

Tropical linear-fractional programming and parametric mean payoff games

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ABSTRACT

Tropical polyhedra have been recently used to represent disjunctive invariants in static analysis. To handle larger instances, tropical analogues of classical linear programming results need to be developed. This motivation leads us to study the tropical analogue of the classical linear-fractional programming problem. We construct an associated parametric mean payoff game problem, and show that the optimality of a given point, or the unboundedness of the problem, can be certified by exhibiting a strategy for one of the players having certain infinitesimal properties (involving the value of the game and its derivative) that we characterize combinatorially. We use this idea to design a Newton-like algorithm to solve tropical linear-fractional programming problems, by reduction to a sequence of auxiliary mean payoff game problems.

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1. Introduction

1.1. Motivation from static analysis

Tropical algebra is the structure in which the set of real numbers, completed with $-\infty$, is equipped with the “additive” law “ $a + b$ ” := $a \vee b = \max(a, b)$ and the “multiplicative” law “ ab ” := $a + b$. The max-plus or tropical analogues of convex sets have been studied by a number of authors (Zimmermann, 1977; Cuninghame-Green, 1979; Gaubert and Plus, 1997; Litvinov et al., 2001; Cohen et al., 2004; Develin and Sturmfels, 2004; Briec and Horvath, 2004; Joswig, 2005), under various names (idempotent spaces, semimodules, \mathbb{B} -convexity, extremal convexity), with different degrees of generality, and various motivations.

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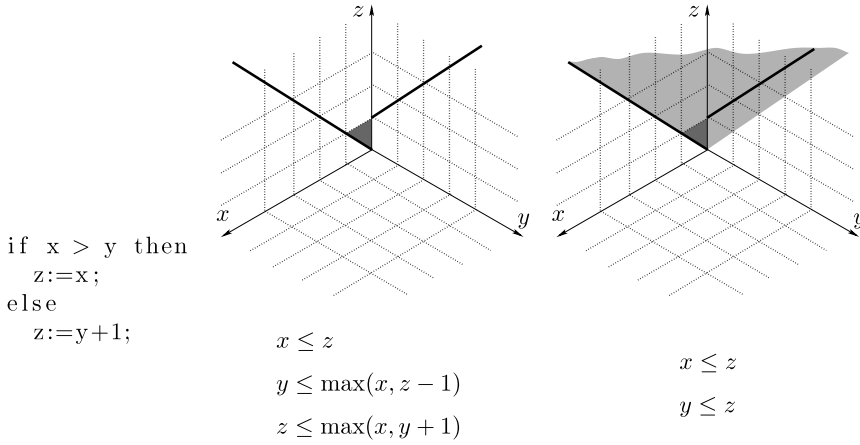


Fig. 1. The tropical polyhedral abstraction of a transfer function, following (Allamigeon et al., 2008), versus the classical polyhedral abstraction. The optimal invariant is represented by the union of two half-lines. The tropical over-approximation of this behavior (union of the triangle in dark gray and of the half-lines, left) is more accurate than the classical polyhedral one (cone in light gray, right).

In a recent work, Allamigeon et al. (2008) have used tropical polyhedra to compute disjunctive invariants in static analysis. A general (affine) tropical polyhedron can be represented as

$$P := \left\{ x \in (\mathbb{R} \cup \{-\infty\})^n \mid \left(\bigvee_{j \in [n]} (a_{ij} + x_j) \right) \vee c_i \leq \left(\bigvee_{j \in [n]} (b_{ij} + x_j) \right) \vee d_i, \forall i \in [m] \right\}. \quad (1)$$

Here, we use the notation $[n] := \{1, \dots, n\}$, and the parameters a_{ij} , b_{ij} , c_i and d_i are given, with values in $\mathbb{R} \cup \{-\infty\}$. The analogy with classical polyhedra becomes clearer with the tropical notation, which allows us to write the constraints as “ $Ax + c \leq Bx + d$ ”, to be compared with classical systems of linear inequalities, $Ax \leq d$ (in the tropical setting, we need to consider affine functions on both sides of the inequality due to the absence of opposite law for addition). The previous representation of P is the analogue of the *external* representation of polyhedra, as the intersection of half-spaces. As in the classical case, tropical polyhedra have a dual (internal) representation, which involves extreme points and extreme rays. The tropical analogue of the Motzkin double description method allows one to pass from one representation to the other (Allamigeon et al., 2010).

Disjunctive invariants arise naturally when analyzing sorting algorithms or in the verification of string manipulation programs. The well known `memcpy` function of C is discussed in (Allamigeon et al., 2008) as a simple illustration: when copying the first n characters of a string buffer `src` to a string buffer `dst`, the length `len_dst` of the latter buffer may differ from the length `len_src` of the former, for if n is smaller than `len_src`, the null terminal character of the buffer `src` is not copied. However, the relation $\min(\text{len_src}, n) = \min(\text{len_dst}, n)$ is valid. This can be expressed geometrically by saying that the vector $(-\text{len_src}, -\text{len_dst})$ belongs to a tropical polyhedron. Several examples of programs of a disjunctive nature, which are analyzed by means of tropical polyhedra, can be found in (Allamigeon et al., 2008; Allamigeon, 2009), in which the tropical analogue of the classical polyhedra-based abstract interpretation method of (Cousot and Halbwachs, 1978) has been developed.

The comparative interest of tropical polyhedra is illustrated in Fig. 1, which gives a simple fragment of code in which the tropical invariant is tighter. Note that there is still an over-approximation in the tropical case, because the transfer function considered here is discontinuous (tropical polyhedra share with classical polyhedra the property of being connected, and therefore cannot represent exactly such discontinuities). Such tropical invariants can be obtained automatically via the methods of (Allamigeon et al., 2008; Allamigeon, 2009), which rely on the tropical analogue of the double description algorithm (Allamigeon et al., 2010), allowing one to obtain the vertices of a tropical polyhedron from a family of defining inequalities, and vice versa.

As in the case of classical polyhedra, the scalability of the approach is inherently limited by the exponential blow up of the size of representations of polyhedra, since the number of vertices or of defining inequalities can be exponential in the size of the input data (Allamigeon et al., 2011a,b).

The complexity of earlier polyhedral approaches led Sankaranarayanan et al. (2005, 2006) to introduce the method of *templates*. In a nutshell, a template consists of a finite set $\mathcal{T} = \{g_1, \dots, g_m\}$ of linear forms on \mathbb{R}^n . The latter define a parametric family of polyhedra

$$P_\alpha = \{x \in \mathbb{R}^n \mid g_k(x) \leq \alpha_k, \quad k \in [m]\}$$

with precisely m degrees of freedom $\alpha_1, \dots, \alpha_m \in \mathbb{R} \cup \{+\infty\}$. The classical domains of boxes or the domain of *zones* (potential constraints) (Miné, 2004) are recovered by incorporating in the template the linear forms $g(x) = \pm x_i$ or $g(x) = x_i - x_j$, respectively. Fixing the template, or changing it dynamically while keeping m bounded, avoids the exponential blow up.

The method of (Sankaranarayanan et al., 2006) relies critically on linear programming, which allows one to evaluate quickly the fixed point functional of abstract interpretation. However, the precision of the invariants remains limited by the linear nature of templates, and it is natural to ask whether the machinery of templates carries over to the non-linear case.

The formalism of templates has been extended to the non-linear case by Adjé et al. (2010), who considered specially quadratic templates, the linear programming methods of (Sankaranarayanan et al., 2006) being replaced by semidefinite programming, thanks to Shor's relaxation. More generally, every tractable subclass of optimization problems yields a tractable template.

In order to compute disjunctive invariants, Allamigeon, Gaubert and Goubault suggested to develop a generalization of the template method to the case of tropical polyhedra. As a preliminary step, the relevant results of linear programming must be tropicalized: this is the object of the present paper.

In particular, comparing the expressions of P and P_α , we see that the classical linear forms must now be replaced by the differences of tropical affine forms

$$g(x) = \left(\bigvee_{j \in [n]} (p_j + x_j) \vee r \right) - \left(\bigvee_{j \in [n]} (q_j + x_j) \vee s \right) \quad (2)$$

where p_j, r, q_j and s are given parameters with values in $\mathbb{R} \cup \{-\infty\}$.

1.2. The problem

In this paper, we study the following *tropical linear-fractional programming problem*:

$$\begin{array}{ll} \text{minimize} & g(x) \\ \text{subject to:} & x \in P \end{array} \quad (3)$$

where P is given by (1) and g is given by (2). This is the tropical analogue of the classical linear-fractional programming problem

$$\begin{array}{ll} \text{minimize} & (px + r)/(qx + s) \\ \text{subject to:} & Ax + c \leq Bx + d, \quad x \geq 0, x \in \mathbb{R}^n \end{array}$$

where p, q are nonnegative vectors, r, s are nonnegative scalars, and A, B, c, d are matrices and vectors. The constraint " $x \geq 0$ " is implicit in (3), since any number is "nonnegative" (i.e., $\geq -\infty$) in the tropical world.

Problem (3) includes as special cases

$$g(x) = \bigvee_{j \in [n]} (p_j + x_j) \quad \text{and} \quad g(x) = - \bigvee_{j \in [n]} (q_j + x_j) \quad (4)$$

(take $q_j \equiv -\infty, r = -\infty$ and $s = 0$, or $p_j \equiv -\infty, s = -\infty$ and $r = 0$). According to the terminology of (Butkovič and Aminu, 2008), the latter may be thought of as the tropical analogues of the linear programming problem. However, optimizing the more general fractional objective function (2) appears to be needed in a number of basic applications. In particular, in static analysis, we need typically to compute the tightest inequality of the form $x_i \leq K + x_j$ satisfied by the elements of P . This fits in the general form (2), but not in the special cases (4).

1.3. Contribution

A basic question in linear programming is to certify the optimality of a given point. This is classically done by exhibiting a feasible solution of the dual problem (i.e., a vector of Lagrange multipliers) with the same value. There is no such a simple result in the tropical setting, because as remarked in (Gaubert and Katz, 2009), there are (tropically linear) inequalities which can be logically deduced from some finite system of (tropically linear) inequalities but which cannot be obtained by taking (tropical) linear combinations of the inequalities of the system. In other words, the usual statement of Farkas lemma is not valid in the tropical setting. However, recently Allamigeon et al. (2011b) established a tropical analogue of Farkas lemma, building on (Akian et al., 2009), in which Lagrange multipliers are replaced by strategies of an associated mean payoff game. We use the same idea here, and show in Section 3.3 (Theorem 12) that the optimality of a solution can be (concisely) certified by exhibiting a strategy of a game, having certain combinatorial properties. Similarly, whether the value of the tropical linear-fractional programming problem is unbounded can also be certified in terms of strategies (Theorem 13).

The second ingredient is to think of the tropical linear-fractional programming problem as a parametric mean payoff game problem. Then, the tropical linear-fractional programming problem reduces to the computation of the minimal parameter for which the value of the game is nonnegative. As a function of the parameter, this value is piecewise-linear and 1-Lipschitz, see Section 3.2 for a more detailed description.

The main contribution of this paper is a Newton-like method, where at each iteration we select a strategy playing the role of derivative (whose existence is implied by the fact that the current feasible point is not optimal). This defines a *one player* parametric game problem, and we show that the smallest value of the parameter making the value zero can be computed in polynomial time for this subgame, by means of shortest path algorithms as described in Section 3.5. The master algorithm (Algorithm 2) requires solving at each iteration an auxiliary mean payoff game, see Sections 3.6 and 3.7. Mean payoff games can be solved either by value iteration, which is pseudo-polynomial, or by policy iteration, for which exponential time instances have been recently constructed by (Friedmann, 2009), although the algorithm is fast on typical examples. The number of Newton type iterations of the Newton-like algorithm has a trivial exponential bound (the number of strategies), and we show that the algorithm is pseudo-polynomial, see Theorem 19. Although this algorithm seems to behave well on typical examples, see in particular Section 4.1, some further work would be needed to assess its worst case complexity (its behavior is likely to be similar to the one of policy iteration, see Section 4.3 for a preliminary account of numerical experiments).

1.4. Related work

Butkovič and Aminu (2008) studied the special cases (4), see also (Butkovič, 2010, Chapter 10). At each iteration, they solve a feasibility problem (whether a tropical polyhedron is non-empty), which is equivalent to checking whether a mean payoff game is winning for one of the players. However, their algorithm does not involve a Newton-like iteration, but rather a dichotomy argument. In Butkovič and Aminu (2008) the number of calls to a mean payoff oracle, whose implementation relies on the alternating method of Cuninghame-Green and Butkovič (2003), depends on the size of the integers in the input, whereas the number of calls in the present algorithm can be bounded independently of these, just in terms of strategies.

A different approach to tropical linear programming was developed previously in the works of Zimmermann (1981) and Zimmermann (2005). This approach, which is based on residuation theory, and works over more general idempotent semirings, identifies important special cases in which the solution can be obtained explicitly (and often, in linear time). However, it cannot be applied to our more general formulation, in which there is little hope to find similar explicit solutions.

Newton methods for finding the least fixed point of nonlinear functions are also closely related to this paper. Such methods were developed in Costan et al. (2005); Gaubert et al. (2007); Gawlitza and Seidl (2007) to solve monotone fixed point problems arising in abstract interpretation. Esparza et al. (2008) develop such methods for monotone systems of min-max-polynomial equations. They seek

for the least fixed point of a function whose components are max-polynomials or min-polynomials, showing that their Newton methods have linear convergence at least. The class of functions considered there is considerably more general than the tropical linear forms appearing here, but the fixed point problems considered in [Esparza et al. \(2008\)](#) appear to be of a different nature. It would be interesting, however, to connect the two approaches.

1.5. Further motivation

Tropical polyhedra have been used in [Katz \(2007\)](#) to determine invariants of discrete event systems. Systems of constraints equivalent to the ones defining tropical polyhedra have also appeared in the analysis of delays in digital circuits, and in the study of scheduling problems with both “and” and “or” constraints ([Möhring et al., 2004](#)). Such systems have been studied by Bezem, Nieuwenhuis, and Rodríguez-Carbonell ([Bezem et al., 2008, 2010](#)), under the name of the “max-atom problem”. The latter is motivated by SAT Modulo theory solving, since conjunctions of max-atoms determine a remarkable fragment of linear arithmetic. Tropical polyhedra also turn out to be interesting mathematical objects in their own right ([Develin and Sturmfels, 2004; Joswig, 2005](#)). A final motivation arises from mean payoff games, the complexity of which is a well known open problem: a series of works show that a number of problems which can be expressed in terms of tropical polyhedra are polynomial time equivalent to mean payoff games problems ([Möhring et al., 2004; Dhingra and Gaubert, 2006; Akian et al., 2009; Bezem et al., 2010; Allamigeon et al., 2011b](#)).

The results of the present paper rely on [Akian et al. \(2009\); Allamigeon et al. \(2011b\)](#).

2. Preliminaries

In this section, we recall some definitions and results needed to describe the present tropical linear-fractional programming algorithm.

