Narrowing the search for generalized eigenvalues in max-algebra

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Abstract

Max-algebra is an analogue of linear algebra developed for the pair of operations $(\oplus, \otimes) = (\max, +)$ over $\mathbb{R} \cup \{-\infty\}$. There is a number of solution methods for solving the eigenproblem $A \otimes x = \lambda \otimes x$. On the other hand little seems to be known about the generalized eigenproblem $A \otimes x = \lambda \otimes B \otimes x$. We present a method for narrowing the search for generalized eigenvalues for a pair of real square matrices. It is based on the solvability conditions for two-sided systems formulated using symmetrized semirings.

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1 Problem formulation

This paper deals with the generalized eigenvalue-eigenvector problem (briefly, generalized eigenproblem) in max-algebra. Max-algebra is an analogue of linear algebra developed for the pair of operations $(\oplus, \otimes) = (\max, +)$ over $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\}$, extended to matrices and vectors. That is if $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$ are matrices of compatible sizes with entries from $\overline{\mathbb{R}}$, we write $C = A \oplus B$ if $c_{ij} = a_{ij} \oplus b_{ij}$ for all i, j and $C = A \otimes B$ if $c_{ij} = \sum_k^{\oplus} a_{ik} \otimes b_{kj} = \max_k (a_{ik} + b_{kj})$ for all i, j. If $\alpha \in \overline{\mathbb{R}}$ then $\alpha \otimes A = (\alpha \otimes a_{ij})$. In max-algebra $-\infty$ plays the role of a neutral element for $\oplus$ and a null for $\otimes$. Throughout the paper we denote $-\infty$ by ε and for convenience we also denote by the same symbol any vector or matrix whose every component is $-\infty$.

The letter I will denote a square matrix whose diagonal entries are 0 and off-diagonal entries are ε. Clearly, $A \otimes I = A$ for every matrix A compatible with I.

One of the most important problems in max-algebra is the eigenproblem (EP):

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Given $A \in \overline{\mathbb{R}}^{n \times n}$, *find all* $\lambda \in \overline{\mathbb{R}}$ *(eigenvalues) and* $x \in \overline{\mathbb{R}}^{n}, x \neq \varepsilon$ *(eigenvectors) such that* $A \otimes x = \lambda \otimes x$.

EP has been studied since the 1960's and can now be efficiently solved [8], [10], [11], [1], [6], [9], [15]. All eigenvalues and bases of all eigenspaces can be found using $O\left(n^3\right)$ operations [8].

On the other hand relatively little seems to be known about the generalized eigenproblem (GEP):

Given $A, B \in \overline{\mathbb{R}}^{m \times n}$, *find all* $\lambda \in \overline{\mathbb{R}}$ (*generalized eigenvalues*) and $x \in \overline{\mathbb{R}}^{n}, x \neq \varepsilon$ (*generalized eigenvectors*) *such that*

$$A \otimes x = \lambda \otimes B \otimes x. \tag{1}$$

Obviously, EP is obtained from GEP when $B = I$. However, it is likely that GEP is profoundly more difficult than EP. This is indicated by the fact that while every $n \times n$ matrix over $\overline{\mathbb{R}}$ has at least one and up to n eigenvalues [8], the GEP for a pair of real matrices may have no generalized eigenvalue, any finite number or a continuum of generalized eigenvalues [13]. To the author's knowledge [2] and [13] are the only papers dealing with GEP. The first of these papers solves the problem completely when $m = 2$ and some special cases for general m and n, the second one solves some other special cases. No solution method seems to have been published either for finding a λ or an $x \neq \varepsilon$ satisfying (1) for general matrices. Note that a two-sided system

$$C \otimes x = D \otimes x \tag{2}$$

with $C, D \in \mathbb{R}^{m \times n}$ can be solved by so-called Alternating Method [12] which has pseudopolynomial complexity when applied to integer matrices. We concentrate on the question of finding all generalized eigenvalues since for any of them the question of finding a generalized eigenvector can be solved using the Alternating Method. We present a method for narrowing the search for generalized eigenvalues for a pair of real square matrices. It is based on the solvability conditions for two-sided systems formulated using symmetrized semirings [1], [14], [17].

A motivation for GEP is given by the following: Consider the *multi-machine interactive production process* (MMIPP) where products $P_1, ..., P_m$ are prepared using n machines (or processors), every machine contributing to the completion of each product by producing a partial product. It is assumed that every machine can work for all products simultaneously and that all these actions on a machine start as soon as the machine starts to work. Let a_{ij} be the duration of the work of the j^{th} machine needed to complete the partial product for P_i ($i = 1, ..., m; j = 1, ..., n$). Let us denote by x_j the starting time of the j^{th} machine ($j = 1, ..., n$). Then all partial products for P_i ($i = 1, ..., m$) will be ready at time $\max(x_1 + a_{i1}, ..., x_n + a_{in})$. Now suppose that independently, n other machines prepare partial products for products $Q_1, ..., Q_m$ and the duration and starting times are b_{ij} and y_j, respectively. Then the *synchronisation problem* is to find starting times of all $2n$ machines so that each pair (P_i, Q_i)

