_i \ \forall (i, j) \in E_c(A)\}.$$

This is a linear subspace of \mathbb{R}^n which contains both $V^*(A)$ (as its convex subcone) and $V(A)$ (as a max subcone of $V^*(A)$). If $B = X^{-1}AX$ with $X = \text{diag}(x)$ and x positive, then, by Proposition 2.10, we have $C(A) = C(B)$, and we infer that $L(C(A)) = \{Xy | y \in L(C(B))\}$.

Let K be a convex cone. The least linear space which contains K will be called the *linear hull* of K and denoted by $\text{Lin}(K)$. This is a special case of the affine hull of a convex set, see [25]. Denote by B_y^ε the open ball with radius $\varepsilon > 0$ and centered at y . The *relative interior* of K , denoted by $\text{ri}(K)$, is the set of points $y \in \mathbb{R}_+^n$ such that for sufficiently small ε we have that $B_y^\varepsilon \cap \text{Lin}(K) \subseteq K$. If $\text{Lin}(K) = \mathbb{R}^n$, then it is the *interior* of K , denoted by $\text{int}(K)$.

The following important “splitting” lemma can be deduced from [42, Lemma 2.9].

Lemma 3.6. *Suppose that $K \subseteq \mathbb{R}_+^n$ is a convex cone which is a solution set of a finite system of linear inequalities S . Let S_1 be composed of the inequalities of S which are satisfied by all points in K with equality, and $S_2 := S \setminus S_1$ be non-empty.*

1. *There exists a point in K by which all inequalities in S_2 are satisfied strictly.*
2. *$\text{Lin}(K)$ is the solution set to S_1 , and $\text{ri}(K)$ is the cone which consists of the points in K by which all inequalities in S_2 are satisfied strictly.*

Now we describe all scalings that give rise to strict visualization.

Theorem 3.7. *Let $A \in \mathbb{R}_+^{n \times n}$ and $\lambda(A) > 0$.*

1. *$L(C(A))$ is the linear hull of the subeigencone $V^*(A)$.*
2. *$x \in \text{ri}(V^*(A))$ if and only if, for $X = \text{diag}(x)$, the matrix $X^{-1}AX$ is strictly visualized.*
3. *$\text{ri}(V^*(A))$ contains the eigenvectors of A if and only if $V^*(A) = V(A)$.*
4. *If A is definite, then any positive linear combination, and, if A is irreducible, also any positive log-convex combination x of all columns of A^* belongs to $\text{ri}(V^*(A))$ and $X^{-1}AX$ with $X = \text{diag}(x)$ is strictly visualized.*

Proof. and 2: Consider Lemma 3.6 with $K = V^*(A)$, then $V^*(A)$ is the solution set to the system of inequalities $a_{ij}x_j \leq x_i$, and we need to show that the inequalities with $(i, j) \in E_c(A)$, and those with $(i, j) \notin E_c(A)$, play the role of S_1 , and S_2 of Lemma 3.6, respectively. For this, we note that by Theorem 2.8 part 6, the inequalities with $(i, j) \in E_c(A)$ are satisfied with equality for all $x \in V^*(A)$, and Proposition 3.4 implies that there is $x \in V^*(A)$ by which all the inequalities with $(i, j) \notin E_c(A)$ are satisfied strictly.

3: The “if” part is obvious. The “only if” part: from Theorem 2.8 it follows that $V^*(A) = V(A)$ if and only if the set of critical nodes is $[n]$. Suppose that $V(A)$ is properly contained in $V^*(A)$, then there is a node i which is not critical. Then for any eigenvector y there is an edge (i, j) for which $a_{ij}y_j = y_i$ and obviously $(i, j) \notin E_c(A)$. Hence $y \notin \text{ri}(V^*(A))$.