We start by introducing deterministic mean payoff games played on a finite bipartite digraph, see Section 2.1. We then summarize some elements of the operator approach to such games, including a theorem of [Kohlberg \(1980\)](#), [Theorem 2](#), implying that their dynamic programming operators f have invariant half-lines (χ, v) . These invariant half-lines determine the ultimate growth of the orbits $f^k(x)/k$, known as the cycle-time vector $\chi(f)$ of f . They also determine a pair of optimal strategies, and the winning nodes of the players $(\{i \mid \chi_i(f) \geq 0\})$ and $(\{i \mid \chi_i(f) < 0\})$ in the associated game.

Then, we recall in Section 2.2 the correspondence between tropical polyhedra and mean payoff games, along the lines of ([Akian et al., 2009](#)): [Theorem 5](#) shows that i is a winning node ($\chi_i(f) \geq 0$) if, and only if, the associated tropical two-sided system of inequalities has a solution x whose i th coordinate is finite ($x_i \neq -\infty$). We also recall the max-plus and min-plus representations of min-max functions, together with the combinatorial characterization, in terms of maximal or minimal cycle means, of the cycle time vector for one-player games. This will be used to construct optimality and unboundedness certificates in Section 3.3.

2.1. Mean payoff games and min-max functions

Consider a two-player deterministic game where the players, called “Max” and “Min”, make alternate moves of a pawn on a weighted bipartite digraph \mathcal{G} . The set of nodes of \mathcal{G} is the disjoint union of the nodes $[m]$ where Max is active, and the nodes $[n]$ where Min is active. When the pawn is in node $k \in [m]$ of Max, he must choose an arc in \mathcal{G} connecting node k with some node $l \in [n]$ of Min, and while moving the pawn along this arc, he receives the weight b_{kl} of the selected arc as payment from Min. When the pawn is in node $j \in [n]$ of Min, she must choose an arc in \mathcal{G} connecting node j with some node $i \in [m]$ of Max, and pays $-a_{ij}$ to Max, where $-a_{ij}$ is the weight of the selected arc. We assume that $b_{kl}, a_{ij} \in \mathbb{R}$. Moreover, certain moves may be prohibited, meaning that the corresponding arcs are not present in \mathcal{G} . Then, we set $b_{kl} = -\infty$ and $a_{ij} = -\infty$. Thus, the whole game is equivalently defined by two $m \times n$ matrices $A = (a_{ij})$ and $B = (b_{kl})$ with entries in $\mathbb{R} \cup \{-\infty\}$. We make the following assumptions, which assure that both players have at least one move allowed in each node.

Assumption 1. For all $k \in [m]$ there exists $l \in [n]$ such that $b_{kl} \neq -\infty$.

Assumption 2. For all $j \in [n]$ there exists $i \in [m]$ such that $a_{ij} \neq -\infty$.

A *general strategy* for a player (Max or Min) is a function that for every finite history of a play ending at some node selects a successor of this node (i.e., a move of the player). A *positional strategy* for a player is a function that selects a unique successor of every node independently of the history of the play.

A strategy for player Max will be usually denoted by σ and a strategy for player Min by τ . Thus, a positional strategy for player Max is a function $\sigma : [m] \mapsto [n]$ such that $b_{i\sigma(i)}$ is finite for all $i \in [m]$, and a positional strategy for player Min is a function $\tau : [n] \mapsto [m]$ such that $a_{\tau(j)j}$ is finite for all $j \in [n]$.

When player Max reveals his positional strategy σ , the play proceeds within the digraph \mathcal{G}^σ where at each node i of Max all but one arc $(i, \sigma(i))$ are removed. When player Min reveals her positional strategy τ , the play proceeds within the digraph \mathcal{G}^τ where at each node j of Min all but one arc $(j, \tau(j))$ are removed.

When player Max reveals his positional strategy σ and player Min her positional strategy τ , the play proceeds within the digraph $\mathcal{G}^{\sigma, \tau}$ where each node has a unique outgoing arc $(i, \sigma(i))$ or $(j, \tau(j))$. Thus, $\mathcal{G}^{\sigma, \tau}$ is a “sunflower” digraph, i.e., such that each node has a unique path to a unique cycle.

The *infinite horizon version* of this mean payoff game starts at a node j of Min² and proceeds according to the strategies (not necessarily positional) of the players, who are interested in the value of average payment. More precisely, player Min wants to minimize

$$\Phi_{A,B}^{\sup}(j, \tau, \sigma) := \limsup_{k \rightarrow \infty} \left(\sum_{t=1}^k -a_{i_t j_{t-1}} + b_{i_t j_t} \right) / k, \quad j_0 = j,$$

while player Max wants to maximize

$$\Phi_{A,B}^{\inf}(j, \tau, \sigma) := \liminf_{k \rightarrow \infty} \left(\sum_{t=1}^k -a_{i_t j_{t-1}} + b_{i_t j_t} \right) / k, \quad j_0 = j,$$

where $j_1 \in [n], i_1 \in [m], j_2 \in [n], i_2 \in [m], \dots$ is the infinite sequence of positions of the pawn resulting from the selected strategies τ and σ of the players. The next theorem shows that this game has a value. An analogue of this theorem concerning stochastic games was obtained by [Liggett and Lippman \(1969\)](#).

Theorem 1 ([Ehrenfeucht and Mycielski \(1979\)](#); [Gurvich et al. \(1988\)](#)). For the mean payoff game whose payments are given by the matrices $A, B \in (\mathbb{R} \cup \{-\infty\})^{m \times n}$, where A, B satisfy [Assumptions 1 and 2](#), there exists a vector $\chi \in \mathbb{R}^n$ and a pair of positional strategies σ^* and τ^* such that

- (i) $\Phi_{A,B}^{\sup}(j, \tau^*, \sigma) \leq \chi_j$ for all (not necessarily positional) strategies σ ,
- (ii) $\Phi_{A,B}^{\inf}(j, \tau, \sigma^*) \geq \chi_j$ for all (not necessarily positional) strategies τ ,

for all nodes j of Min.

In other words, player Max has a positional strategy σ^* which secures a mean profit of at least χ_j whatever is the strategy of player Min, and player Min has a positional strategy τ^* which secures a mean loss of no more than χ_j whatever player Max does.

A *finite duration version* of the mean payoff game considered above can be also formulated. Again, it starts at a certain node j of Min and proceeds according to the strategies of the players (not necessarily positional), but stops immediately when a cycle $j_0 \in [m] \rightarrow i_1 \in [n] \rightarrow j_1 \in [m] \rightarrow \dots \rightarrow i_k \in$

² The game can start also at a node i of Max, the requirement to be started by Min is for better consistency with min–max functions and two-sided tropical systems, see (7) and (14).

$[n] \rightarrow j_k = j_0$ is formed. Then, the outcome of the game is the mean weight per turn (so the length of a cycle may be seen as the number of nodes of Max or Min it contains) of that cycle:

$$\Phi_{A,B}(j, \tau, \sigma) := \left(\sum_{t=1}^k -a_{i_t j_{t-1}} + b_{i_t j_t} \right) / k. \quad (5)$$

The ambition of Max is to maximize $\Phi_{A,B}(j, \tau, \sigma)$ while Min is seeking to minimize it.

It was shown in [Ehrenfeucht and Mycielski \(1979, Theorem 2\)](#) that this finite duration version of the game has the same value as the infinite horizon version described above, and that there are positional optimal strategies which secure this value for both versions of the game. It follows by standard arguments that χ_j is determined uniquely by

$$\chi_j = \Phi_{A,B}(j, \tau^*, \sigma^*) = \min_{\tau} \max_{\sigma} \Phi_{A,B}(j, \tau, \sigma) = \max_{\sigma} \min_{\tau} \Phi_{A,B}(j, \tau, \sigma), \quad (6)$$

where τ and σ range over the sets of all strategies (not necessarily positional) for players Min and Max, respectively.

The dynamic programming operator $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ associated with the infinite horizon mean payoff game is defined by:

$$f_j(x) = \bigwedge_{k \in [m]} \left(-a_{kj} + \bigvee_{l \in [n]} (b_{kl} + x_l) \right). \quad (7)$$

This function, combining min-plus and max-plus linearity (see below in Section 2.2), is known as a *min-max function* ([Cochet-Terrasson et al., 1999](#)). Min-max functions are isotone ($x \leq y \Rightarrow f(x) \leq f(y)$) and additively homogeneous ($f(\lambda + x) = \lambda + f(x)$). Hence, they are nonexpansive in the sup-norm. Moreover, they are piecewise affine (\mathbb{R}^n can be covered by a finite number of polyhedra on which f is affine). We are interested in the following limit (*cycle-time vector*):

$$\chi(f) = \lim_{k \rightarrow \infty} f^k(x)/k. \quad (8)$$

The j th entry of the vector $\chi(f)$ can be interpreted as the limit of the mean value of the game per turn, as the horizon k tends to infinity, when the starting node is j . The existence of $\chi(f)$ follows from a theorem of Kohlberg.

Theorem 2 ([Kohlberg \(1980\)](#)). *Let $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ be a nonexpansive and piecewise affine function. Then, there exist $v \in \mathbb{R}^n$ and $\chi \in \mathbb{R}^n$ such that*

$$f(v + t\chi) = v + (t + 1)\chi, \quad \forall t \geq T,$$

where T is a large enough real number.

The function $t \mapsto v + t\chi$ is known as an *invariant half-line*. Using the nonexpansiveness of f , one deduces that the limit (8) exists, is the same for all $x \in \mathbb{R}^n$ and is equal to the growth rate χ of any invariant half-line.

Given fixed positional strategies τ and σ for players Min and Max, respectively, we can consider the dynamic operators corresponding to the partial digraphs \mathcal{G}^τ and \mathcal{G}^σ . These operators are max-only and min-only functions:

$$\begin{aligned} f_j^\tau(x) &= -a_{\tau(j)j} + \bigvee_{l \in [n]} (b_{\tau(j)l} + x_l), \\ f_j^\sigma(x) &= \bigwedge_{i \in [m]} (-a_{ij} + b_{i\sigma(i)} + x_{\sigma(i)}). \end{aligned} \quad (9)$$

They are the main subject of tropical linear algebra, see Section 2.2, where in particular we recall how their cycle-time vectors can be computed. Eq. (10) relates these cycle-time vectors with the cycle time vector of the min-max function (7).

The following result can be derived as a standard corollary of Kohlberg's theorem. Indeed, we define a positional strategy τ of Min and a positional strategy σ of Max by the condition that $f(v + t\chi) = f^\sigma(v + t\chi) = f^\tau(v + t\chi)$ for t large enough, where $t \mapsto v + t\chi$ is an invariant half-line. These strategies are easily seen to be optimal for the mean payoff game.

Theorem 3 (Corollary of [Kohlberg \(1980\)](#)). For $f(x)$ given by (7), the j th coordinate of $\chi(f)$ is the value of the mean payoff game which starts at node j of Min .

In what follows, we shall use the following form of the value existence result (6), which was proved in ([Gaubert and Gunawardena, 1998](#)) as a corollary of the termination of the policy iteration algorithm of ([Gaubert and Gunawardena, 1998](#); [Cochet-Terrasson et al., 1999](#)), see ([Dhingra and Gaubert, 2006](#)) for a more recent presentation. Alternatively, it can be quickly derived from [Kohlberg \(1980\)](#) (the derivation can be found in ([Akian et al., 2009](#), Theorem 2.13)). This result has been known as the “duality theorem” in the discrete event systems literature.

Theorem 4 (Coro. of [Kohlberg \(1980\)](#), [Gaubert and Gunawardena \(1998\)](#)). Let $A, B \in (\mathbb{R} \cup \{-\infty\})^{m \times n}$ satisfy [Assumptions 1 and 2](#), and let S and T be the sets of all positional strategies for players Max and Min, respectively. Then,

$$\max_{\sigma \in S} \chi(f^\sigma) = \chi(f) = \min_{\tau \in T} \chi(f^\tau). \quad (10)$$

This characterization of $\chi(f)$ should be compared with (6). The latter shows that the infinite horizon version of the game with limsup/liminf payoff has a value, whereas (10) concerns the limit of the value of the finite horizon version. Thus, in loose terms, the “limit” and “value” operations commute.

2.2. Tropical linear systems and mean payoff games

Max-only and min-only functions of the form (9) belong to tropical linear algebra. Max-only functions are linear in the *max-plus semiring* \mathbb{R}_{\max} , which is the set $\mathbb{R} \cup \{-\infty\}$ equipped with the operations of “addition” “ $a + b$ ” := $a \vee b$ and “multiplication” “ ab ” := $a + b$. For min-only functions, we use the *min-plus semiring* \mathbb{R}_{\min} , i.e., the set $\mathbb{R} \cup \{+\infty\}$ equipped with the operations of “addition” “ $a + b$ ” := $a \wedge b$ and the same “additive” multiplication. The setting in which both structures are considered simultaneously has been called minimax algebra in ([Cuninghame-Green, 1979](#)). Then, we need to allow the scalars to belong to the enlarged set $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$. Note that in $\overline{\mathbb{R}}$, $(-\infty) + (+\infty) = -\infty$ if the max-plus convention is considered and $(-\infty) + (+\infty) = +\infty$ if the min-plus convention is considered.

The tropical operations are extended to matrices and vectors in the usual way. In particular, for any matrix $E = (e_{ij})$ and any vector x of compatible dimensions:

$$“(Ex)_i” = \bigvee_j e_{ij} + x_j \quad (\text{max-plus}), \quad “(Ex)_i” = \bigwedge_j e_{ij} + x_j \quad (\text{min-plus}). \quad (11)$$

Max-plus and min-plus linear functions are mutually adjoint, or *residuated*. Recall that for a max-plus linear function from $\overline{\mathbb{R}}^n$ to $\overline{\mathbb{R}}^m$, given by $E \in \mathbb{R}_{\max}^{m \times n}$, the *residuated operator* E^\sharp from $\overline{\mathbb{R}}^m$ to $\overline{\mathbb{R}}^n$ is defined by

$$(E^\sharp y)_j := \bigwedge_{i \in [m]} (-e_{ij} + y_i), \quad (12)$$

with the convention $(-\infty) + (+\infty) = +\infty$. Note that this residuated operator, also known as *Cuninghame-Green inverse*, is given by the multiplication of $-E^T$ by y with the min-plus operations (here E^T denotes the transposed of E), and that it sends \mathbb{R}_{\max}^m to \mathbb{R}_{\max}^n whenever E does not have columns identically equal to $-\infty$.

In what follows, concatenations such as Ex should be understood as the multiplication of E by x with the max-plus operations, and concatenations such as $E^\sharp y$ should be understood as the multiplication of $-E^T$ by y with the min-plus operations (and the corresponding conventions for $(-\infty) + (+\infty)$).

The term “residuated” refers to the property

$$Ex \leq y \Leftrightarrow x \leq E^\sharp y, \quad (13)$$

where \leq is the partial order on \mathbb{R}_{\max}^m or \mathbb{R}_{\max}^n , which can be deduced from

$$e_{ij} + x_j \leq y_i \quad \forall i, j \Leftrightarrow x_j \leq -e_{ij} + y_i \quad \forall i, j.$$

As a consequence, the residuated operator is crucial for max-plus two-sided systems of inequalities, because (13) implies:

$$Ax \leq Bx \Leftrightarrow x \leq A^\sharp Bx. \quad (14)$$

Writing the last inequality explicitly, we have

$$x_j \leq \bigwedge_{k \in [m]} \left(-a_{kj} + \bigvee_{l \in [n]} (b_{kl} + x_l) \right), \quad \forall j \in [n]. \quad (15)$$

Thus, we obtain the same min-max function as in (7).