$(i=1,...,m)$ is completed at the same time. This task is equivalent to finding $x_1,...,x_n,y_1,...,y_n \in \mathbb{R}$ satisfying the system

$$\max(x_1+a_{i1},...,x_n+a_{in}) = \max(y_1+b_{i1},...,y_n+b_{in})$$

for $i=1,...,m$. If the machines are linked it may also be required that the starting times x_j, y_j of each pair of machines $(j=1,...,n)$ differ by the same value. If we denote this value by λ then the equations read

$$\max(x_1+a_{i1},...,x_n+a_{in}) = \max(\lambda+x_1+b_{i1},...,\lambda+x_n+b_{in}) \tag{3}$$

for $i=1,...,m$. In max-algebraic notation this system gets the form

$$\sum_{j=1,\ldots,n}^{\oplus} a_{ij}\otimes x_j = \lambda\otimes \sum_{j=1,\ldots,n}^{\oplus} b_{ij}\otimes x_j \quad (i=1,...,m) \tag{4}$$

which is essentially (1).

We will use the following notation: If $a_1,...,a_n \in \overline{\mathbb{R}}$ then the expression $a_1\oplus...\oplus a_n$ will be denoted by $\sum_{i=1,\ldots,n}^{\oplus} a_i$. The iterated expression $a\otimes a\otimes...\otimes a$ where the symbol a appears k-times $(k\geq 1)$ will be denoted $a^{(k)}$ and $a^{(0)}=0$ by definition.

We assume everywhere that $m,n\geq 1$ are integers. P_n will stand for the set of permutations of the set $N=\{1,...,n\}$. Given $A,B\in\overline{\mathbb{R}}^{m\times n}$ it will be convenient to denote

$$\Lambda(A,B)=\left\{\lambda\in\overline{\mathbb{R}};\left(\exists x\in\overline{\mathbb{R}}^n-\{\varepsilon\}\right)A\otimes x=\lambda\otimes B\otimes x\right\}.$$

Notice that in the GEP the case when $\lambda=\varepsilon$ is trivial (a corresponding generalized eigenvector exists if and only if A contains an ε column).

Finally, the following simple property may be helpful when solving two-sided systems in max-algebra:

Lemma 1.1 *(Cancellation Law) Let $v,w,a,b\in\mathbb{R}, a>b$. Then for any real x we have*

$$v\oplus a\otimes x=w\oplus b\otimes x \tag{5}$$

if and only if

$$v\oplus a\otimes x=w. \tag{6}$$

Proof. If x satisfies (5) then LHS $\geq a\otimes x>b\otimes x$. Hence RHS $=w$ and (6) follows. If (6) holds then $w\geq a\otimes x>b\otimes x$ and thus $w=w\oplus b\otimes x$. ■

Since in what follows we only deal with generalized eigenvalues and generalized eigenvectors we will omit the word "generalized". A solution x to (2) will be called *trivial* if $x=\varepsilon$ and *nontrivial* otherwise.

2 Regularisation

Let $C = (c_{ij}), D = (d_{ij}) \in \mathbb{R}^{m\times n}$. The system (2) is called *regular* if

$$c_{ij} \neq d_{ij}$$

for all i, j. For the method presented in this paper it is crucial that λ is such that (1) is regular. There are at most mn values of λ in (1) for which this requirement is not satisfied. More precisely, we have:

Let $A = (a_{ij}), B = (b_{ij}) \in \mathbb{R}^{m\times n}$ and

$$L = \{\lambda \in \mathbb{R}; a_{ij} = \lambda \otimes b_{ij} \text{ for some } i, j\}.$$

Then obviously $|L| \leq mn$ and (1) is regular for all $\lambda \in \mathbb{R} - L$. Every real number in L will be called a *singular value* (of λ). Recall that solvability of (1) can be checked for each fixed and in particular singular value of λ using the Alternating Method. Note that the elements of L appear as entries of $A - B$.

It is also easily seen that if for some i we have $c_{ij} > d_{ij}$ for all j then (2) has no nontrivial solution. Therefore (1) has no nontrivial solution if λ is too big or too small, in particular for $\lambda > \max L$ and $\lambda < \min L$. These two conditions may be slightly refined as $a_{ij} > \lambda \otimes b_{ij}$ for all j or $a_{ij} < \lambda \otimes b_{ij}$ for all j must not hold for any $i = 1, ..., m$. Hence (1) has no nontrivial solution for $\lambda < \lambda'$ and $\lambda > \lambda''$ where λ' is the m^{th} smallest value in L and λ'' is the m^{th} greatest value in L (both considered with multiplicities). So actually only at most $mn - 2m$ singular values of λ need to be checked.

We can now assume that λ is from an open interval J whose ends are singular values but $J \cap L = \emptyset$. Thus the endpoints are singular values but there are no singular values inside J. We will call such intervals *regular* and we will also call every real number *regular* if it belongs to a regular interval. It follows that there are at most $mn - 2m - 1$ regular intervals to be considered. In the rest of the paper we assume that one such interval, say J, has been fixed, and we consider (1) only for $\lambda \in J$.

3 A necessary condition for solving two-sided systems

Symmetrized semirings [1], [17], [14] are useful to study two-sided systems of equations in max-algebra. We now give a brief account of this theory.