4: Follows from Theorem 3.3 and part 2. \square

Note that as $V^*(A)$ is the *max algebraic* column span of A^* , its relative interior may also contain vectors which are not positive linear combinations or positive log-convex combinations of the columns of A^* . However, the relative interior of $V^*(A)$, or the set of vectors which lead to strict visualization, is exactly the set of vectors that can be represented as positive combinations of all *convex* extremals in $\text{ext}_{\oplus}(V^*(A))$, see [25, Section 2.3].

We also remark here that a bijection between $\text{ri}(V^*(A))$ and $\text{ri}(V^*(A^T))$ is given by $x \mapsto x^{-1}$, since $\lambda(A) = \lambda(A^T)$ and if x is positive, then $a_{ij}x_j = \lambda(A)x_i$ (resp. $a_{ij}x_j < \lambda(A)x_i$) holds if and only if $a_{ij}x_i^{-1} = \lambda(A)x_j^{-1}$ (resp. $a_{ij}x_i^{-1} < \lambda(A)x_j^{-1}$). In particular, positive linear combinations of *rows* of Kleene stars also lead, after the inversion, to strict visualization scalings.

If A is strongly definite (that is, $\lambda(A) = 1$ and $a_{ii} = 1$ for all $i \in [n]$), then by Proposition 2.6 we have $V^*(A) = V(A)$, so that $V(A)$ is convex and the maximum cycle geometric mean can be strictly

visualized by eigenvectors in $\text{ri}(V(A))$. We note that in the case when, in addition, the weights of all non-trivial cycles are strictly less than 1, the strict visualization scalings have been described in [8].

Strongly definite matrices are related to the *assignment problem*. By this we understand the following task: given $A \in \mathbb{R}_+^{n \times n}$ find a permutation $\pi \in P_n$ such that its *weight* $a_{1,\pi(1)} \cdot a_{2,\pi(2)} \cdots a_{n,\pi(n)}$ is maximal. A permutation π of maximal weight will be also called a *maximal* permutation.

Again, our aim is to precisely identify (“visualize”) matrix entries belonging to an optimal solution using matrix scaling. That is, for a matrix A with nonzero permutations, find diagonal matrices X and Y such that all entries of XAY on maximal permutations are equal to 1 and that all other entries are strictly less than 1.

To do this, we first find a maximal permutation π and define the corresponding permutation matrix D^π by

$$D_{ij}^\pi = \begin{cases} a_{ij}, & \text{if } j = \pi(i), \\ 0, & \text{if } j \neq \pi(i). \end{cases}$$

Using this matrix, we scale A to one of its strongly definite forms $(D^\pi)^{-1}A$. In a strongly definite matrix, any maximal permutation is decomposed into critical cycles. Conversely, any critical cycle can be extended to a maximal permutation, using the diagonal entries. Therefore, scalings X which visualize the maximal permutations of $(D^\pi)^{-1}A$ are scalings which visualize the critical cycles, and these are given by Theorem 3.7. After we have done this diagonal similarity scaling, we need permutation matrix $E^{\pi^{-1}} = (\delta_{i\pi^{-1}(i)})$ to bring all permutations again to their right place. Thus we get scaling $E^{\pi^{-1}}X^{-1}(D^\pi)^{-1}AX$ which visualizes all maximal permutations.

Numerically, solving visualization problems by the methods described above, relies on the following three standard problems: finding the maximal cycle mean, computing the Kleene star of a matrix, and finding a maximal permutation. The first problem can be solved by Karp’s method [4,33,26], the second problem can be solved by the Floyd–Warshall algorithm [34] and the third problem can be solved by the Hungarian method [34]. All of these methods are polynomial and require $O(n^3)$ operations, which also gives a complexity bound for the visualization problems.