Moreover, positional strategies $\sigma : [m] \mapsto [n]$ and $\tau : [n] \mapsto [m]$ correspond to affine functions B^σ and A_τ defined by

$$(A_\tau)_{ij} = \begin{cases} a_{ij} & \text{if } i = \tau(j), \\ -\infty & \text{otherwise,} \end{cases} \quad (B^\sigma)_{ij} = \begin{cases} b_{ij} & \text{if } j = \sigma(i), \\ -\infty & \text{otherwise.} \end{cases} \quad (16)$$

Recasting (10) in max(min)-plus algebra, we obtain

$$\max_{\sigma \in S} \chi(A^\sharp B^\sigma) = \chi(A^\sharp B) = \min_{\tau \in T} \chi(A_\tau^\sharp B). \quad (17)$$

The following result, obtained by Akian, Gaubert and Guterman, relates solutions of $Ax \leq Bx$ and nonnegative coordinates of $\chi(A^\sharp B)$. These coordinates correspond to *winning nodes* of the mean payoff game: if the game starts in these nodes, then Max can ensure nonnegative profit with any strategy of Min.

Theorem 5 (Akian et al., 2009, Theorem 3.2). *Let $A, B \in \mathbb{R}_{\max}^{m \times n}$ satisfy Assumptions 1 and 2. Then, $\chi_i(A^\sharp B) \geq 0$ if and only if there exists $x \in \mathbb{R}_{\max}^n$ such that $Ax \leq Bx$ and $x_i \neq -\infty$.*

This is derived in Akian et al. (2009) from Kohlberg's theorem. The vector x is constructed by taking an invariant half-line, $t \mapsto v + t\chi$, setting $x_i = v_i + t\chi_i$ for t large enough if $\chi_i \geq 0$, and $x_i = -\infty$ otherwise.

Theorem 5 shows that to decide whether $Ax \leq Bx$ can be satisfied by a vector x such that $x_i \neq -\infty$, we can exploit a *mean payoff oracle*, which decides whether i is a winning node of the associated mean payoff game and gives a winning strategy for player Max. This oracle can be implemented either by using the value iteration method, which is pseudo-polynomial (Zwick and Paterson, 1996), by the approach of Puri (solving an associated discounted game for a discount factor close enough to 1 by policy iteration (Puri, 1995)), by using the policy iteration algorithm for mean payoff games of Cochet-Terrasson et al. (1999); Gaubert and Gunawardena (1998); Dhingra and Gaubert (2006), or the one of Bjorklund and Vorobyov (2007).

In tropical linear algebra, there is no obvious subtraction. However, for any $E \in \overline{\mathbb{R}}^{n \times n}$ we can define the *Kleene star*

$$E^* := "(I - E)^{-1}" = I \vee E \vee E^2 \vee \dots \quad (\text{max-plus}), \quad (18)$$

and analogously with \wedge in the min-plus case. In (18), I is the max-plus identity matrix with 0 entries on the main diagonal and $-\infty$ off the diagonal, and the powers are understood in the tropical (max-plus) sense. Due to the order completeness of $\overline{\mathbb{R}}$, the series in (18) is well-defined for all matrices. Note that in $\overline{\mathbb{R}}^{n \times n}$, $X = E^*$ is a solution of the matrix Bellman equation $X = EX \vee I$. Similarly, $x = E^*h$ is a solution of $x = Ex \vee h$. Indeed, if $z \geq Ez \vee h$, then we also have

$$z \geq Ez \vee h \geq E^2z \vee Eh \vee h \geq \dots \geq E^{k+1}z \vee E^k h \vee E^{k-1}h \vee \dots \vee h,$$

so that $z \geq E^*h$ for all such z . We sum this up in the following standard proposition.

Proposition 6 (See e.g. (Baccelli et al., 1992, Theorem 3.17)). *Let $E \in \overline{\mathbb{R}}^{n \times n}$ and $h \in \overline{\mathbb{R}}^n$. Then, E^*h is the least solution of $z \geq Ez \vee h$.*

For $E \in \mathbb{R}^{n \times n}$, consider the associated digraph $\mathcal{D}(E)$, with set of nodes $[n]$ and an arc connecting node i with node j whenever e_{ij} is finite, in which case e_{ij} is the weight of this arc. We shall say that *node i accesses node j* if there exists a path from i to j in $\mathcal{D}(E)$.

The maximal (minimal) cycle mean is another important object of tropical algebra. For $E \in \mathbb{R}_{\max}^{n \times n}$ ($E \in \mathbb{R}_{\min}^{n \times n}$), it is defined as

$$\begin{aligned}\mu^{\max}(E) &= \max_{k \in [n]} \max_{i_1, \dots, i_k} \frac{e_{i_1 i_2} + \dots + e_{i_k i_1}}{k} \quad (\text{max-plus}), \\ \mu^{\min}(E) &= \min_{k \in [n]} \min_{i_1, \dots, i_k} \frac{e_{i_1 i_2} + \dots + e_{i_k i_1}}{k} \quad (\text{min-plus}).\end{aligned}\tag{19}$$

Denote by $\mu_i^{\max}(E)$ ($\mu_i^{\min}(E)$) the maximal (minimal) cycle mean of the strongly connected component of $\mathcal{D}(E)$ to which i belongs. These numbers are given by the same expressions as in (19), but with i_1, \dots, i_k restricted to that strongly connected component. Using $\mu_i^{\max}(E)$ ($\mu_i^{\min}(E)$), we can write explicit expressions for the cycle-time vector of a max-plus (min-plus) linear function $x \mapsto Ex$:

$$\begin{aligned}\chi_i^{\max}(E) &= \max\{\mu_j^{\max}(E) \mid i \text{ accesses } j\} \quad (\text{max-plus}), \\ \chi_i^{\min}(E) &= \min\{\mu_j^{\min}(E) \mid i \text{ accesses } j\} \quad (\text{min-plus}).\end{aligned}\tag{20}$$

See (Cochet-Terrasson et al., 1998) or (Heidergott et al., 2005) for proofs. Importantly, these cycle-time vectors of max-plus and min-plus linear functions appear in (10).

Finally, note that (19) and (20) can be deduced from (6) if σ or τ is fixed.

Remark 1. Observe that any entry (i, j) of E^k in max-plus (resp. min-plus) algebra expresses the maximal (resp. minimal) weight of paths with k arcs connecting node i with node j in $\mathcal{D}(E)$. It follows then from (18) that any entry (i, j) of E^* , for $i \neq j$, expresses the maximal (or minimal) weight of paths connecting node i with node j without restrictions on the number of arcs. Further we can add to $\mathcal{D}(E)$ a new node and, whenever b_i is finite, an arc of weight b_i connecting node i of $\mathcal{D}(E)$ with this new node. Then, $(E^*b)_i$ provides the maximal (or minimal) weight of paths connecting node i with the new node. Therefore, computing E^*b is equivalent to solving a single destination shortest path problem, which can be done in $O(n^3)$ time (for instance by the Bellman-Ford algorithm).

Remark 2. We note that $\chi^{\max}(E)$ and $\chi^{\min}(E)$ can also be computed in $O(n^3)$ time. To do this, decompose first the digraph $\mathcal{D}(E)$ in strongly connected components, and apply Karp's algorithm to compute the maximal or minimal cycle mean of each component.

3. Tropical linear-fractional programming

This is the main section of the paper. Here we solve the tropical linear-fractional programming problem (3), i.e., the problem

$$\begin{aligned}\text{minimize} \quad & (px \vee r) - (qx \vee s) \\ \text{subject to:} \quad & Ax \vee c \leq Bx \vee d, \quad x \in \mathbb{R}_{\max}^n\end{aligned}\tag{21}$$

where $p, q \in \mathbb{R}_{\max}^n$, $c, d \in \mathbb{R}_{\max}^m$, $r, s \in \mathbb{R}_{\max}$ and $A, B \in \mathbb{R}_{\max}^{m \times n}$.

In Section 3.1, we apply Theorem 5 to reduce (21) to the problem of finding the smallest zero of a function giving the value of a parametric game (the *spectral function*).

In Section 3.2, we show that the spectral function is 1-Lipschitz and piecewise linear, and that it can be written as a finite supremum or infimum of *partial* spectral functions, corresponding to one-player games. We also prove a number of technical statements about the piecewise-linear structure of the spectral functions, which will be used in the complexity analysis.

In Section 3.3, we provide certificates of optimality and unboundedness (these certificates are given by strategies for the players). This generalizes the result of (Allamigeon et al., 2011b), concerning the tropical analogue of Farkas lemma. We recover as a special case the unboundedness certificates of (Butkovič and Aminu, 2008).

The rest of the section is devoted to finding the least zero of the spectral function. With this aim, we introduce a bisection method, as well as a Newton-type method, in which partial spectral functions play the role of derivatives, see Section 3.4.

Each Newton iteration consists of

- (i) Computing a derivative, i.e., choosing a strategy for player Max (or dually Min) which satisfies a local optimality condition;
- (ii) Finding the smallest zero of the tangent map, which represents the parametric spectral function of a one-player game in which the strategy for player Max (or dually Min) is already fixed.

The iteration in the space of strategies for player Max has an advantage: the second subproblem can be reduced to a shortest-path problem (Section 3.5). The first subproblem is discussed in Section 3.6, where the overall worst-case complexity of the Newton method is given. Section 3.7, which can be skipped by the reader, gives an alternative approach to the first subproblem in which the computation of (left) optimal strategies is rather algebraic and not relying on the integrality.

3.1. The spectral function method

In this subsection we recast (21) as a parametric two-sided tropical system and a mean payoff game, introducing the key concept of spectral function. However, before doing this we need to mention special cases in which there exists a feasible x (i.e., satisfying $Ax \vee c \leq Bx \vee d$) such that $px \vee r = -\infty$ or $qx \vee s = -\infty$. For these cases we assume the following rules:

$px \vee r$	$qx \vee s$	$(px \vee r) - (qx \vee s)$
$-\infty$	finite	$-\infty$
finite	$-\infty$	$+\infty$
$-\infty$	$-\infty$	$-\infty$

(22)

which are formally consistent with the rules of $\overline{\mathbb{R}}_{\max}$. Then, it is easy to check that

$$(px \vee r) - (qx \vee s) = \min\{\lambda \in \mathbb{R}_{\max} \mid px \vee r \leq \lambda + (qx \vee s)\}. \quad (23)$$

Introducing the notation

$$U = \begin{pmatrix} A & c \\ p & r \end{pmatrix} \quad \text{and} \quad V(\lambda) = \begin{pmatrix} B & d \\ \lambda + q & \lambda + s \end{pmatrix}, \quad (24)$$

we reformulate the tropical linear-fractional programming problem in terms of a spectral function, which gives the value of a parametric mean payoff game: the payments are given by the matrices U and $V(\lambda)$, and the initial node is $n + 1$.

Proposition-Definition 7. *With assumption (22), the tropical linear-fractional programming problem (21) is equivalent to*

$$\min\{\lambda \in \mathbb{R}_{\max} \mid \phi(\lambda) \geq 0\} \quad (25)$$

where the spectral function ϕ is given by

$$\phi(\lambda) := \chi_{n+1}(U^\sharp V(\lambda)).$$

Proof. We first show that (21) is equivalent to the following problem:

$$\begin{aligned} &\text{minimize} \quad \lambda \\ &\text{subject to:} \quad px \vee r \leq \lambda + (qx \vee s), \quad Ax \vee c \leq Bx \vee d, \quad x \in \mathbb{R}_{\max}^n, \quad \lambda \in \mathbb{R}_{\max} \end{aligned} \quad (26)$$

Indeed, denoting $P = \{x \in \mathbb{R}_{\max}^n \mid Ax \vee c \leq Bx \vee d\}$, we verify that

$$\begin{aligned} \min_{x \in P} \{(px \vee r) - (qx \vee s)\} &= \min_{x \in P} \min_{\lambda} \{\lambda \mid px \vee r \leq \lambda + (qx \vee s)\} \\ &= \min_{\lambda} \{\exists x \in P \mid px \vee r \leq \lambda + (qx \vee s)\}. \end{aligned}$$

Every problem concerning affine polyhedra has an equivalent “homogeneous” version concerning cones, which is obtained by adding to the system of inequalities defining an affine polyhedron a

new variable whose coefficients are the free terms of this system. Then, the original polyhedron is recovered by setting this new variable to 0. The homogeneous equivalent version of (26) reads:

$$\begin{aligned} & \text{minimize } \lambda \\ & \text{subject to: } uy \leq \lambda + vy, \quad Cy \leq Dy, \quad y_{n+1} \neq -\infty, \quad y \in \mathbb{R}_{\max}^{n+1}, \quad \lambda \in \mathbb{R}_{\max} \end{aligned} \quad (27)$$

where we set $u = [p, r]$, $v = [q, s]$, $C = [A, c]$ and $D = [B, d]$.

We can still reformulate (27) in a more compact way:

$$\min\{\lambda \in \mathbb{R}_{\max} \mid Uy \leq V(\lambda)y, \ y_{n+1} \neq -\infty \text{ is solvable}\},$$

with U and $V(\lambda)$ defined in (24). Finally, by Theorem 5, it follows that $Uy \leq V(\lambda)y$ is solvable with finite y_{n+1} if, and only if, $\chi_{n+1}(U^\sharp V(\lambda)) \geq 0$. \square

Remark 3. Butkovič and Aminu (2008) considered the following special cases of (21):

$$\begin{aligned} & \text{minimize } px \quad (\text{resp. maximize } qx) \\ & \text{subject to: } Ax \vee c \leq Bx \vee d, \quad x \in \mathbb{R}^n \end{aligned} \quad (28)$$

where $p, q \in \mathbb{R}^n$, $c, d \in \mathbb{R}^m$, $r, s \in \mathbb{R}$ and $A, B \in \mathbb{R}^{m \times n}$ have only finite entries. Clearly, (21) becomes (28) if we set $r = -\infty$, $q = -\infty$ and $s = 0$ for minimization, or respectively $s = -\infty$, $p = -\infty$ and $r = 0$ for maximization, where the opposite (tropical inverse) of the minimal value of λ equals the maximum of qx .

In this connection, formulation (27) (or equivalently (21)) has a good geometric insight, meaning optimization for general tropical half-spaces (or hyperplanes) defined by bivectors $(u, \lambda + v)$, see (Gaubert and Katz, 2009; Allamigeon et al., 2011b) for more background.