Denote $\mathbb{S} = \overline{\mathbb{R}} \times \overline{\mathbb{R}}$ and extend $\oplus$ and $\otimes$ to $\mathbb{S}$ as follows:

$$(a, a') \oplus (b, b') = (a \oplus b, a' \oplus b'),$$

$$(a, a') \otimes (b, b') = (a \otimes b \oplus a' \otimes b', a \otimes b' \oplus a' \otimes b).$$

It is easy to check that $(\varepsilon, \varepsilon) \in \mathbb{S}$ is the neutral element wrt $\oplus$ and $(0, \varepsilon)$ is the neutral element wrt to $\otimes$ in $\mathbb{S}$.

If $x = (a, b)$ then $\ominus x$ stands for $(b, a), x \ominus y$ means $x \oplus (\ominus y)$, the *modulus* of $x \in \mathbb{S}$ is $|x| = a \oplus b$, the *balance operator* is $x^\bullet = x \ominus x = (|x|, |x|)$. The following identities hold:

$$\ominus(\ominus x) = x,$$

$$\ominus(x \oplus y) = (\ominus x) \oplus (\ominus y),$$

$$\ominus(x \otimes y) = (\ominus x) \otimes y.$$

We will need the following properties:

Lemma 3.1 *Let $x, y \in \mathbb{S}$. Then the following hold:*

(a) $|x \oplus y| = |x| \oplus |y|$,

(b) $|x \otimes y| = |x| \otimes |y|$,

(c) $|\ominus x| = |x|$.

Proof. Let $x = (a, b), y = (c, d)$. Then $|x \oplus y| = a \oplus c \oplus b \oplus d$ and $|x| \oplus |y| = a \oplus b \oplus c \oplus d = |x \oplus y|$, hence the first identity. Also, we have

$$\begin{aligned} |x \otimes y| &= (a \otimes c \oplus b \otimes d) \oplus (a \otimes d \oplus b \otimes c) \\ &= (a \oplus b) \otimes (c \oplus d) \\ &= |x| \otimes |y|. \end{aligned}$$

Part (c) is trivial. ■

Let $x = (a, b), y = (c, d) \in \mathbb{S}$. We say that x *balances* y (notation $x \bigtriangledown y$) if $a \oplus d = b \oplus c$. Note that although $\bigtriangledown$ is reflexive and symmetric, it is not transitive.

If $x = (a, b) \in \mathbb{S}$ then x is called *sign-positive* [*sign-negative*], if $a > b$ [$a < b$] or $x = \varepsilon; x$ is called *balanced* if $a = b$, otherwise it is called *unbalanced*. Thus, ε is the only element of $\mathbb{S}$ that is both signed and balanced.

Due to the bijective semiring morphism $t \longrightarrow (t, \varepsilon)$ we will identify, when appropriate, the elements of $\overline{\mathbb{R}}$ and the sign-positive elements of $\mathbb{S}$ of the form (t, ε). Conversely, a sign-positive element (a, b) may be identified with $a \in \overline{\mathbb{R}}$. So for instance 3 may denote the real number as well as the element $(3, \varepsilon)$ of $\mathbb{S}$. By these conventions we may write $3 \ominus 2 = 3, 3 \ominus 7 = \ominus 7, 3 \ominus 3 = 3^\bullet$.

The following are easily proved:

$x \bigtriangledown y, u \bigtriangledown v \Longrightarrow x \oplus u \bigtriangledown y \oplus v$

$x \bigtriangledown y \Longrightarrow x \otimes u \bigtriangledown y \otimes u$

$x \bigtriangledown y$ and $x = (a, b), y = (c, d)$ are sign-positive $\Longrightarrow a = c$

The operations $\oplus$ and $\otimes$ are extended to matrices and vectors over $\mathbb{S}$ in the same way as in conventional linear algebra. A vector is called *sign-positive* [*sign-negative*, *signed*], if all its components are sign-positive [sign-negative, signed]. The properties mentioned above hold if they are appropriately modified for vectors. For more details see [17]. We will now formulate two results presented in [17] of particular relevance for the present paper.

Proposition 3.1 *[17] To every solution of the system $A \otimes x = B \otimes x, x \in \overline{\mathbb{R}}^n, x \neq \varepsilon$ there exists a sign-positive solution to the system of linear balances $(A \ominus B) \otimes x \,\nabla\, \varepsilon, x \in \mathbb{S}^n, x \neq \varepsilon$ and conversely.*

For the second result we define the determinant of matrices in symmetrized semirings. The sign of a permutation σ is $sgn(\sigma) = 0$ if σ is even and it is $\ominus 0$ if σ is odd. The determinant of $A = (a_{ij}) \in \mathbb{S}^{n \times n}$ is

$$det(A) = \sum_{\sigma \in P_n}^{\oplus} \left(sgn\,(\sigma) \otimes \prod_{i \in N}^{\otimes} a_{i,\sigma(i)} \right).$$

Theorem 3.1 *[17] Let $A \in \mathbb{S}^{n \times n}$. Then the system of balances $A \otimes x \,\nabla\, \varepsilon$ has a signed nontrivial (i.e. $\neq \varepsilon$) solution if and only if $det(A) \,\nabla\, \varepsilon$.*

$A \in \mathbb{S}^{n \times n}$ is said to have *balanced determinant* if $det(A) \,\nabla\, \varepsilon$, otherwise it is said to have *unbalanced determinant.* Note that the system of balances $A \otimes x \,\nabla\, \varepsilon$ may not have a nontrivial sign-positive solution if A has balanced determinant.