Finally we note that the problem of strict visualization is related to the problem of max balancing considered in [35,37,38]. A matrix B is max balanced if and only if each non-zero element lies on a cycle on which it is a minimal element. It follows that B is strictly visualized. It was shown in [35,37,38] that for each irreducible nonnegative A there is an essentially unique diagonal matrix X such that the scaling $B = X^{-1}AX$ is max balanced, and hence there is a unique max balanced matrix $MB(A)$ diagonally similar to A . Importantly, the matrix $MB(A)$ is canonical for diagonal similarity of irreducible nonnegative matrices, that is A is diagonally similar to C if and only if $MB(A) = MB(C)$. A complexity bound for max balancing which follows from [35,37,38], is $O(n^4)$, see also [41] for a faster version of the max balancing algorithm.

4. Diagonal similarity scalings which leave a matrix visualized

Another approach to describing the visualization scalings is to start with a visualized matrix and describe all scalings which leave it visualized.

We first describe the Kleene star of a definite visualized matrix $A \in \mathbb{R}_+^{n \times n}$. Let $C^*(A)$ have m strongly connected components C_μ , where $\mu \in [m]$, and denote by N_μ the set of nodes in C_μ . Denote by $A_{\mu\nu}$ the (μ, ν) -submatrix of A extracted from the rows with indices in N_μ and from the columns with indices in N_ν . Let $A^C \in \mathbb{R}_+^{m \times m}$ be the $m \times m$ matrix with entries $\alpha_{\mu\nu} = \max\{a_{ij} \mid i \in N_\mu, j \in N_\nu\}$, and let $E \in \mathbb{R}_+^{n \times n}$ be the $n \times n$ matrix with all entries equal to 1.

Proposition 4.1. *Let $A \in \mathbb{R}_+^{n \times n}$ be a definite visualized (resp. strictly visualized) matrix, let m be the number of strongly connected components of $C^*(A)$ and let $A^C = (\alpha_{\mu\nu})$, $A_{\mu\nu}^*$ and $E_{\mu\nu}$ be as defined above. Then*

1. $\alpha_{\mu\mu} = 1$ for all $\mu \in [m]$ and $\alpha_{\mu\nu} \leq 1$ (resp. $\alpha_{\mu\nu} < 1$ for $\mu \neq \nu$), where $\mu, \nu \in [m]$;
2. Any (μ, ν) -submatrix of A^* is equal to $A_{\mu\nu}^* = \alpha_{\mu\nu}^* E_{\mu\nu}$, where $\alpha_{\mu\nu}^*$ is the (μ, ν) -entry of $(A^C)^*$, and $E_{\mu\nu}$ is the (μ, ν) -submatrix of E .

Proof. 1: Immediate from the definitions.

2: Take any $i \in N_\mu, j \in N_\nu$, and any path $\pi = (i_1, \dots, i_k)$ with $i_1 := i$ and $i_k := j$. Then π can be decomposed as $\pi = \tau_1 \circ \sigma_1 \circ \tau_2 \circ \dots \circ \sigma_{l-1} \circ \tau_l$, where τ_i , for $i \in [l]$, are (possibly trivial) paths which entirely belong to some critical component C_{μ_i} , with $\mu_1 := \mu$ and $\mu_l = \nu$, and σ_i , for $i \in [l-1]$, are edges between the strongly connected components. Then $w(\pi, A) \leq w(\pi', A)$, where $\pi' = \tau'_1 \circ \sigma'_1 \circ \tau'_2 \circ \dots \circ \sigma'_{l-1} \circ \tau'_l$ is also a path from i to j such that τ'_i entirely belong to the same critical components as τ_i , and σ'_i are edges connecting the same critical components as σ_i , but $w(\sigma'_i, A) = \max\{a_{ij} | i \in N_{\mu_i}, j \in N_{\mu_{i+1}}\}$ and $w(\tau'_i, A) = 1$. Such a path exists, since in a visualized matrix, there exists a path of weight 1 between any nodes in the same component of the critical digraph. Thus a_{ij}^* is the greatest weight over all such paths π' . As π' bijectively correspond to the paths in the weighted digraph associated with A^C , the claim follows. \square