Example 1. Assume we want to maximize $(1 + x_1) \vee (3 + x_2)$ over the tropical polyhedron of \mathbb{R}_{\max}^2 defined by the system $Ax \vee c \leq Bx \vee d$, where

$$A = \begin{pmatrix} -\infty & -1 \\ -2 & -2 \\ -1 & -\infty \\ 0 & -\infty \end{pmatrix}, \quad c = \begin{pmatrix} -\infty \\ -\infty \\ -\infty \\ -\infty \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -\infty \\ -\infty & -\infty \\ -\infty & 0 \\ -\infty & 2 \end{pmatrix}, \quad d = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This tropical polyhedron is displayed on the left-hand side of Fig. 6 below. This maximization problem is equivalent to minimizing λ subject to $0 \leq \lambda + ((1 + x_1) \vee (3 + x_2))$, $Ax \vee c \leq Bx \vee d$. Indeed, the value of the latter problem is the opposite (tropical inverse) of the value of the maximization problem. The homogeneous version of this minimization problem reads:

$$\begin{aligned} & \text{minimize } \lambda \\ & \text{subject to: } uy \leq \lambda + vy, \quad Cy \leq Dy, \quad y_3 \neq -\infty, \quad y \in \mathbb{R}_{\max}^3, \quad \lambda \in \mathbb{R}_{\max} \end{aligned} \quad (29)$$

where $C = [A, c]$, $D = [B, d]$, $u = (-\infty, -\infty, 0)$, and $v = (1, 3, -\infty)$.

3.2. Partial spectral functions and piecewise linearity

We have shown that solving the tropical linear-fractional programming problem (21) is equivalent to finding the least zero of the spectral function $\phi(\lambda) := \chi_{n+1}(U^\sharp V(\lambda))$. Here we will analyze the graph of this spectral function, after introducing analogues of derivatives, the partial spectral functions.

Given a strategy $\sigma \in S$ for player Max and a strategy $\tau \in T$ for player Min, we respectively define the min-plus linear function $U^\sharp V^\sigma(\lambda)$ and the max-plus linear function $U_\tau^\sharp V(\lambda)$, see (9) and (16). We introduce the *partial spectral functions* $\phi^\sigma(\lambda) := \chi_{n+1}(U^\sharp V^\sigma(\lambda))$ and $\phi_\tau(\lambda) := \chi_{n+1}(U_\tau^\sharp V(\lambda))$. With this notation, (17) yields

$$\phi(\lambda) = \max_{\sigma \in S} \phi^\sigma(\lambda) = \min_{\tau \in T} \phi_\tau(\lambda). \quad (30)$$

Partial spectral functions can be represented as in (6), where one of the strategies is fixed, see also (20):

$$\phi^\sigma(\lambda) = \min_{\tau \in T} \Phi_{U, V(\lambda)}(n+1, \tau, \sigma), \quad \phi_\tau(\lambda) = \max_{\sigma \in S} \Phi_{U, V(\lambda)}(n+1, \tau, \sigma). \quad (31)$$

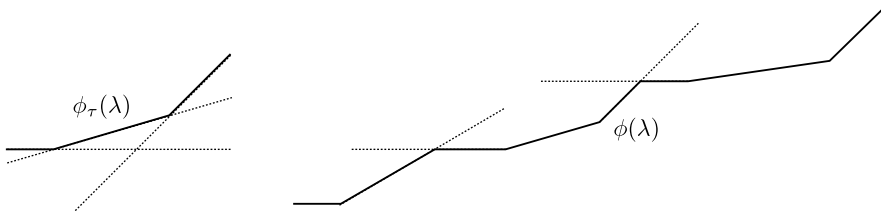


Fig. 2. A partial spectral function (left) and a spectral function (right).

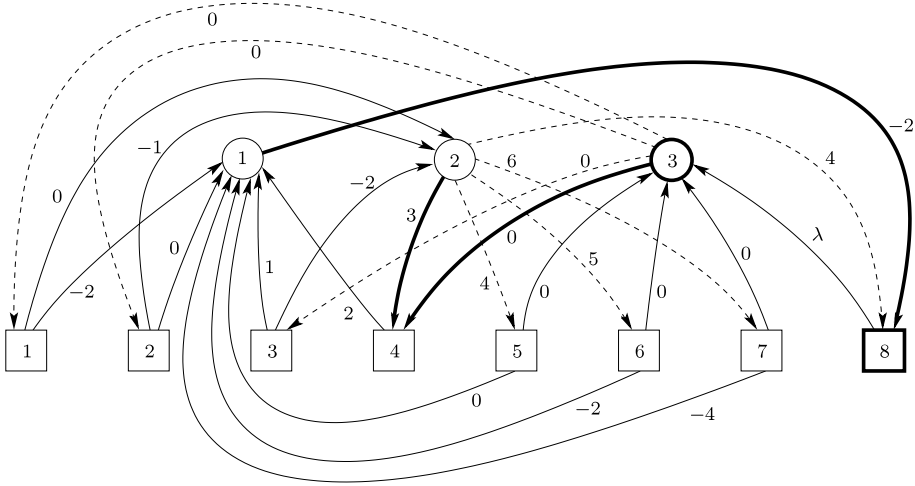


Fig. 3. The bipartite digraph \mathcal{G}_λ of the mean payoff game associated with the tropical linear programming problem of Example 2. Special nodes $n+1$ and $m+1$ are in bold as well as a strategy τ for player Min certifying the optimality of $\lambda^* = 0$. The subdigraph $\mathcal{G}_{\lambda^*}^\tau$ is obtained by deleting the dashed arcs.

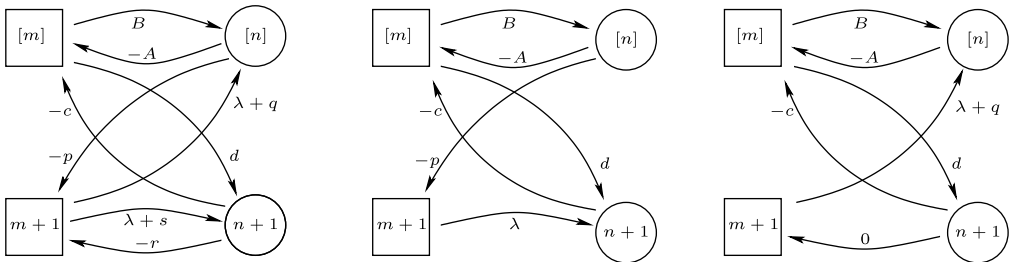


Fig. 4. On the left: the bipartite digraph of the mean payoff game associated with the tropical linear programming problem (26). The nodes of Max are represented by squares and the nodes of Min by circles. In the middle and on the right: the bipartite digraphs of the special cases (28), for the minimization problem and the maximization problem, respectively.

A graphical presentation of (31) and (30) using only ϕ_τ is given in Fig. 2.

Let \mathcal{G}_λ be the bipartite digraph of the mean payoff game whose payments are given by the matrices U and $V(\lambda)$, see Figs. 3 and 4 for illustrations. Observe that in this digraph, only the weight of the arcs connecting node $m+1$ of Max with nodes $l \in [n+1]$ of Min depend on λ . Recall that given a strategy σ for player Max (resp. τ for player Min), $\mathcal{G}_\lambda^\sigma$ (resp. \mathcal{G}_λ^τ) denotes the bipartite subdigraph of \mathcal{G}_λ obtained by deleting from \mathcal{G}_λ all the arcs (i, j) such that $i \in [m+1]$ and $j \neq \sigma(i)$ (resp. the arcs (j, i) such that $j \in [n+1]$ and $i \neq \tau(j)$).

We next investigate the properties of spectral functions.

Theorem 8. Let $\sigma : [m+1] \mapsto [n+1]$ be a positional strategy for player Max and $\tau : [n+1] \mapsto [m+1]$ be a positional strategy for player Min. Then,

- (i) $\phi(\lambda)$, $\phi_\tau(\lambda)$ and $\phi^\sigma(\lambda)$ are 1-Lipschitz nondecreasing piecewise-linear functions, whose linear pieces are of the form $(\alpha + \beta\lambda)/k$, where $k \in [\min(m, n) + 1]$ and $\beta \in \{0, 1\}$.
- (ii) If the absolute values of all the finite coefficients in (27) are bounded by M , then $|\alpha/k| \leq 2M$.
- (iii) $\phi_\tau(\lambda)$ is convex and $\phi^\sigma(\lambda)$ is concave. Both functions consist of no more than $\min(m, n) + 2$ linear pieces.

Proof. As follows from (30) and (31), spectral functions are built from the finite number of functions $\lambda \mapsto \Phi_{U, V(\lambda)}(n+1, \sigma, \tau)$, each of which is given by the mean weight per turn of the only elementary cycle in $\mathcal{G}_{\lambda}^{\sigma, \tau}$ accessible from node $n+1$ of Min. Recall that $\mathcal{G}_{\lambda}^{\sigma, \tau}$ is the subdigraph of \mathcal{G}_{λ} where all arcs at all nodes except for those chosen by player Max (strategy σ) and player Min (strategy τ) are removed. As a function of λ , this mean weight per turn is a line $(\alpha + \beta\lambda)/k$. Here $k \in [\min(m, n) + 1]$ since the length (i.e., the number of nodes of Max it contains) of any elementary cycle in the bipartite digraph \mathcal{G}_{λ} does not exceed both $m+1$ and $n+1$. Also $\beta \in \{0, 1\}$, because an elementary cycle can contain node $m+1$ of Max no more than once. If the absolute values of all the payments in the game are bounded by M , then $|\alpha/k| \leq 2M$ since the arithmetic mean of payments, counted per turn, does not exceed the greatest sum of two consecutive payments.

Thus, the functions $\lambda \mapsto \Phi_{U, V(\lambda)}(n+1, \sigma, \tau)$ satisfy all the properties of (i) and (ii). Using (30) and (31) we conclude that $\phi(\lambda)$, $\phi_\tau(\lambda)$ and $\phi^\sigma(\lambda)$ also satisfy these properties.

Convexity (resp. concavity) of ϕ_τ (resp. ϕ^σ) follows from (31). In a convex or concave piecewise-linear function, each slope can appear only once, while the possible slopes are $0, 1, 1/2, \dots, 1/(\min(m, n) + 1)$. This shows (iii). \square

Some useful facts can be deduced further from this description of spectral functions.

Corollary 9. Spectral functions satisfy the following properties:

- (i) If the absolute values of all coefficients in (27) are either infinite or bounded by M , then $\phi(\lambda)$, $\phi^\sigma(\lambda)$ and $\phi_\tau(\lambda)$ are linear for $\lambda \leq -4M(\min(m, n) + 1)^2$ and for $\lambda \geq 4M(\min(m, n) + 1)^2$.
- (ii) If the absolute values of all coefficients in (27) are either infinite or bounded by M , then the solutions to the problems $\min\{\lambda \mid \phi(\lambda) \geq 0\}$, $\min\{\lambda \mid \phi^\sigma(\lambda) \geq 0\}$ and $\min\{\lambda \mid \phi_\tau(\lambda) \geq 0\}$ lie (if finite) in $[-2M(\min(m, n) + 1), 2M(\min(m, n) + 1)]$. Moreover, if all the finite coefficients are integers, then the solutions to all these problems are integers as well.
- (iii) If the finite coefficients in (27) are integers, then the breaking points of $\phi(\lambda)$, $\phi^\sigma(\lambda)$ or $\phi_\tau(\lambda)$ are rational numbers whose denominators do not exceed $\min(m, n) + 1$.
- (iv) If the finite coefficients in (27) are integers with absolute values bounded by M , then $\phi(\lambda)$ consists of no more than $8M(\min(m, n) + 1)^4 + 2$ linear pieces.

Proof. (i) Consider the intersection point μ of one linear piece $(\alpha_1 + \beta_1\lambda)/k_1$ with another linear piece $(\alpha_2 + \beta_2\lambda)/k_2$. By Theorem 8, $k_1, k_2 \leq \min(m, n) + 1$ and $|\alpha_1/k_1|, |\alpha_2/k_2| \leq 2M$, and we obtain from

$$|\beta_1/k_1 - \beta_2/k_2| \geq \frac{1}{(\min(m, n) + 1)^2}, \quad |\alpha_1/k_1 - \alpha_2/k_2| \leq 4M,$$

that $|\mu| \leq 4M(\min(m, n) + 1)^2$. This means that $\phi(\lambda)$ is linear for $\lambda \geq 4M(\min(m, n) + 1)^2$ and $\lambda \leq -4M(\min(m, n) + 1)^2$. (Note that this part did not impose the integrality of coefficients.)

(ii) Note that due to piecewise-linearity, the solution to each of these problems (if finite) is given by the intersection point of a certain linear piece of the form $(\alpha + \lambda)/k$ with zero. Then, since $|\alpha/k| \leq 2M$ and $k \leq \min(m, n) + 1$ by Theorem 8, we conclude that this intersection point $-\alpha$ lies in $[-2M(\min(m, n) + 1), 2M(\min(m, n) + 1)]$. Moreover, if the finite coefficients in (27) are integers, then this solution $-\alpha$ is also integer.

(iii) By Theorem 8, spectral functions are piecewise linear and the linear pieces are of the form $(\alpha + \beta\lambda)/k$, where in particular $k \in [\min(m, n) + 1]$ and $\beta \in \{0, 1\}$. Considering the intersection point μ of one such piece $(\alpha_1 + \beta_1\lambda)/k_1$ with another piece $(\alpha_2 + \beta_2\lambda)/k_2$ and assuming the integrity of α_1, α_2 we obtain that $\mu = (k_1\alpha_2 - k_2\alpha_1)/(k_2\beta_1 - k_1\beta_2)$ is a rational number with denominator not exceeding $\min(m, n) + 1$.

(iv) The denominators of breaking points do not exceed $\min(m, n) + 1$, and hence the difference between their inverses is not less than $1/(\min(m, n) + 1)^2$. This is a lower bound for the difference between two consecutive breaking points. We get the claim applying part (i). \square

Note that to determine the slope of $\phi(\lambda)$ at $+\infty$, meaning for $\lambda \geq 4M(\min(m, n) + 1)^2$, or at $-\infty$, meaning for $\lambda \leq -4M(\min(m, n) + 1)^2$, we can set all the finite coefficients in (27) to 0. Then, we “play” the mean payoff game at $\lambda = 1$ or at $\lambda = -1$, respectively.

Denote by $\text{MPGI}(m, n, M)$ the worst-case complexity of an oracle computing the value of mean payoff games with integer payments whose absolute values are bounded by M , with m nodes of Max and n nodes of Min. There exist pseudo-polynomial algorithms computing the value of mean payoff games. For instance, in (Zwick and Paterson, 1996) the authors describe a value iteration algorithm with $O(mn^4M)$ complexity. Using this we now show that all the linear pieces of a spectral function can be identified in pseudo-polynomial time. Note that we do not require the oracle to compute optimal strategies here.

Proposition 10. *Let all the finite coefficients in (27) be integer with absolute values not exceeding M . Then, all the linear pieces that constitute the graph of $\phi(\lambda)$ can be identified in*

$$O(M \min(m, n)^4) \times \text{MPGI}(m + 1, n + 1, M(\min(m, n) + 1)(1 + 4(\min(m, n) + 1)^2))$$

operations.

Proof. By Corollary 9 part (iii), the breaking points of $\phi(\lambda)$ are rational numbers whose denominators do not exceed $\min(m, n) + 1$. To identify the linear pieces that constitute the graph of the spectral function, we only need to evaluate $\phi(\lambda)$ on such rational points in the interval $[-4M(\min(m, n) + 1)^2, 4M(\min(m, n) + 1)^2]$, the number of which does not exceed $O(M \min(m, n)^4)$.