Corollary 3.1 *Let $A, B \in \overline{\mathbb{R}}^{n \times n}$ and $C = A \ominus B$. Then a necessary condition that the system $A \otimes x = B \otimes x$ have a nontrivial solution is that C has balanced determinant.*

Corollary 3.2 *Let $A, B \in \mathbb{R}^{n \times n}, \lambda \in \mathbb{R}$ and $C\,(\lambda) = A \ominus \lambda \otimes B$. Then a necessary condition that the system $A \otimes x = \lambda \otimes B \otimes x$ have a nontrivial solution is that $C\,(\lambda)$ has balanced determinant.*

The idea of narrowing the search for the eigenvalues is based on Corollary 3.2: We show how to find all λ for which $C\,(\lambda)$ has balanced determinant. It turns out that this can be done using a polynomial number of operations in terms of n. This method may in some cases identify all eigenvalues, see Examples 7.1 and 7.2.

4 Verifying that a determinant is balanced

In this section we show how to convert the question of checking that the determinant of a square matrix over a symmetrized semiring is balanced into a polynomially solvable problem. For this purpose we define the *max-algebraic permanent* of $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}$ as

$$maper(A) = \sum_{\pi \in P_n}^{\oplus} \prod_{i \in N}^{\otimes} a_{i,\pi(i)} = \max_{\pi \in P_n} \sum_{i \in N} a_{i,\pi(i)}.$$

Obviously, $maper(A)$ is the optimal value in the classical (linear) assignment problem for A. We therefore denote the set of optimal permutations by $ap(A)$, that is

$$ap(A) = \left\{ \sigma \in P_n; maper(A) = \sum_{i \in N} a_{i,\sigma(i)} \right\}.$$

Note that the assignment problem is a basic combinatorial optimization problem studied in many books and papers. We mention at least [5] and [16]. Perhaps the best known solution method is the Hungarian method of computational complexity $O\left(n^3\right)$. The method transforms A to a nonpositive matrix B with $ap(A) = ap(B)$ and $maper(B) = 0$. We will refer to the matrix $B = (b_{ij})$ derived from A in this way as to the *normal form of* A. Thus for $\pi \in ap(B)$ we have $b_{i,\pi(i)} = 0$ for all $i \in N$. If $b_{ij} = 0$ for some $i, j \in N$ then a $\pi \in ap(B)$ with $j = \pi(i)$ may or may not exist. But this can easily be decided by checking that $maper\left(B_{ij}\right) = 0$ where B_{ij} is the matrix obtained from B by removing row i and column j.

The link between the determinant of a matrix over a symmetrized semiring and the max-algebraic permanent is shown in the following proposition. If $C = (c_{ij}) \in \mathbb{S}^{n \times n}$ then we denote $|C| = (|c_{ij}|) \in \overline{\mathbb{R}}^{n \times n}$.

Proposition 4.1 *For every $C = (c_{ij}) \in \mathbb{S}^{n \times n}$ we have:*

$$|\det(C)| = maper(|C|).$$

Proof. By a repeated use of Lemma 3.1 we have

$$\begin{aligned}
|\det(C)| &= \left| \sum_{\sigma \in P_n}^{\oplus} \left(sgn(\sigma) \otimes \prod_{i \in N}^{\otimes} c_{i,\sigma(i)} \right) \right| \\
&= \sum_{\sigma \in P_n}^{\oplus} \left| sgn(\sigma) \otimes \prod_{i \in N}^{\otimes} c_{i,\sigma(i)} \right| \\
&= \sum_{\sigma \in P_n}^{\oplus} \left| \prod_{i \in N}^{\otimes} c_{i,\sigma(i)} \right| \\
&= \sum_{\sigma \in P_n}^{\oplus} \prod_{i \in N}^{\otimes} \left| c_{i,\sigma(i)} \right| \\
&= maper(|C|).
\end{aligned}$$

■

A square $(0, 1, -1)$ matrix is called *sign-nonsingular* (SNS) if at least one term of its standard determinant expansion is non-zero and all non-zero terms have the same sign [3]. The problem of checking whether a $(0, 1, -1)$ matrix is SNS or not is equivalent to the even cycle problem in digraphs [3], [14] and therefore polynomially solvable [18].

Given $C = (c_{ij}) \in \mathbb{S}^{n\times n}$ we define $\widetilde{C} = (\tilde{c}_{ij})$ to be the $n \times n$ $(0, 1, -1)$ matrix satisfying

$$\begin{aligned}
\tilde{c}_{ij} &= \quad 1 \text{ if } j = \sigma(i) \text{ for some } \sigma \in ap(|C|) \text{ and } c_{ij} \text{ is sign-positive} \\
\tilde{c}_{ij} &= -1 \text{ if } j = \sigma(i) \text{ for some } \sigma \in ap(|C|) \text{ and } c_{ij} \text{ is sign-negative} \\
\tilde{c}_{ij} &= \quad 0 \text{ else}
\end{aligned}$$

The matrix $\widetilde{C}$ can easily be constructed since (as already explained) it is straightforward to check whether $j = \sigma(i)$ for some $\sigma \in ap(|C|)$. The next theorem was proved in [14].