Note that, after a convenient simultaneous permutation of rows and columns, we have that if A is a definite visualized matrix, then

$$A^* = \begin{pmatrix} E_{11} & \alpha_{12}^* E_{12} & \dots & \alpha_{1n}^* E_{1n} \\ \alpha_{21}^* E_{21} & E_{22} & \dots & \alpha_{2n}^* E_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1}^* E_{m1} & \alpha_{m2}^* E_{m2} & \dots & E_{mm} \end{pmatrix}. \tag{1}$$

Note that A^C does not contain critical cycles except for the loops, otherwise C_μ are not the components of $C^*(A)$. Hence $L(A^C) = \mathbb{R}^m$, and we can speak of the interior of $V^*(A^C)$.

Given a strictly visualized matrix A as above, denote by I_μ , $\mu \in [m]$, the matrix such that $(I_\mu)_{ij} = 1$ whenever $i = j$ belongs to N_μ and $(I_\mu)_{ij} = 0$ elsewhere, and by $A+B$ the direct sum of matrices A and B .

Theorem 4.2. Let $A \in \mathbb{R}_+^{n \times n}$ be a definite visualized matrix and let m be the number of strongly connected components of $C^*(A)$. Let A^C and I_μ be as defined above. Then $X^{-1}AX$, where $X = \text{diag}(x)$ with $x \in \mathbb{R}_+^n$ positive, is visualized (resp. strictly visualized) if and only if X has the form

$$X = \tilde{x}_1 I_1 \dot{+} \dots \dot{+} \tilde{x}_m I_m,$$

where \tilde{x} is a vector satisfying $\alpha_{\mu\nu} \tilde{x}_\nu \leq \tilde{x}_\mu$ (resp. $\alpha_{\mu\nu} \tilde{x}_\nu < \tilde{x}_\mu$), where $\mu \neq \nu, \nu \in [m]$. In other words, $\tilde{x} \in V^*(A^C)$ (resp. $\tilde{x} \in \text{int}(V^*(A^C))$).

Proof. The “if” part: Let x be as described, then the elements a_{ij} , for $i, j \in N_\mu$, do not change after the scaling, so each block $A_{\lambda\lambda}$ remains unchanged, and hence visualized (resp. strictly visualized). For a_{ij} with $i \in N_\mu, j \in N_\nu, \mu \neq \nu$, we have that $a_{ij}x_j \leq x_i$ (resp. $a_{ij}x_j < x_i$), as $x_i = \tilde{x}_\mu, x_j = \tilde{x}_\nu$, and α_{ij} is the maximum over these a_{ij} . Hence $X^{-1}AX$ is visualized (resp. strictly visualized).

The “only if” part: Suppose that scaling by X leaves A visualized (resp. makes A strictly visualized). As A is initially visualized, all critical edges have weights equal to 1, and x should be such that $x_i = x_j = \tilde{x}_\mu$ whenever i, j belong to the same N_μ . For $i \in N_\mu, j \in N_\nu, \mu \neq \nu$, we should have that $a_{ij}\tilde{x}_\nu \leq \tilde{x}_\mu$ (resp. $a_{ij}\tilde{x}_\nu < \tilde{x}_\mu$). Taking maximum over these a_{ij} , we obtain that this is equivalent to $\alpha_{\mu\nu}\tilde{x}_\nu \leq \tilde{x}_\mu$ (resp. $\alpha_{\mu\nu}\tilde{x}_\nu < \tilde{x}_\mu$).

It remains to apply Lemma 3.6 (with $S_1 = \emptyset$), to obtain that the same is equivalent to $\tilde{x} \in V^*(A^C)$ (resp. $\tilde{x} \in \text{int}(V^*(A^C))$). \square

In the following we discuss some issues concerning linear algebraic properties of Kleene cones and Kleene stars. In this context, Kleene stars are known as path product matrices, see [28–30].