Further, when computing $\phi(\lambda)$, the payments in the mean payoff games that the oracle works with are either a or $a + \lambda$, where a is an integer satisfying $|a| \leq M$ and λ is a rational number in $[-4M(\min(m, n) + 1)^2, 4M(\min(m, n) + 1)^2]$ whose denominator does not exceed $\min(m, n) + 1$. The properties of the game will not change if we multiply all the payments by this denominator, obtaining a new game in which the payments are integers with absolute values bounded by $(\min(m, n) + 1)(M + 4M(\min(m, n) + 1)^2)$. Then, the complexity of the mean payoff oracle will not exceed $\text{MPGI}(m + 1, n + 1, (\min(m, n) + 1)(M + 4M(\min(m, n) + 1)^2))$. Multiplying by $O(M \min(m, n)^4)$ we get the claim. \square

Remark 4. It follows that the tropical linear-fractional programming problem (27) can be solved in pseudo-polynomial time by reconstructing all the linear pieces that constitute the graph of $\phi(\lambda)$. However, more efficient methods will be described in Section 3.4.

Remark 5. A similar spectral function has been introduced in (Gaubert and Sergeev, 2010) to compute the set of solutions λ of the two-sided eigenproblem $Ax = \lambda Bx$. The present approach can be extended to a larger class of parametric games, in which the payments are piecewise affine functions of the parameter λ , with integer slopes. See (Sergeev, 2010).

3.3. Strategies as certificates

In the classical simplex method, the optimality of a feasible solution is certified by the sign of Lagrange multipliers. In the tropical case, following the idea of (Allamigeon et al., 2011b), we shall show that the certificate is of a different nature: it is a strategy. We shall also use such strategies to guide the next iteration of the Newton method in Section 3.4, when the current feasible solution is not optimal.

Definition 11 (Left and Right Optimal Strategies). A strategy σ – for player Max (resp. τ – for player Min) is *left optimal* at $\lambda \in \mathbb{R}$, if there exists $\epsilon > 0$ such that

$$\phi(\mu) = \phi^{\sigma-}(\mu) \quad (\text{resp. } \phi(\mu) = \phi_{\tau-}(\mu)) \quad \forall \mu \in [\lambda - \epsilon, \lambda].$$

Right optimal strategies $\sigma+$ and $\tau+$ are defined in a similar way, replacing $[\lambda - \epsilon, \lambda]$ by $[\lambda, \lambda + \epsilon]$.

The existence of left and right optimal strategies at each point follows readily from (30), together with the finiteness of the number of strategies and the piecewise affine character of each function $\phi^{\sigma}(\lambda)$ and $\phi_{\tau}(\lambda)$.

Theorem 12. *The tropical linear-fractional programming problem (27) has the optimal value $\lambda^* \in \mathbb{R}$ if, and only if, $\phi(\lambda^*) \geq 0$ and there exists a strategy τ for player Min such that the digraph $\mathcal{G}_{\lambda^*}^{\tau}$ satisfies the following conditions:*

- (i) all cycles accessible from node $n + 1$ of Min have nonpositive weight,
- (ii) any cycle of zero weight accessible from node $n + 1$ of Min passes through node $m + 1$ of Max.

Moreover, these conditions are always satisfied when τ is left optimal at λ^* .

Proof. The tropical linear-fractional programming problem (27) has the optimal value λ^* if, and only if, $\phi(\lambda^*) = 0$ and $\phi(\lambda) < 0$ for all $\lambda < \lambda^*$. If τ is any left optimal strategy at λ^* , then the previous conditions are satisfied if, and only if, $\phi_{\tau}(\lambda^*) = 0$ and $\phi_{\tau}(\lambda)$ has nonzero left derivative at λ^* .

By (31), or (19) and (20), we know that $\phi_{\tau}(\lambda)$ is the maximal cycle mean (per turn) over all cycles in $\mathcal{G}_{\lambda}^{\tau}$ accessible from node $n + 1$ of Min. It follows that $\phi_{\tau}(\lambda^*) = 0$ if, and only if, all cycles in $\mathcal{G}_{\lambda^*}^{\tau}$ accessible from node $n + 1$ of Min have nonpositive weight and at least one of them has zero weight. Moreover, $\phi_{\tau}(\lambda)$ has nonzero left derivative at λ^* if, and only if, any zero-weight cycle in $\mathcal{G}_{\lambda^*}^{\tau}$ accessible from node $n + 1$ of Min has arcs with weights depending on λ , which can only occur if it passes through node $m + 1$ of Max. Thus, the conditions of the theorem are necessary and they are satisfied by any left optimal strategy τ at λ^* .

Assume now that there exists a strategy τ satisfying the conditions of the theorem. Then, the argument above shows that $\phi_{\tau}(\lambda^*) \leq 0$ and $\phi_{\tau}(\lambda) < 0$ for all $\lambda < \lambda^*$. Since $\phi(\lambda^*) \geq 0$ and by (30) we have $\phi(\lambda) \leq \phi_{\tau}(\lambda)$ for all λ , it follows that $\phi(\lambda^*) = 0$ and $\phi(\lambda) < 0$ for all $\lambda < \lambda^*$. Therefore, λ^* is the optimal value of the tropical linear-fractional programming problem (27). \square

In the same way, we can certify when the tropical linear-fractional programming problem (27) is unbounded.

Theorem 13. *The tropical linear-fractional programming problem (27) is unbounded if, and only if, there exists a strategy σ for player Max such that all cycles in the digraph \mathcal{G}_0^{σ} accessible from node $n + 1$ of Min do not contain node $m + 1$ of Max and have nonnegative weight.*

Proof. We know that the tropical linear-fractional programming problem (27) is unbounded if, and only if, $\phi(\lambda) \geq 0$ for all λ . By the first equality in (30), the latter condition is satisfied if, and only if, there exists a strategy σ for player Max such that $\phi^{\sigma}(\lambda) \geq 0$ for all λ . Note that the weight of a cycle in $\mathcal{G}_{\lambda}^{\sigma}$ that passes through node $m + 1$ of Max can be made arbitrarily small by decreasing λ , because this cycle must contain an arc whose weight depends on λ . Therefore, using the fact that $\phi^{\sigma}(\lambda)$ is the minimal cycle mean (per turn) over all cycles in $\mathcal{G}_{\lambda}^{\sigma}$ accessible from node $n + 1$ of Min (see (31), or (19) and (20)), it follows that $\phi^{\sigma}(\lambda) \geq 0$ for all λ if, and only if, all cycles in \mathcal{G}_0^{σ} accessible from node $n + 1$ of Min have nonnegative weight and do not pass through node $m + 1$ of Max. \square

Remark 6. Theorems 12 and 13 are inspired by Theorem 18 and Corollary 20 of (Allamigeon et al., 2011b), in which similar certificates are given for the problem of checking whether an implication of the form $Ax \leq Bx \implies px \leq qx$ holds. The latter can be cast as a special tropical linear-fractional programming problem.

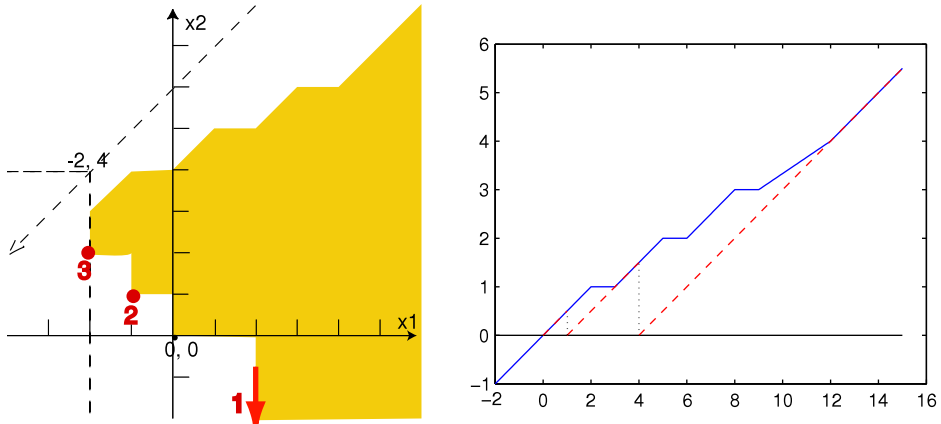


Fig. 5. The minimization problem of Example 2: the tropical polyhedron and the spectral function.

Example 2. Consider the tropical linear programming problem given by the minimization of $(2+x_1) \vee (-4+x_2)$ over the tropical polyhedron of \mathbb{R}_{\max}^2 defined by the system of inequalities $Ax \vee c \leq Bx \vee d$, where

$$A = \begin{pmatrix} -\infty & -\infty \\ -\infty & -\infty \\ -\infty & -\infty \\ -\infty & -3 \\ -\infty & -4 \\ -\infty & -5 \\ -\infty & -6 \end{pmatrix}, \quad c = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\infty \\ -\infty \\ -\infty \end{pmatrix}, \quad B = \begin{pmatrix} -2 & 0 \\ 0 & -1 \\ 1 & -2 \\ 2 & -\infty \\ 0 & -\infty \\ -2 & -\infty \\ -4 & -\infty \end{pmatrix}, \quad d = \begin{pmatrix} -\infty \\ -\infty \\ -\infty \\ -\infty \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This polyhedron is displayed on the left-hand side of Fig. 5. The direction of minimization of $(2+x_1) \vee (-4+x_2)$ is shown there by a dotted line above the polyhedron, together with the optimal tropical hyperplane $(2+x_1) \vee (-4+x_2) = 0$. The bipartite digraph \mathcal{G}_λ corresponding to this problem is depicted in Fig. 3, where the nodes of Max are represented by squares and the nodes of Min by circles. Note that in this case we have $m = 7$ and $n = 2$.

The equivalent homogeneous version of this problem (as described in Section 3.1) is to minimize λ subject to $uy \leq \lambda + vy$, $Cy \leq Dy$, and $y_3 \neq -\infty$, where $C = [A, c]$, $D = [B, d]$, $u = (2, -4, -\infty)$ and $v = (-\infty, -\infty, 0)$.

Thanks to Theorem 12, it is possible to certify that $\lambda^* = 0$ is the optimal value of this problem. To show this, consider the strategy τ for player Min defined by: $\tau(1) = 8$, $\tau(2) = 4$ and $\tau(3) = 4$, which is represented in bold in Fig. 3. Observe that the resulting subdigraph $\mathcal{G}_{\lambda^*}^\tau$ contains only one cycle, which is accessible from node $n+1$ of Min (indeed it passes through this node), has zero weight and passes through node $m+1$ of Max. Moreover, by Theorem 5, we have $\phi(\lambda^*) \geq 0$ because $y = (-2, 2, 0)^T$ satisfies $Cy \leq Dy$ and $uy \leq \lambda^* + vy = vy$. Therefore, by Theorem 12, $\lambda^* = 0$ is the optimal value.

The special cases (28) of the tropical linear-fractional programming problem (27) have been studied in (Butkovič and Aminu, 2008), where necessary and sufficient conditions for these problems to be unbounded were in particular given. We next show that under the assumptions of (Butkovič and Aminu, 2008), which require the entries of all vectors and matrices to be finite, these conditions turn out to be equivalent to the one given in Theorem 13.

Theorem 3.3 of (Butkovič and Aminu, 2008) shows that, when only finite entries are considered, the minimization problem in (28) is unbounded if, and only if, $c \leq d$. Under the finiteness assumption, this condition is equivalent to the one given in Theorem 13. To show this, in the first place observe that in this case the associated digraph \mathcal{G}_λ (see Fig. 4) contains arcs connecting any node of Min $[n+1]$ with any the node of Max $[m+1]$, with exception of the arc connecting node $n+1$ with node $m+1$, and arcs

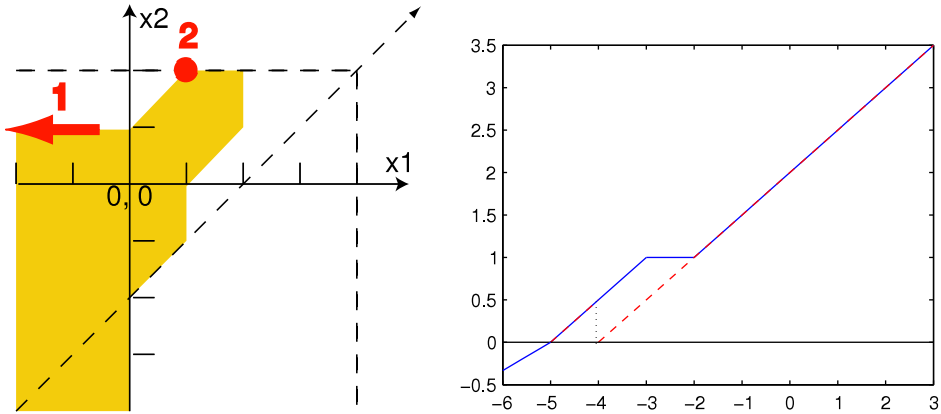


Fig. 6. The maximization problem of Example 1: the tropical polyhedron and the spectral function.

connecting any node of $\text{Max}[m+1]$ with any node of $\text{Min}[n+1]$, with exception of the arcs connecting node $m+1$ with nodes in $[n]$. Thus, if we define the strategy σ for player Max by $\sigma(i) = n+1$ for all $i \in [m+1]$, it can be checked that the only cycles in \mathcal{G}_0^σ accessible from node $n+1$ are of the form $n+1 \rightarrow i_1 \rightarrow n+1 \rightarrow \dots \rightarrow n+1 \rightarrow i_k \rightarrow n+1$ for some $i_1, \dots, i_k \in [m]$. Since the weight of such a cycle is $d_{i_1} - c_{i_1} + \dots + d_{i_k} - c_{i_k}$, the strategy σ satisfies the conditions in Theorem 13 if $c \leq d$. Conversely, assume that a strategy σ for player Max satisfies the conditions in Theorem 13. Then, the only possible value for $\sigma(m+1)$ is $n+1$, and we must also have $\sigma(i) = n+1$ for all $i \in [m]$, because if $\sigma(i) = j \neq n+1$ for some $i \in [m]$, \mathcal{G}_0^σ would contain the cycle $m+1 \rightarrow n+1 \rightarrow i \rightarrow j \rightarrow m+1$, contradicting the fact that no cycle accessible from node $n+1$ of Min passes through node $m+1$ of Max. Now, since \mathcal{G}_0^σ contains the cycles $n+1 \rightarrow i \rightarrow n+1$ for $i \in [m]$, which are accessible from node $n+1$ of Min, the weights of these cycles $d_i - c_i$ must be nonnegative, implying that $c \leq d$.