Theorem 4.1 *Let C be a square matrix over a symmetrized semiring. A sufficient condition that C have balanced determinant is that $\widetilde{C}$ is not SNS. If C has no balanced entry then this condition is also necessary.*

If the system λ is regular then $C = A \ominus \lambda \otimes B$ has no balanced entry. Hence we have:

Corollary 4.1 *Let $A, B \in \mathbb{R}^{n\times n}$, λ be regular. Then the question whether $C(\lambda)$ has balanced determinant can be checked using a polynomial number of operations by checking that $\widetilde{C(\lambda)}$ is not SNS.*

These results enable us to decide for any fixed regular value of λ whether the determinant of $C(\lambda)$ is balanced by checking that $\widetilde{C(\lambda)}$ is not SNS. This is important and will be used later. However, $C(\lambda)$ may have balanced determinant for a continuum of values of λ (see Example 7.2) and therefore we also need to develop a tool which enables us to make the same decision for a whole regular interval. This tool will be presented in Section 6. As a preparation we first show in Section 5 how to find *maper* $|C(\lambda)|$ as a function of $\lambda \in J$.

5 Finding *maper* $|C(\lambda)|$

In this section we show how to efficiently find *maper* $|C(\lambda)|$. This will then be used in the next section to produce a method for finding all regular values of λ for which $\widetilde{C(\lambda)}$ is not SNS.

Recall first that $|C(\lambda)| = (a_{ij} \oplus \lambda \otimes b_{ij}) = (c_{ij}(\lambda))$ and for every $\lambda \in J$ we have

$$a_{ij} \neq \lambda \otimes b_{ij}$$

for all $i, j \in N$. Therefore for every $\lambda \in J$ and for all $i, j \in N$ the entry $c_{ij}(\lambda) = a_{ij} \oplus \lambda \otimes b_{ij}$ is equal to exactly one of a_{ij} and $\lambda \otimes b_{ij}$. Note that $f(\lambda) =$ *maper* $|C(\lambda)|$ is the maximum of $n!$ terms. Each term is a $\otimes$ product of n entries $c_{ij}(\lambda)$, hence of the form $b \otimes \lambda^{(k)}$, where $b \in \mathbb{R}$ and k is a natural number between 0 and n. Since $b \otimes \lambda^{(k)}$ in conventional notation is simply $k\lambda + b$, we deduce that $f(\lambda)$ is the maximum of a finite number of linear functions and therefore a piecewise linear convex function. Note that slopes of all linear pieces of $f(\lambda)$ are natural numbers between 0 and n. Recall that $f(\lambda)$ for any particular λ can easily be found by solving the assignment problem for $|C(\lambda)|$. It follows that all linear pieces can therefore efficiently be identified. We now describe one possible way of finding these linear functions: Assume first that the linear pieces of smallest and greatest slope are known, let us denote them $f_l(\lambda) = a_l \otimes \lambda^{(l)}$ and $f_h(\lambda) = a_h \otimes \lambda^{(h)}$, respectively. If $l = h$ then there is nothing to do, so assume $l \neq h$. We start by finding the intersection point of f_l and f_h, that is, say, λ_1 satisfying $f_l(\lambda_1) = f_h(\lambda_1)$. Calculate $f(\lambda_1) = maper\,|C(\lambda_1)|$. If $f(\lambda_1) = f_l(\lambda_1) = f_h(\lambda_1)$ then there is no linear piece other than f_l and f_h. Otherwise $f(\lambda_1) > f_l(\lambda_1) = f_h(\lambda_1)$. Let r be the number of λ terms appearing in an optimal permutation (if there are several optimal permutations with various numbers of λ appearances then take any). Since r is the slope of the linear piece we have $l < r < h$. Then $a_r = f(\lambda_1) - r\lambda_1$ and $f_r(\lambda) = a_r \otimes \lambda^{(r)}$. This term is a new linear piece and we then repeat this procedure with f_l and f_r and f_r and f_h, and so on. At every step a new linear piece is discovered unless all linear pieces have already been found. Hence the number of iterations is at most $n-1$.

For finding f_l and f_h it will be convenient to use the independent ones problem (IOP) for $0-1$ square matrices:

Given a $0-1$ matrix $M = (m_{ij}) \in \mathbb{R}^{n \times n}$, find the greatest number of ones in M so that no two are from the same row or column or, equivalently, so that there is a $\pi \in P_n$ selecting all these ones.

Clearly, IOP is a special case of the assignment problem, and therefore easily solvable. Note that in graph terminology IOP is known as the maximum cardinality bipartite matching problem solvable in $O\left(n^{2.5}\right)$ time. In general we say that a set of positions in a matrix are independent if no two of them belong to the same row or column.