For a matrix $A \in \mathbb{R}_+^{n \times n}$ with $\lambda(A) > 0$, we proved that

$$L(C(A)) = \{x \in \mathbb{R}^n \mid a_{ij}x_j = \lambda(A)x_i, (i, j) \in E_c(A)\} \tag{2}$$

is the linear hull of $V^*(A)$. Note that in the case when A is definite and strictly visualized, $a_{ij} = 1$ for all $(i, j) \in E_c(A)$ and $\lambda(A) = 1$. Also see Section 2 for the definition of $n(C(A))$ and $|\overline{N_c(A)}|$.

Proposition 4.3. *Let $A \in \mathbb{R}_+^{n \times n}$ have $\lambda(A) > 0$.*

1. *The dimension of $L(C(A))$ is equal to the number of strongly connected components in $C^*(A)$, that is, to $n(C(A)) + |\overline{N_c(A)}|$;*
2. *If A is definite, then $C^*(A) = C(A^*)$ and $L(C(A)) = L(C(A^*))$.*

Proof. Let N_μ , for $\mu \in [m]$ where $m = n(C(A)) + |\overline{N_c(A)}|$, be the set of nodes of C_μ , a strongly connected component of $C^*(A)$. In the case when A is definite and strictly visualized, $C^*(A) = C(A^*)$ is seen from (1), where $\alpha_{\mu\nu}^* < 1$ for all $\mu \neq \nu$, and it is also seen from (1) that $L(C(A^*))$ is the linear space comprising all vectors $x \in \mathbb{R}_+^n$ such that $x_i = x_j$ whenever i and j belong to the same N_μ . As $L(C(A))$ is also equal to that space by (2), we have that $L(C(A)) = L(C(A^*))$. We can take, as a basis of this space, the vectors e^μ , for $\mu \in [m]$, such that $e_j^\mu = 1$ if $j \in N_\mu$ and $e_j^\mu = 0$ if $j \notin N_\mu$, and hence the dimension of $L(C(A))$ is $n(C(A)) + |\overline{N_c(A)}|$. The general case can be obtained using diagonal similarity. \square

Proposition 4.3 enables us to present the following result.

Theorem 4.4. *For any matrix A with $\lambda(A) > 0$, the max algebraic dimension of $V^*(A)$ is equal to the (linear algebraic) dimension of $L(C(A))$, which is the linear hull of $V^*(A)$.*

Proof. It follows from Proposition 2.9 and Proposition 4.3 part 1. that both dimensions are equal to the number of strongly connected components in $C(A)$. \square

When A is strongly definite and the weights of all nontrivial cycles are strictly less than 1, Theorem 4.4 implies that $V^*(A)$ contains n linearly independent vectors. This result has been obtained by Butkovič [7, Theorem 4.1]. One could also conjecture that in this case the columns of A^* should be linearly independent in the usual sense. However, this is not so in general as we show by modifying Example 3.11 in Johnson-Smith [28]. Let

$$A = A^* = \begin{pmatrix} 1 & 5/11 & 5/11 & 7/11 & 7/11 & 7/11 \\ 5/11 & 1 & 5/11 & 7/11 & 7/11 & 7/11 \\ 5/11 & 5/11 & 1 & 7/11 & 7/11 & 7/11 \\ 7/11 & 7/11 & 7/11 & 1 & 5/11 & 5/11 \\ 7/11 & 7/11 & 7/11 & 5/11 & 1 & 5/11 \\ 7/11 & 7/11 & 7/11 & 5/11 & 5/11 & 1 \end{pmatrix}.$$

Then the linear algebraic rank of A^* is 5, however, by Theorem 4.4 (or [7, Theorem 4.1]) the max algebraic dimension of $V^*(A)$, and therefore the linear algebraic dimension of $L(C(A))$, are 6. We observe that $x = [7/11, 7/11, 7/11, 1, 1, 1]^T$ is a max eigenvector of A^* (hence in $V^*(A)$) but it is not in the linear algebraic span of the columns of A^* . Finally we note that the original form of Example 3.11 in [28] provides a Kleene star with negative determinant.

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