Regarding the maximization problem in (28), Theorem 3.4 of (Butkovič and Aminu, 2008) shows that this problem is unbounded if, and only if, the system $Ax \leq Bx$ has a finite solution. In this case, due to the finiteness assumption, it follows that the associated digraph \mathcal{G}_λ (see Fig. 4) contains arcs connecting any node of $\text{Max}[m+1]$ with any node of $\text{Min}[n+1]$, with exception of the arc connecting node $m+1$ with node $n+1$, and arcs connecting any node of $\text{Min}[n+1]$ with any node of $\text{Max}[m+1]$, with exception of the arcs connecting nodes in $[n]$ with node $m+1$. If the system $Ax \leq Bx$ has a finite solution, from Theorem 5 and (17) it follows that there exists a strategy $\bar{\sigma} : [m] \mapsto [n]$ such that $\chi(A^\# B^{\bar{\sigma}}) = \chi(A^\# B) \geq 0$. By (19) and (20), this implies that any cycle in $\bar{\mathcal{G}}^{\bar{\sigma}}$ has nonnegative weight, where $\bar{\mathcal{G}}$ is the bipartite digraph of the mean payoff game associated with the matrices A and B . If we define the strategy $\sigma(i) = \bar{\sigma}(i)$ for all $i \in [m]$ and $\sigma(m+1) = j$ for some $j \in [n]$, then σ satisfies the conditions of Theorem 13 because the cycles accessible from node $n+1$ of Min in \mathcal{G}_0^σ are precisely the cycles in $\bar{\mathcal{G}}^{\bar{\sigma}}$ and there is no cycle containing node $m+1$ of Max in \mathcal{G}_0^σ . Conversely, if a strategy σ for player Max satisfies the conditions in Theorem 13, then necessarily we have $\sigma(i) \in [n]$ for all $i \in [m]$, because if $\sigma(i) = n+1$ for some $i \in [m]$, \mathcal{G}_0^σ would contain the cycle $n+1 \rightarrow m+1 \rightarrow j \rightarrow i \rightarrow n+1$ where $j = \sigma(m+1) \in [n]$, contradicting the fact that there is no cycle in \mathcal{G}_0^σ accessible from node $n+1$ of Min passing through node $m+1$ of Max. Now, if we define $\bar{\sigma}(i) = \sigma(i)$ for all $i \in [m]$, the cycles accessible from node $n+1$ of Min in \mathcal{G}_0^σ are precisely the cycles in $\bar{\mathcal{G}}^{\bar{\sigma}}$, which therefore have nonnegative weight. Then, by (19) and (20) we have $\chi(A^\# B^{\bar{\sigma}}) \geq 0$, and so from Theorem 5 and (17) we conclude that the system $Ax \leq Bx$ has a finite solution.

Remark 7. If the strategies σ or τ and the scalar λ^* are fixed (considered as inputs) the conditions of Theorems 12 and 13, i.e., the validity of the certificates, can be checked in polynomial time.

To see this, in the first place assume that τ and λ^* are given. Using Karp's algorithm, compute the maximal cycle mean of each strongly connected component of $\mathcal{G}_{\lambda^*}^\tau$ that is accessible from node $n+1$ of Min. The certificate is valid only if these maximal cycle means are nonpositive and one of them is zero. To check the second condition of Theorem 12, delete node $m+1$ of Max (and the arcs adjacent to

it) from $\mathcal{G}_{\lambda^*}^\tau$ and compute for the resulting digraph (using again Karp's algorithm) the maximal cycle mean of each strongly connected component accessible from node $n + 1$ of Min. To be valid, all these maximal cycle means must be negative. Observe that in [Theorem 12](#) we also assume that $\phi(\lambda^*) \geq 0$. By (30), this can be certified by a strategy σ for player Max such that the minimal cycle mean of any strongly connected component of $\mathcal{G}_{\lambda^*}^\sigma$ accessible from node $n + 1$ of Min is nonnegative, which can be checked by applying Karp's algorithm to each of these components. By [Theorem 5](#), another possibility is to exhibit a vector y such that $Cy \leq Dy$, $uy \leq \lambda^* + vy$ and $y_{n+1} \neq -\infty$.

Assume now that σ is given. To check the validity of the certificate in [Theorem 13](#), decompose first \mathcal{G}_0^σ in strongly connected components and see whether the component containing node $m + 1$ of Max is trivial (i.e., contains just this node) or it is not accessible from node $n + 1$ of Min. If this is the case, compute the minimal cycle mean of each strongly connected component of \mathcal{G}_0^σ accessible from node $n + 1$ of Min by applying Karp's algorithm. Then, the certificate is valid if each of these minimal cycle means is nonnegative.

3.4. Bisection and Newton methods for tropical linear-fractional programming

In (25), we need to find the least λ such that $\phi(\lambda) \geq 0$, where $\phi(\lambda)$ is nondecreasing and Lipschitz continuous. Thus, we can consider certain classical methods for finding zeros of “good enough” functions of one variable. In particular, the bisection method for $\phi(\lambda)$ corresponds to the approach of ([Butkovič and Aminu, 2008](#)). More specifically, it can be formulated as follows, when the finite coefficients in (27) are integers.

Algorithm 1. Bisection method

Start. A point $\bar{\lambda}_0$ such that $\phi(\bar{\lambda}_0) \geq 0$ and a point $\underline{\lambda}_0$ such that $\phi(\underline{\lambda}_0) < 0$.

Iteration k . Let $\lambda = \lceil (\bar{\lambda}_{k-1} + \underline{\lambda}_{k-1})/2 \rceil$. If $\phi(\lambda) \geq 0$, then set $\bar{\lambda}_k = \lambda$ and $\underline{\lambda}_k = \underline{\lambda}_{k-1}$. Otherwise, set $\bar{\lambda}_k = \bar{\lambda}_{k-1}$ and $\underline{\lambda}_k = \lambda$.

Stop. Verify $\bar{\lambda}_k - \underline{\lambda}_k = 1$. If true, return $\bar{\lambda}_k$.

For this method, which uses that tropical linear-fractional programming preserves integrity ([Corollary 9](#) part (iv)), it is not important to know the actual value of $\phi(\lambda)$, but just whether $\phi(\lambda) \geq 0$, i.e., whether $Uy \leq V(\lambda)y$ is solvable with $y_{n+1} \neq -\infty$.

Further, the concept of (left, right) optimal strategy, see [Definition 11](#), yields an analogue of (left, right) derivative, and leads to the following analogue of the Newton method, which does not have any integer restriction.

Algorithm 2. Positive Newton method

Start. A point λ_0 such that $\phi(\lambda_0) \geq 0$.

Iteration k . Find a left optimal strategy σ for player Max at λ_{k-1} and compute $\lambda_k = \min\{\lambda \in \mathbb{R}_{\max} \mid \phi^\sigma(\lambda) \geq 0\}$.

Stop. Verify $\lambda_k = \lambda_{k-1}$ or $\lambda_k = -\infty$. If true, return λ_k .

It remains to explain how each step of this algorithm can be implemented. We shall see that λ_k can be easily computed (reduction to a shortest path problem) and that finding left optimal strategies can be done by existing algorithms for mean payoff games.

For the sake of comparison, we state a dual version of [Algorithm 2](#).

Algorithm 3. Negative Newton method

Start. A point λ_0 such that $\phi(\lambda_0) < 0$.

Iteration k . Find a (right) optimal strategy τ for player Min at λ_{k-1} and compute $\lambda_k = \min\{\lambda \in \mathbb{R}_{\max} \mid \phi_\tau(\lambda) \geq 0\}$.

Stop. Verify $\phi(\lambda_k) = 0$ or $\lambda_k = +\infty$. If true, return λ_k .

Remark 8. Note that in [Algorithm 3](#) we can use optimal strategies instead of right optimal ones, because if τ is optimal at λ_{k-1} , we have $\phi_\tau(\lambda_{k-1}) = \phi(\lambda_{k-1}) < 0$ and so $\lambda_k > \lambda_{k-1}$ by the definition of λ_k (recall that by [Theorem 8](#) all the spectral functions are nondecreasing and piecewise-linear). This means that all the strategies considered in the iterations of [Algorithm 3](#) are different, and as the

number of strategies is finite, this algorithm must terminate in a finite number of steps. A similar argument shows that we can also use optimal strategies in [Algorithm 2](#) at points λ_{k-1} where the spectral function ϕ is strictly positive, because in that case we have $\lambda_k < \lambda_{k-1}$ even if σ is just optimal and not left optimal (however, when $\phi(\lambda_{k-1}) = 0$ and λ_{k-1} is not optimal, only a left optimal strategy at λ_{k-1} guarantees $\lambda_k < \lambda_{k-1}$).

Remark 9. Due to [Corollary 9](#) part (ii), the values $\lambda^+ := 2M(\min(m, n) + 1)$ and $\lambda^- := -2M(\min(m, n) + 1)$ can be first checked in the case of the positive and negative Newton methods, respectively. We recall that M is a bound on the absolute value of the coefficients in [\(27\)](#).

If $\phi(\lambda^+) < 0$ then the problem is infeasible, and if $\phi(\lambda^-) > 0$ then the problem is unbounded. If $\phi(\lambda^+) \geq 0$ and $\phi(\lambda^-) < 0$, then the problem is both feasible and bounded. The case $\phi(\lambda^-) = 0$ requires a left optimal strategy for player Max at λ^- to decide that either this point is optimal, or the problem is unbounded.

This rule of starting with $\pm 2M(\min(m, n) + 1)$, as we shall see, secures pseudo-polynomiality of the instances of the mean payoff games generated by the bisection and Newton methods.

The following logarithmic bound on the complexity of the bisection method is standard and its proof will be omitted.

Proposition 14. *If the finite coefficients in [\(27\)](#) are integers with absolute values bounded by M , then the number of iterations of the bisection method does not exceed $\log(4M(\min(m, n) + 1))$ if it is started as in [Remark 9](#). Hence, the computational complexity of the bisection method in this case does not exceed*

$$\log(4M(\min(m, n) + 1)) \times \text{MPGI}(m + 1, n + 1, M + 2M(\min(m, n) + 1)).$$

Remark 10. [Butkovič and Aminu \(2008\)](#) give better initial values $\bar{\lambda}_0$ and $\underline{\lambda}_0$ for the bisection method than $\pm 2M(\min(m, n) + 1)$, but only for the special cases [\(28\)](#), where all the coefficients are assumed to be finite. These initial values depend on the input data and lie in the interval $[-3M, 3M]$. As oracle, they exploit the alternating method of [Cuninghame-Green and Butkovič \(2003\)](#), which requires $O(mn(m+n)M)$ operations, being related to the value iteration of [Zwick and Paterson \(1996\)](#). Hence, in this case, the complexity of the bisection method is no more than $O(mn(m+n)M \log M)$. In [Sergeev \(2010\)](#), the same kind of initial values were obtained for the general formulation [\(21\)](#), leading to a similar complexity, but with the same finiteness restriction on the coefficients. The initial values of [Butkovič and Aminu \(2008\)](#) and [Sergeev \(2010\)](#) will be exploited in the numerical experiments, see [Section 4.3](#).

As observed above, in the case of the bisection method the mean payoff oracle is only required to check whether $\phi(\lambda) \geq 0$.

In the case when the finite coefficients in [\(27\)](#) are real, the bisection method computes $(\bar{\lambda}_{k-1} + \underline{\lambda}_{k-1})/2$ without rounding and it yields only an approximate solution to the problem. However, Newton methods always converge in a finite number of steps.

Proposition 15. *Denote by $|S|$ and $|T|$ the number of available strategies for players Max and Min, respectively. Then,*

- (i) *[Algorithms 2](#) and [3](#) terminate in a finite number of steps, the number of which does not exceed $|S|$ and $|T|$, respectively.*
- (ii) *If the finite coefficients in [\(27\)](#) are integers with absolute values bounded by M , then the values λ_k produced by [Algorithm 2](#) are also integer, and the number of iterations does not exceed $4M(\min(m, n) + 1)$ if it is started as in [Remark 9](#).*

Proof. (i) At different iterations of [Algorithm 2](#) we have different strategies, because $\lambda_k = \min\{\lambda \in \mathbb{R}_{\max} \mid \phi^\sigma(\lambda) \geq 0\}$ are different for all k . Similarly for [Algorithm 3](#), with τ instead of σ . Thus, the number of steps is limited by the number of strategies, which is finite.

(ii) The numbers λ_k generated by [Algorithm 2](#) are integers due to [Corollary 9](#) part (ii). By [Remark 9](#), we can start the algorithm at $2M(\min(m, n) + 1)$, and it will finish before it reaches $-2M(\min(m, n) + 1)$. \square

A worst-case complexity bound, different from the number of strategies, will be given below in [Theorem 19](#), and it is worse than that of the bisection method above (see also [Remark 10](#)). First, Newton iterations require more sophisticated oracles which compute the value of the game and a left optimal strategy. Second, we have only used the integrality of the method in [Proposition 15](#), so the bound on the number of iterations is rough. However, the positive Newton method is an interesting alternative to the bisection method, since it preserves feasibility. Therefore, it may be more sensitive to the geometry of the feasible set, which is especially convenient if this set has only few generators or its dimension is small. The experiments of [Section 4.3](#) indicate that this is indeed the case, and the worst-case complexity bound of [Theorem 19](#) (using [Proposition 15](#)) is often too pessimistic. The main reason to give the result of [Theorem 19](#) is that it shows the method is pseudo-polynomial.

Not aiming to obtain a better overall worst-case complexity result, in the next subsection we will rather consider the implementation of the positive Newton method, reducing the computation of λ_k to a (polynomial-time solvable) shortest path problem. [Section 3.6](#) will be devoted to the computation of left optimal strategies in the integer case by means of perturbed mean payoff games. As noticed in [Proposition 15](#), Newton iterations should work also in the case of real coefficients. For this we propose the algebraic approach of [Section 3.7](#), encoding a perturbed game as a game over the semiring of germs.

3.5. Newton iterations by means of Kleene star

In this subsection we show that in the case of [Algorithm 2](#) the steps of the Newton method can be performed by calculating least solutions of inequalities of the form $z \geq Ez \vee h$, as in [Proposition 6](#).

Assume that we are at iteration k of [Algorithm 2](#), so that we need to compute $\lambda_k = \min\{\lambda \in \mathbb{R}_{\max} \mid \phi^\sigma(\lambda) \geq 0\}$, where σ is a left optimal strategy for player Max at λ_{k-1} . If we set $V^\sigma(\lambda)$ instead of $V(\lambda)$ in $Uy \leq V(\lambda)y$, by [Theorem 5](#) the minimal zero λ_k of $\phi^\sigma(\lambda)$ is exactly the least value of λ for which this system is satisfied by some $y \in \mathbb{R}_{\max}^{n+1}$ with $y_{n+1} \neq -\infty$, i.e., we have

$$\lambda_k = \min\{\lambda \in \mathbb{R}_{\max} \mid Uy \leq V^\sigma(\lambda)y, y_{n+1} \neq -\infty \text{ is solvable}\}. \quad (32)$$

The main idea is to compute this minimal zero by considering the system $Uy \leq V^\sigma(\lambda)y$ directly. With this aim, we shall need the following observation.

Lemma 16. Assume that at iteration k of [Algorithm 2](#) we have $l := \sigma(m+1) \neq n+1$, where σ is a left optimal strategy for player Max at λ_{k-1} . Then, if

$$(Uy \leq V^\sigma(\lambda)y \text{ and } y_{n+1} \neq -\infty) \implies y_l \neq -\infty, \quad (33)$$

for all λ , it follows that

$$\lambda_k = \min\{\lambda \in \mathbb{R}_{\max} \mid Uy \leq V^\sigma(\lambda)y, y_l \neq -\infty \text{ is solvable}\}. \quad (34)$$

Otherwise, i.e., if [Condition \(33\)](#) does not hold, $\lambda_k = -\infty$.

Proof. [Condition \(33\)](#) implies that the minimum in [\(34\)](#) is less than or equal to that in [\(32\)](#).