Now we discuss how to find f_l and f_h. The values of l and h are obviously the smallest and biggest number of independent entries in $|C(\lambda)|$ containing λ and these can be found by solving the corresponding IOP. For h this problem can be described by the matrix $M = (m_{ij})$ with $m_{ij} = 1$ when $|c_{ij}(\lambda)| = \lambda \otimes b_{ij}$ and 0 otherwise and for l by $E - M$, where E is the all-one matrix.

Now we show how to find a_l and a_h. Let $d_{ij} = b_{ij}$ if $c_{ij}(\lambda) = \lambda \otimes b_{ij}$ and $d_{ij} = a_{ij}$ if $c_{ij}(\lambda) = a_{ij}$ (note that by regularity of λ only one of these two possibilities occurs for $\lambda \in J$). For finding a_l and a_h we need to determine permutations π and σ that maximize $\sum_{i \in N} d_{i,\pi(i)}$ and $\sum_{i \in N} d_{i,\sigma(i)}$ and select l and h entries containing λ, respectively. To achieve this we interpret the two above mentioned IOPs as assignment problems and describe their solution sets using normal matrices M_h and M_l (that is nonpositive matrices with at least one set of n independent zeros) obtained by the Hungarian method. It remains then to replace all entries in $D = (d_{ij})$ corresponding to nonzero entries in M_h and M_l by $-\infty$ and solve the assignment problem for the obtained matrices.

6 Narrowing the search for eigenvalues

In this section we show how to efficiently find the set of all regular values of λ for which $\det(C(\lambda))$ is balanced. This set will be denoted by S. We use essentially the fact that the decision whether $\det(C(\lambda))$ is balanced can be made efficiently for any individual value of λ (Corollary 4.1). The following will be useful:

Lemma 6.1 *Let $f(x), g(x), h(x)$ be piecewise linear convex functions on $\mathbb{R}$, $f(x) = g(x) \oplus h(x)$ for all $x \in \mathbb{R}$. Suppose $a, b \in \mathbb{R}$ are such that f is linear on $[a, b]$. If $g(x) = h(x)$ for at least one $x \in (a, b)$ then $g(x) = h(x)$ for all $x \in [a, b]$.*

Proof. Suppose $g(x_0) = h(x_0), x_0 \in (a, b)$. Hence $g(x_0) = h(x_0) = f(x_0)$. If $g(x) < f(x)$ for an $x \in [a, b]$, without loss of generality for $x \in [a, x_0)$, then by convexity of g and linearity of f we have that $g(x) > f(x)$ for all $x \in (x_0, b)$, a contradiction. Therefore $g(x) = f(x)$ for all $x \in [a, b]$ and similarly $h(x) = f(x)$ for all $x \in [a, b]$. ■

Recall that as before J is a regular interval. Let us denote $\det(C(\lambda)) = (d^+(C(\lambda)), d^-(C(\lambda)))$ or just $(d^+(\lambda), d^-(\lambda))$. Then $C(\lambda)$ for $\lambda \in J$ has balanced determinant if and only if

$$d^+(\lambda) = d^-(\lambda). \tag{7}$$

It follows from the results of the previous section that the piecewise linear convex function

$$|\det(C(\lambda))| = d^+(\lambda) \oplus d^-(\lambda) = maper\,|C(\lambda)|$$

can efficiently be found. By the same argument as for $maper\,|C(\lambda)|$ we see that both $d^+(\lambda)$ and $d^-(\lambda)$ are max-algebraic polynomials in λ (hence piecewise linear and convex functions) containing at most $n+1$ powers of λ between 0 and n. No method other than exhaustive search (requiring $n!$ permutation evaluations) seems to be known for finding $d^+(\lambda)$ and $d^-(\lambda)$ separately for any particular λ [7], however for a fixed $\lambda \in \mathbb{R}-L$ by Theorem 4.1 we can decide in polynomial time whether $d^+(\lambda) = d^-(\lambda)$ or not. Since $d^+(\lambda) \oplus d^-(\lambda) = maper\,|C(\lambda)|$ then if $maper\,|C(\lambda)|$ is known using Lemma 6.1 we can easily find ALL values of $\lambda \in J$ satisfying $d^+(\lambda) = d^-(\lambda)$ by checking this equality for any point strictly between any two consecutive breakpoints and for the breakpoints of $maper\,|C(\lambda)|$. We summarize this in the following:

Theorem 6.1 *If the set $S = \{\lambda \in J; d^+(\lambda) = d^-(\lambda)\}$ is nonempty then it consists of some breakpoints of* maper $|C(\lambda)|$ *and a number (possibly zero) of closed intervals whose endpoints are pairs of adjacent breakpoints of* maper $|C(\lambda)|$*. All these can be identified in polynomial time.*

Proof. The statement is essentially proved by Lemma 6.1. We only need to add that each interval whose endpoints are adjacent breakpoints of $maper\,|C(\lambda)|$ can be decided by checking $d^+(\lambda) = d^-(\lambda)$ for one internal point of the interval and that the number of breakpoints is at most n and therefore the number of intervals is at most $n-1$. The equality $d^{+}(\lambda)=d^{-}(\lambda)$ for a fixed λ can be decided in polynomial time by Theorem 4.1. ■

We summarize our work in the following procedure for finding all regular values of λ for which $\det(C(\lambda))$ is balanced:

ALGORITHM *NARROWING THE EIGENVALUE SEARCH*

Input: $A, B \in \mathbb{R}^{n\times n}$ and a regular interval J.