To show the converse, suppose that for some $\lambda \in \mathbb{R}$ there exists $\hat{y} \in \mathbb{R}_{\max}^{n+1}$ such that $\hat{y}_l \neq -\infty$ and $U\hat{y} \leq V^\sigma(\lambda)\hat{y}$. Since $\phi^\sigma(\lambda_{k-1}) = \phi(\lambda_{k-1}) \geq 0$, by [Theorem 5](#) there exists a solution \tilde{y} of $Uy \leq V^\sigma(\lambda_{k-1})y$ (and so, in particular, of the first m inequalities of this system, i.e., $Cy \leq D^\sigma y$) such that $\tilde{y}_{n+1} \neq -\infty$. Then, for any $\beta \in \mathbb{R}$ the combination $\bar{y} = \hat{y} \vee \tilde{y} - \beta$ satisfies the first m inequalities in $Uy \leq V^\sigma(\lambda)y$ (in other words, we have $C\bar{y} \leq D^\sigma \bar{y}$) as a tropical linear combination of solutions of this system of tropically linear inequalities. Moreover, if β is sufficiently large, \bar{y} also satisfies the last inequality $uy \leq \lambda + v^\sigma y$ of the system $Uy \leq V^\sigma(\lambda)y$ because $u\hat{y} \leq \lambda + v^\sigma \hat{y}$ and $v^\sigma \hat{y} = v_l + \hat{y}_l > -\infty$. But then \bar{y} satisfies $U\bar{y} \leq V^\sigma(\lambda)\bar{y}$ and $\bar{y}_{n+1} \geq \tilde{y}_{n+1} - \beta \neq -\infty$. This shows that the minimum in [\(32\)](#) is less than or equal to that in [\(34\)](#).

Finally, if [Condition \(33\)](#) does not hold, for some $\bar{\lambda}$ there exists a solution \bar{y} of the system $Uy \leq V^\sigma(\bar{\lambda})y$ such that $\bar{y}_{n+1} \neq -\infty$ but $\bar{y}_l = -\infty$. Since $u\bar{y} \leq \bar{\lambda} + v^\sigma \bar{y} = \bar{\lambda} + v_l + \bar{y}_l$, this can only happen if $u\bar{y} = -\infty$, which implies that \bar{y} satisfies $U\bar{y} \leq V^\sigma(\lambda)\bar{y}$ for any $\lambda \in \mathbb{R}$, and so $\lambda_k = -\infty$. \square

We next show how to make sure that [Condition \(33\)](#) is satisfied. Note that this condition is not satisfied if, and only if, for some λ the system $Uy \leq V^\sigma(\lambda)y$ has a solution \bar{y} with $\bar{y}_{n+1} \neq -\infty$

but $\bar{y}_l = -\infty$. The latter implies the existence of a solution \bar{y} of $Cy \leq Dy$ such that $\bar{y}_i = -\infty$ for all $i \in \text{supp}(u)$, but $\bar{y}_{n+1} \neq -\infty$ (so in particular Condition (33) is satisfied if $n+1 \in \text{supp}(u)$). Eliminating from the system $Cy \leq Dy$ the columns corresponding to the indices in $\text{supp}(u)$, the existence of such a solution is reduced to the solvability of a two-sided homogeneous system with the condition $y_{n+1} \neq -\infty$, which can be decided using a mean payoff game oracle. If this problem has no solution, then Condition (33) is satisfied. Otherwise, the value of the original tropical linear-fractional programming problem is $-\infty$.

As a consequence of the previous discussion, in what follows we assume that it has already been checked that Condition (33) is satisfied, and we explain how to perform Newton iterations in that case.

Suppose that we are at iteration k of Algorithm 2, and let σ be a left optimal strategy at λ_{k-1} . Then, if we set $l := \sigma(m+1)$, by Lemma 16 we have

$$\lambda_k = \min\{\lambda \in \mathbb{R}_{\max} \mid Uy \leq V^\sigma(\lambda)y, y_l \neq -\infty \text{ is solvable}\}.$$

Since the system $Uy \leq V^\sigma(\lambda)y$ is satisfied by some \bar{y} with $\bar{y}_l \neq -\infty$ if, and only if, it is satisfied by some \hat{y} with $\hat{y}_l = 0$ (it is enough to define $\hat{y}_i = \bar{y}_i - \bar{y}_l$ for all $i \in [n+1]$), it follows that

$$\begin{aligned} \lambda_k &= \min\{\lambda \in \mathbb{R}_{\max} \mid Uy \leq V^\sigma(\lambda)y, y_l = 0 \text{ is solvable}\} \\ &= \min\{\lambda \in \mathbb{R}_{\max} \mid Cy \leq D^\sigma y, uy \leq \lambda + v_l + y_l, y_l = 0 \text{ is solvable}\}. \end{aligned}$$

Thus, setting $y_l = 0$ we are in the situation of problem (28), because $\lambda_k + v_l$ is given by:

$$\begin{aligned} &\text{minimize} \quad px \vee r \\ &\text{subject to:} \quad Ax \vee c \leq B^\sigma x \vee d^\sigma, x \in \mathbb{R}_{\max}^n \end{aligned} \quad (35)$$

where $p \in \mathbb{R}_{\max}^n$, $r \in \mathbb{R}_{\max}$, $c, d^\sigma \in \mathbb{R}_{\max}^m$ and $A, B^\sigma \in \mathbb{R}_{\max}^{m \times n}$ are such that

$$U = \begin{pmatrix} A & c \\ p & r \end{pmatrix} \quad \text{and} \quad V^\sigma(\lambda) = \begin{pmatrix} B^\sigma & d^\sigma \\ -\infty & \lambda + v_l \end{pmatrix}.$$

Here, just for the simplicity of the presentation, column l is in the place of column $n+1$ when $l \neq n+1$ (in other words, the columns of A and B^σ are respectively the columns of C and D^σ with exception of column l , c is column l of C , d^σ is column l of D^σ , $r = u_l$ and $p_i = u_i$ for $i \neq l$).

We claim that the system of constraints (the second line) in (35) has a least solution, which then minimizes $px \vee r$, and we explain how to find it. First note that the constraints in (35) can be written as:

$$\begin{aligned} (Ax)_i \vee c_i &\leq b_{i\sigma(i)} + x_{\sigma(i)}, \quad \text{if } \sigma(i) \neq l, \\ (Ax)_i \vee c_i &\leq d_i, \quad \text{if } \sigma(i) = l. \end{aligned} \quad (36)$$

In order to find the least solution of this system, observe that the second subsystem can be dispensed with. Indeed, since $\phi^\sigma(\lambda_{k-1}) = \phi(\lambda_{k-1}) \geq 0$, by Theorem 5 there exists a solution \tilde{y} of $Uy \leq V^\sigma(\lambda_{k-1})y$ such that $\tilde{y}_{n+1} \neq -\infty$, and so this solution also satisfies $\tilde{y}_l \neq -\infty$ (recall we assume that Condition (33) holds). Then, if $\tilde{x} \in \mathbb{R}_{\max}^n$ is the vector defined by $\tilde{x}_i := \tilde{y}_i - \tilde{y}_l$ for $i \neq l$, it follows that \tilde{x} is a solution of (36). Hence, if the first subsystem has the least solution \underline{x} , we have $\underline{x} \leq \tilde{x}$ and so \underline{x} is also a solution of the second subsystem.

To show that the first subsystem in (36) has a least solution, first note that a system of two inequalities of the form

$$\begin{aligned} r_{11} + x_1 \vee \cdots \vee r_{1n} + x_n &\leq x_1 \\ r_{21} + x_1 \vee \cdots \vee r_{2n} + x_n &\leq x_1 \end{aligned}$$

is equivalent to just one inequality:

$$s_1 + x_1 \vee \cdots \vee s_n + x_n \leq x_1,$$

where $s_i = r_{1i} \vee r_{2i}$ for $i = 1, \dots, n$. Using this kind of reduction, the first subsystem can be transformed in no more than $m(n+1)$ operations to an equivalent system of the form

$$Ex_l \vee Fx_j \vee h \leq x_l, \quad (37)$$

where x_i is the sub-vector whose coordinates appear on the right-hand side of the first subsystem in (36), and x_j is the sub-vector corresponding to the rest of the coordinates which are present in that system. Since we are interested in the least solution, we can set $x_j \equiv -\infty$, and then the remaining system is just of the form

$$Ez \vee h \leq z, \quad (38)$$

where $z = x_i$. By Proposition 6, the least solution to this system in $\mathbb{R}^{|I|}$ is given by

$$\underline{z} = E^*h = h \vee Eh \vee E^2h \vee E^3h \vee \dots$$

As \tilde{x}_i satisfies (38), we have $\underline{z} \leq \tilde{x}_i$ and so $\underline{z} \in \mathbb{R}_{\max}^{|I|}$.

Thus, we have the following method:

Algorithm 4. Solving (35)

Step 1. Split the system $Ax \vee c \leq B^\sigma x \vee d^\sigma$ in two subsystems as in (36) and transform the first subsystem to the form (37).

Step 2. Compute $\underline{z} = E^*h$. Set $x_i = \underline{z}$ and $x_j \equiv -\infty$.

Step 3. Return $p\tilde{x} \vee r$.

We also conclude the following.

Proposition 17. The problems $\min\{\lambda \mid \phi^\sigma(\lambda) \geq 0\}$ can be solved in $O(mn) + O(n^3)$ time.

Note that in general, a system of the form (36) is solvable in \mathbb{R}_{\max}^n if, and only if, the least solution of the first subsystem belongs to \mathbb{R}_{\max}^n and satisfies the second subsystem.

Example 3. Consider the following tropical linear-fractional programming problem:

minimize λ

subject to: $uy \leq \lambda + vy, Cy \leq Dy, y_4 \neq -\infty, y \in \mathbb{R}_{\max}^4, \lambda \in \mathbb{R}_{\max}$

where

$$C = \begin{pmatrix} -3 & -4 & -\infty & -\infty \\ -1 & -\infty & -\infty & 1 \\ -\infty & -\infty & -\infty & 0 \\ 1 & -\infty & 0 & -\infty \end{pmatrix}, \quad D = \begin{pmatrix} -\infty & -\infty & -\infty & 0 \\ -\infty & 0 & -\infty & -\infty \\ 0 & -\infty & -\infty & -\infty \\ 0 & -\infty & -\infty & 3 \end{pmatrix},$$

$u = (-\infty, 0, -\infty, -\infty)$ and $v = (3, -\infty, -\infty, -\infty)$, so in this case we have $m = 4$ and $n = 3$ with the notation of Problem (27).

Before performing Newton iterations, we need to check whether the system $Cy \leq Dy$ has solutions with $y_2 = -\infty$ but $y_4 \neq -\infty$ (since $\text{supp}(u) = \{2\}$). By the second inequality of this system, it follows that this is impossible. Hence, Condition (33) is satisfied and so we can use (34) in order to compute λ_k . Moreover, in this example the Newton method requires no more than two iterations, since player Max has only two strategies, which correspond to the two finite entries in the last row of D .

Assume that we start with $\lambda_0 = 0$. Then, an optimal strategy σ for player Max at λ_0 is given by: $\sigma(1) = 4, \sigma(2) = 2, \sigma(3) = 1, \sigma(4) = 4$ and $\sigma(5) = 1$. Since $l = \sigma(m+1) = \sigma(5) = 1$, we have

$$\lambda_1 = \min\{\lambda \in \mathbb{R}_{\max} \mid Cy \leq D^\sigma y, y_2 = uy \leq \lambda + v_1 + y_1, y_1 = 0 \text{ is solvable}\},$$

and so, setting $y_1 = 0, \lambda_1 + v_1 = \lambda_1 + 3$ is given by:

$$\begin{aligned} &\text{minimize } y_2 \\ &\text{subject to: } (y_2 - 4) \vee (-3) \leq y_4, (y_4 + 1) \vee (-1) \leq y_2, y_4 \leq 0, y_3 \vee 1 \leq y_4 + 3. \end{aligned} \quad (39)$$

Note that the system of constraints in (39), obtained by setting $y_1 = 0$ in $Cy \leq D^\sigma y$ (i.e., in (36) the first column plays the role of free term), reduces to

$$E \begin{pmatrix} y_2 \\ y_4 \end{pmatrix} \vee Fy_3 \vee h \leq \begin{pmatrix} y_2 \\ y_4 \end{pmatrix}, \quad y_4 \leq 0,$$

where

$$E = \begin{pmatrix} -\infty & 1 \\ -4 & -\infty \end{pmatrix}, \quad F = \begin{pmatrix} -\infty \\ -3 \end{pmatrix}, \quad h = \begin{pmatrix} -1 \\ -2 \end{pmatrix}.$$

Since

$$E^*h = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix},$$

the least solution of the system of constraints in (39) is $(y_2, y_3, y_4)^T = (-1, -\infty, -2)$. Therefore, the value of problem (39) is -1 , and thus $\lambda_1 = -4$. It can be checked that this is the optimal solution of the tropical programming problem (and in particular, that σ is still a left optimal strategy for player Max at $\lambda_1 = -4$).

3.6. Computing left optimal strategies

In order to compute left optimal strategies, consider the mean payoff game associated with the tropical linear-fractional programming problem (27), and let the weights $\lambda + v$ of the arcs connecting node $m + 1$ of Max with nodes of Min be replaced by $\lambda - \epsilon + v$, where $\epsilon \in \mathbb{R}$. In this way, we obtain a *perturbed mean payoff game*. For small enough $\epsilon > 0$, the optimal strategies for this game are the left optimal strategies required by the positive Newton iterations. Here, we will require the mean payoff oracle to find optimal strategies, not just the value of the game. The complexity of such oracle will be denoted by $\text{MPGI}^*(m, n, M)$, for mean payoff games with integer payments whose absolute values are bounded by M , with m nodes of Max and n nodes of Min. A pseudo-polynomial algorithm for computing optimal strategies of such games is described in (Zwick and Paterson, 1996).

Proposition 18. *If the finite coefficients in (27) are integers with absolute values bounded by M , then at each iteration of the positive Newton method, started and finished as in Remark 9, a left optimal strategy can be found in*

$$\text{MPGI}^*(m + 1, n + 1, (\min(m, n) + 2)(M + 2M(\min(m, n) + 1)) + 1)$$

operations.