Output: The set $S=\{\lambda \in J; d^{+}(\lambda)=d^{-}(\lambda)\}$.

1. $S := \emptyset$
2. $C(\lambda) := A \ominus \lambda \otimes B$
3. Calculate $f(\lambda) = maper\,|C(\lambda)|$, that is find all breakpoints and linear pieces of $f(\lambda)$
4. For every breakpoint of $f(\lambda)$ do: If $\widetilde{C(\lambda)}$ is not SNS then $S := S \cup \{\lambda\}$
5. For any two consecutive breakpoints a, b and arbitrarily taken $\lambda \in (a,b)$ do: If $\widetilde{C(\lambda)}$ is not SNS then $S := S \cup (a,b)$

7 Examples

In the two examples below we demonstrate that the described method for narrowing the search for eigenvalues may actually find all eigenvalues. Note that in these examples all matrices are of small sizes and therefore the functions $d^{+}(\lambda)$ and $d^{-}(\lambda)$ are explicitly evaluated, however for matrices of nontrivial sizes this would not be practical and the above mentioned method provides an efficient tool for finding all regular values of λ for which $d^{+}(\lambda)=d^{-}(\lambda)$.

Example 7.1 *Let* $A=\begin{pmatrix} 3 & 8 & 2 \\ 7 & 1 & 4 \\ 0 & 6 & 3 \end{pmatrix}$, $B=\begin{pmatrix} 4 & 4 & 3 \\ 2 & 3 & 4 \\ 3 & 2 & 1 \end{pmatrix}$. *Then* $A-B=\begin{pmatrix} -1 & 4 & -1 \\ 5 & -2 & 0 \\ -3 & 4 & 2 \end{pmatrix}$ *and* $L=\{-3,-2,-1,0,2,4,5\}$. *For* $\lambda<-1$ *all terms on the RHS of the first equation in* $A\otimes x=\lambda\otimes B\otimes x$ *are strictly less than the corresponding terms on the left and therefore there is no nontrivial solution to* $A\otimes x=\lambda\otimes B\otimes x$. *For* $\lambda>4$ *similarly all these terms are greater than their counterparts on the left. Hence we only need to investigate regular intervals* $(-1,0),(0,2)$ *and* $(2,4)$ *and the singular points* $-1,0,2,4$.

For $\lambda\in(-1,0)$ *we have*

$$|C(\lambda)|=\begin{pmatrix} 4+\lambda & 8 & 3+\lambda \\ 7 & 3+\lambda & 4 \\ 3+\lambda & 6 & 3 \end{pmatrix},$$

$$d^{+}(\lambda) = \max(10+2\lambda, 14+\lambda, 9+3\lambda),$$
$$d^{-}(\lambda) = \max(16+\lambda, 15+\lambda, 18),$$
$$maper\,|C(\lambda)| = 18.$$

Since $d^{+}(\lambda) \neq d^{-}(\lambda)$ *for* $\lambda \in (-1,0)$*, there are no eigenvalues in this interval.*

For $\lambda \in (0,2)$ *we have*

$$|C(\lambda)| = \begin{pmatrix} 4+\lambda & 8 & 3+\lambda \\ 7 & 3+\lambda & 4+\lambda \\ 3+\lambda & 6 & 3 \end{pmatrix},$$

$$d^{+}(\lambda) = \max(10+2\lambda, 15+2\lambda, 9+3\lambda),$$
$$d^{-}(\lambda) = \max(16+\lambda, 14+2\lambda, 18),$$
$$maper\,|C(\lambda)| = \max(18, 16+\lambda, 15+2\lambda, 9+3\lambda).$$

For $\lambda \in (0,2)$ *there is only one breakpoint for maper* $|C(\lambda)|$ *at* $\lambda_0 = 3/2$*. Since* $d^{+}(\lambda) = d^{-}(\lambda)$ *for* $\lambda = \lambda_0$*, this value is the only candidate for an eigenvalue in* $(0,2)$*. It is not difficult to find that* $x = (2,0,3.5)^T$ *is a corresponding eigenvector.*

For $\lambda \in (2,4)$ *we have*

$$|C(\lambda)| = \begin{pmatrix} 4+\lambda & 8 & 3+\lambda \\ 7 & 3+\lambda & 4+\lambda \\ 3+\lambda & 6 & 1+\lambda \end{pmatrix},$$

$$d^{+}(\lambda) = \max(15+2\lambda, 16+\lambda, 9+3\lambda),$$
$$d^{-}(\lambda) = \max(16+\lambda, 14+2\lambda, 8+3\lambda),$$
$$maper\,|C(\lambda)| = 15+2\lambda$$

Since $d^{+}(\lambda) \neq d^{-}(\lambda)$ *for* $\lambda \in (2,4)$*, there are no eigenvalues in this interval.*

Let us consider the singular point $\lambda = 0:$ *In this small example we solve the system* $A \otimes x = B \otimes x$ *by direct analysis but note that in general the Alternating Method would be used. By the cancellation law (Lemma 1.1) the two-sided system* $A \otimes x = B \otimes x$ *is equivalent to the one with* $A = \begin{pmatrix} \varepsilon & 8 & \varepsilon \\ 7 & \varepsilon & 4 \\ \varepsilon & 6 & 3 \end{pmatrix}$*,* $B = \begin{pmatrix} 4 & \varepsilon & 3 \\ \varepsilon & 3 & 4 \\ 3 & \varepsilon & \varepsilon \end{pmatrix}$*. Here from the first equation either* $x_2 = -4+x_1$ *or* $x_2 = -5+x_3$*. In the first case the third equation yields* $\max(2+x_1, 3+x_3) = 3+x_1$*, thus* $x_1 = x_3$*. Substituting in the second equation then* $x_1 = -4+x_2$*, a contradiction. In the second case the third equation yields again* $x_1 = x_3$ *which implies a contradiction in the same way. Hence* $\lambda = 0$ *is not an eigenvalue and a similar analysis would show that neither are the remaining three singular values.*

We conclude that $\Lambda(A,B) = \{3/2\}$*.*

Example 7.2 *Let* $A = \begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 3 & 1 \end{pmatrix}$. *It is easily seen that* $J = (4,5)$ *is the unique regular interval. For* $\lambda \in (4,5)$ *we have*

$$|C(\lambda)| = \begin{pmatrix} \lambda & 6 \\ 3+\lambda & 9 \end{pmatrix}$$

and

$$maper\,|C(\lambda)| = \max(9+\lambda, 9+\lambda) = 9+\lambda = d^{-}(\lambda) = d^{+}(\lambda).$$

Hence every $\lambda \in J$ *satisfies the necessary condition. In fact all these values are eigenvalues as* $x = (6,\lambda)^T$ *is a corresponding eigenvector (for every* $\lambda \in J$*). This vector is also an eigenvector for* $\lambda \in \{4,5\}$ *and thus* $\Lambda(A,B) = [4,5]$.

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References

[1] F.L. Baccelli, G. Cohen, G.-J. Olsder and J.-P. Quadrat, Synchronization and Linearity, Chichester, J.Wiley and Sons, New York, 1992.

[2] P.A. Binding, H. Volkmer, A generalized eigenvalue problem in the max algebra. Linear Algebra and Its Applications 422 (2007) 360–371.

[3] R.A. Brualdi, H. Ryser, Combinatorial Matrix Theory, Cambridge University Press, 1991.

[4] R.E. Burkard, P. Butkovic, Finding all essential terms of a characteristic maxpolynomial, Discrete Applied Mathematics 130 (2003) 367-380.

[5] R.E. Burkard, M. Dell'Amico, S. Martello, Assignment problems, SIAM, Philadelphia, 2009.

[6] P. Butkovic, Max-algebra: the linear algebra of combinatorics?, Linear Algebra and Its Applications 367 (2003) 313-335.

[7] P. Butkovic, A note on the parity assignment problem, Optimization, 56(4) (2007) 419 - 424.

[8] P. Butkovic, R.A. Cuninghame-Green, S. Gaubert, Reducible spectral theory with applications to the robustness of matrices in max-algebra, Preprint 2007/16, School of Mathematics, University of Birmingham.

[9] G. Cohen, D. Dubois, J.-P. Quadrat, M. Viot, A Linear-System-Theoretic View of Discrete-Event Processes and Its Use for Performance Evaluation in Manufacturing, IEEE Transactions on Automatic Control, Vol. AC-30 (1985) 210-220.

[10] R.A. Cuninghame-Green, Minimax Algebra, Lecture Notes in Economics and Math. Systems 166, Springer, Berlin, 1979.

[11] R.A. Cuninghame-Green, Minimax algebra and applications, Fuzzy Sets and Systems 41 (1991) 251-267.

[12] R.A. Cuninghame-Green, P. Butkovic, The equation Ax = By over (max, +), Theoretical Computer Science 293 (2003) 3-12.

[13] R.A. Cuninghame-Green, P. Butkovic, Generalised eigenproblem in max algebra, IEEE Xplore, Discrete Event Systems (2008) 236-241. http://ieeexplore.ieee.org/stamp/stamp.jsp?arnumber=4605951&isnumber=4605912.

[14] S. Gaubert, P. Butkovic, Sign-nonsingular matrices and matrices with unbalanced determinant in symmetrized semirings, Linear Algebra and Its Applications 301 (1999) 195-201.

[15] B. Heidergott, G.J. Olsder, J. van der Woude (2005), Max Plus at Work: Modeling and Analysis of Synchronized Systems - A Course on Max-Plus Algebra, PUP, 2005.

[16] C.H. Papadimitriou, K. Steiglitz, Combinatorial Optimization-Algorithms and Complexity, Dover Publications, 1998.

[17] M. Plus, Linear systems in (max,+) algebra, in: Proceedings of 29th Conference on Decision and Control Honolulu, 1990.

[18] N. Robertson, P.D. Seymour and R. Thomas, Permanents, Pfaffian orientations and even directed circuits, Ann. of Math. 150 (1999) 929–975.