Proof. By Corollary 9, it follows that each λ_k is integer and the breaking points of $\phi(\lambda)$ are rational numbers whose denominators do not exceed $\min(m, n) + 1$. Therefore, optimal strategies for the game at $\lambda^* = \lambda_k - 1/(\min(m, n) + 2)$ are left optimal strategies at λ_k . Then, we only need to apply a mean payoff oracle in order to compute optimal strategies for the perturbed mean payoff game with $\epsilon = 1/(\min(m, n) + 2)$. Multiplying (in the usual sense) the payments by $\min(m, n) + 2$ we obtain a mean payoff game with integer payments and with the same optimal strategies (in this sense, equivalent to the perturbed game). The computation of optimal strategies in the latter game takes no more than $\text{MPGI}^*(m + 1, n + 1, (\min(m, n) + 2)(M + 2M(\min(m, n) + 1)) + 1)$ operations, because the payments in the perturbed mean payoff game are either of the form a or $a + \lambda_k - \epsilon$, where $|a| \leq M$ and $|\lambda_k| \leq 2M(\min(m, n) + 1)$, and these payments are multiplied by $\min(m, n) + 2$. \square

Example 4. In order to find a left optimal strategy σ for player Max at $\lambda_{k-1} = 0$ in Example 3, we only need to compute an optimal strategy for the associated game at $\lambda^* = \lambda_{k-1} - 1/(\min(m, n) + 2) = -1/5$. This can be done by solving the game whose payments are given by the matrices

$$\begin{pmatrix} -15 & -20 & -\infty & -\infty \\ -5 & -\infty & -\infty & 5 \\ -\infty & -\infty & -\infty & 0 \\ 5 & -\infty & 0 & -\infty \\ -\infty & 0 & -\infty & -\infty \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -\infty & -\infty & -\infty & 0 \\ -\infty & 0 & -\infty & -\infty \\ 0 & -\infty & -\infty & -\infty \\ 0 & -\infty & -\infty & 15 \\ 14 & -\infty & -\infty & -\infty \end{pmatrix},$$

which are obtained by multiplying (in the usual sense) the payments for the game at λ^* by 5.

Remark 11. Proposition 18 was necessary to establish the pseudo-polynomiality of the positive Newton method, which regularly uses left optimal strategies. However, as observed in Remark 8, the use of left optimal strategies is not necessary when $\phi(\lambda_k) > 0$. Moreover, when $\phi(\lambda_k) = 0$ and the coefficients are integers, an alternative to computing a left optimal strategy is to use an optimal strategy, checking whether $\phi(\lambda_k - 1) < 0$ when $\lambda_{k+1} = \lambda_k$. In that case, Corollary 9 part (ii) guarantees that λ_k is optimal. Otherwise, proceed with $\lambda_{k+1} := \lambda_k - 1$. With this modification, the complexity of the computation of optimal strategies falls to

$$\text{MPGI}^*(m + 1, n + 1, M + 2M(\min(m, n) + 1)),$$

instead of the bound of Proposition 18.

We are now ready to sum up the computational complexity of the positive Newton method with left optimal strategies.

Theorem 19. *If the finite coefficients in (27) are integers with absolute values bounded by M , then the positive Newton method, started and finished as in Remark 9, takes no more than*

$$\begin{aligned} &O(M \min(m, n)) \times (O(mn) + O(n^3)) \\ &+ \text{MPGI}^*(m + 1, n + 1, (\min(m, n) + 2)(M + 2M(\min(m, n) + 1)) + 1)) \end{aligned}$$

operations. In particular, the positive Newton method is pseudo-polynomial.

Proof. As the numbers λ_k generated by the positive Newton method are integers and lie within $[-2M(\min(m, n) + 1), 2M(\min(m, n) + 1)]$, the number of iterations does not exceed $4M(\min(m, n) + 1) + 1$. Combining this with Propositions 17 and 18, we get the claim. \square

Note that Remark 11 can be used to reduce the bound of Theorem 19, if the left optimal strategies are not used, getting rid of the factor $(\min(m, n) + 2)$ in the third argument of MPGI^* .

3.7. Perturbed mean payoff games as mean payoff games over germs

Next we discuss an alternative to the perturbation technique of the previous subsection: instead of considering the mean payoff game for several values of the perturbation parameter, we may consider a mean payoff game the payments of which belong to a lattice ordered group of *germs*. In a nutshell, the elements of this group encode infinitesimal perturbations of the payments. This algebraic structure allows one to deal more generally with one-parameter perturbed games (not only the ones arising from tropical linear-fractional programming). A similar structure appeared in (Gaubert and Gunawardena, 1998). This is somehow analogous to the perturbation methods used to avoid degeneracy in linear programming. We hope to develop this further in a subsequent work. The materials of this subsection are not used in the rest of the paper. However, this alternative can be useful in two respects: (1) to develop the present Newton method in the case of real coefficients, (2) to improve the complexity result of the previous subsection.

Consider a *mean payoff game over germs*, finite-duration version, where the weights of arcs in \mathcal{G} are pairs of real numbers (a, b) endowed with lexicographic order:

$$(a^1, b^1) \leq_{\text{lex}} (a^2, b^2) \Leftrightarrow \begin{cases} a^1 < a^2, & \text{or} \\ a^1 = a^2, & b^1 \leq b^2, \end{cases} \quad (40)$$

and the componentwise addition is used to calculate the weights of paths (or cycles). These games correspond to two-sided tropical linear systems over the semiring of germs $\mathbb{G}_{\max} := \mathbb{R}^2 \cup \{(-\infty, -\infty)\}$, where for $g^1 = (a^1, b^1)$, $g^2 = (a^2, b^2) \in \mathbb{G}_{\max}$ we define “ $g^1 + g^2$ ” := $\max(g^1, g^2)$ following (40) and “ $g^1 g^2$ ” := $(a^1 + a^2, b^1 + b^2)$.

With a game over germs we associate an ϵ -perturbed mean payoff game, for $\epsilon \geq 0$, in which the weights (a, b) of the arcs are replaced by $a + \epsilon b$. If the payments in a mean payoff game over germs are given by the matrices A and B (with entries in \mathbb{G}_{\max}), then the matrices associated with the corresponding ϵ -perturbed mean payoff game will be denoted by $A(\epsilon)$ and $B(\epsilon)$, respectively.

Proposition 20. Suppose that the matrices of payments in a mean payoff game over germs (finite-duration version) satisfy [Assumptions 1](#) and [2](#). Then this game has a value and positional optimal strategies (meaning that [\(6\)](#) holds for mean payoff games over germs). Moreover, if (χ_i, κ_i) is the value of such a game, then there exists $\bar{\epsilon} > 0$ such that for any $0 < \epsilon < \bar{\epsilon}$ the associated ϵ -perturbed mean payoff game has value $\chi_i + \epsilon\kappa_i$ and these games have common positional optimal strategies.

Proof. Note that if the matrices of payments A and B in a mean payoff game over germs satisfy [Assumptions 1](#) and [2](#), then for any ϵ -perturbed mean payoff game the corresponding matrices $A(\epsilon)$ and $B(\epsilon)$ also satisfy these assumptions.

Let δ be the minimal absolute value of nonzero differences between cycle means in the mean payoff game with payments given by $A(0)$ and $B(0)$, and let M be the greatest absolute value of the second component of germs. Define $\bar{\epsilon} := \delta/4M$ and consider any ϵ such that $0 < \epsilon < \bar{\epsilon}$.

By [\(6\)](#), for the ϵ -perturbed mean payoff game there exist positional strategies σ^* and τ^* such that

$$\Phi_{A(\epsilon), B(\epsilon)}(j, \tau^*, \sigma) \leq \Phi_{A(\epsilon), B(\epsilon)}(j, \tau^*, \sigma^*) \leq \Phi_{A(\epsilon), B(\epsilon)}(j, \tau, \sigma^*)$$

for all (not necessarily positional) strategies σ and τ . Let σ be any strategy for player Max and assume that $\Phi_{A(\epsilon), B(\epsilon)}(j, \tau^*, \sigma) = a + \epsilon b$ and $\Phi_{A(\epsilon), B(\epsilon)}(j, \tau^*, \sigma^*) = \chi_j + \epsilon\kappa_j$. If $a = \chi_j$, we have $b \leq \kappa_j$ because $a + \epsilon b \leq \chi_j + \epsilon\kappa_j$. Otherwise (i.e., if $a \neq \chi_j$), since $|b|, |\kappa_j| \leq 2M$, $|a - \chi_j| \geq \delta$, $\epsilon < \bar{\epsilon} = \delta/4M$, and $a + \epsilon b \leq \chi_j + \epsilon\kappa_j$, it follows that $a < \chi_j$. Therefore, we conclude that $\Phi_{A,B}(j, \tau^*, \sigma) = (a, b) \leq_{\text{lex}} (\chi_j, \kappa_j) = \Phi_{A,B}(j, \tau^*, \sigma^*)$. The same argument shows that $\Phi_{A,B}(j, \tau^*, \sigma^*) \leq_{\text{lex}} \Phi_{A,B}(j, \tau, \sigma^*)$ for any strategy τ for player Min. This proves that the finite duration version of the mean payoff game over germs has a value, given by (χ_j, κ_j) , and that positional optimal strategies for the ϵ -perturbed mean payoff game (with $\epsilon < \bar{\epsilon}$) are also optimal for the game over germs.

Assume now that σ^* and τ^* are positional optimal strategies for the finite duration version of the mean payoff game over germs, and let (χ_j, κ_j) be its value. Then, $\Phi_{A,B}(j, \tau^*, \sigma) \leq_{\text{lex}} (\chi_j, \kappa_j) \leq_{\text{lex}} \Phi_{A,B}(j, \tau, \sigma^*)$ for all strategies σ and τ (not necessarily positional).

Let σ be any strategy for player Max and assume that $\Phi_{A,B}(j, \tau^*, \sigma) = (a, b)$. If $a = \chi_j$, we conclude $b \leq \kappa_j$ because $\Phi_{A,B}(j, \tau^*, \sigma) \leq_{\text{lex}} (\chi_j, \kappa_j)$, and thus $\Phi_{A(\epsilon), B(\epsilon)}(j, \tau^*, \sigma) = a + \epsilon b \leq \chi_j + \epsilon\kappa_j$ for any $\epsilon \geq 0$. Suppose now that $a < \chi_j$. Since $|b|, |\kappa_j| \leq 2M$ and $\chi_j - a \geq \delta$, it follows that $\Phi_{A(\epsilon), B(\epsilon)}(j, \tau^*, \sigma) = a + \epsilon b \leq \chi_j + \epsilon\kappa_j$ for any ϵ such that $0 < \epsilon < \bar{\epsilon}$. The same argument shows that $\Phi_{A(\epsilon), B(\epsilon)}(j, \tau, \sigma^*) \geq \chi_j + \epsilon\kappa_j$ for any strategy τ for player Min and any ϵ such that $0 < \epsilon < \bar{\epsilon}$. Therefore, we conclude that σ^* and τ^* are positional optimal strategies for the ϵ -perturbed mean payoff game and that its value is $\chi_j + \epsilon\kappa_j$. This proves the claim. \square

Remark 12. [Proposition 20](#) opens the way to using mean payoff games over germs in order to find right or left optimal strategies in the Newton methods. In that case, note that the second component of all finite weights must be set to 0 except for the arcs connecting node $m+1$ of Max with nodes of Min, where it is set to 1 (for right optimality) or to -1 (for left optimality). This raises the issue of developing a direct combinatorial algorithm to solve mean payoff games over germs. Such an algorithm would avoid the perturbation technique of the previous subsection. This will be discussed elsewhere.

Example 5. By [Proposition 20](#), in [Example 3](#) we could find a left optimal strategy σ for player Max at $\lambda_{k-1} = 0$ computing an optimal strategy for the mean payoff game over germs whose matrices of payments are:

$$\begin{pmatrix} (-3, 0) & (-4, 0) & -\infty_{\mathbb{G}} & -\infty_{\mathbb{G}} \\ (-1, 0) & -\infty_{\mathbb{G}} & -\infty_{\mathbb{G}} & (1, 0) \\ -\infty_{\mathbb{G}} & -\infty_{\mathbb{G}} & -\infty_{\mathbb{G}} & (0, 0) \\ (1, 0) & -\infty_{\mathbb{G}} & (0, 0) & -\infty_{\mathbb{G}} \\ -\infty_{\mathbb{G}} & (0, 0) & -\infty_{\mathbb{G}} & -\infty_{\mathbb{G}} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -\infty_{\mathbb{G}} & -\infty_{\mathbb{G}} & -\infty_{\mathbb{G}} & (0, 0) \\ -\infty_{\mathbb{G}} & (0, 0) & -\infty_{\mathbb{G}} & -\infty_{\mathbb{G}} \\ (0, 0) & -\infty_{\mathbb{G}} & -\infty_{\mathbb{G}} & -\infty_{\mathbb{G}} \\ (0, 0) & -\infty_{\mathbb{G}} & -\infty_{\mathbb{G}} & (3, 0) \\ (3, -1) & -\infty_{\mathbb{G}} & -\infty_{\mathbb{G}} & -\infty_{\mathbb{G}} \end{pmatrix},$$

where $-\infty_{\mathbb{G}} := (-\infty, -\infty)$.

Remark 13. [Proposition 20](#) extends [\(6\)](#) to germs. Note that this equation provides a very crude algorithm for computing values and optimal strategies. Further idea is to allow more general algorithms, showing that they can be applied to mean payoff games over germs. This will be investigated elsewhere.

4. Examples

4.1. Minimization

We next apply the positive Newton method to the minimization problem of [Example 2](#). Recall that the equivalent homogeneous version of this problem is to minimize λ subject to $uy \leq \lambda + vy$, $Cy \leq Dy$, and $y_3 \neq -\infty$, with $C = [A, c]$, $D = [B, d]$, $u = (2, -4, -\infty)$ and $v = (-\infty, -\infty, 0)$, where the matrices A, B and the vectors c, d are given in [Example 2](#) (also recall that in this example, $m = 7$ and $n = 2$). Note that in this case, for any strategy σ for player Max, the only possible value for $l := \sigma(m + 1) = \sigma(8)$ is $n + 1 = 3$. Then, at each iteration k of the positive Newton method applied to this problem, in order to compute $\lambda_k + v_l = \lambda_k$ we need to minimize $(2 + y_1) \vee (-4 + y_2)$ subject to the system obtained by setting $y_3 = 0$ in $Cy \leq D^\sigma y$, as explained in [Section 3.5](#). The latter is Problem (35) for this particular case.

We start the positive Newton method with $\lambda_0 = 15$, where $\phi(\lambda_0) = 5.5$. The function $\sigma(1) = 1$, $\sigma(2) = 1, \dots, \sigma(7) = 1$ and $\sigma(8) = 3$ is an optimal strategy for player Max at λ_0 . To perform the first Newton iteration, we find the minimal solution of the system

$$\begin{aligned} 0 \leq x_1 - 2, \quad 0 \leq x_1, \quad 0 \leq 1 + x_1, \quad x_2 - 4 \leq x_1, \\ (x_2 - 3) \vee 0 \leq x_1 + 2, \quad x_2 - 3 \leq x_1, \quad x_2 - 2 \leq x_1, \end{aligned}$$

which is $(\underline{x}_1, \underline{x}_2) = (2, -\infty)$. The next value is $\lambda_1 = (2 + \underline{x}_1) \vee (-4 + \underline{x}_2) = 4$. Then, $\phi(\lambda_1) = 1.5$ and $\sigma(1) = 2, \sigma(2) = 2, \sigma(3) = 1, \sigma(4) = 1, \sigma(5) = 1, \sigma(6) = 3, \sigma(7) = 3$ and $\sigma(8) = 3$ is a new optimal strategy for player Max. For the next Newton iteration, we find the minimal solution of the system

$$\begin{aligned} 0 \leq x_2, \quad 0 \leq x_2 - 1, \quad 0 \leq 1 + x_1, \quad x_2 - 4 \leq x_1, \\ (x_2 - 3) \vee 0 \leq x_1 + 2, \quad x_2 - 5 \leq 0, \quad x_2 - 6 \leq 0, \end{aligned}$$

which is $(\underline{x}_1, \underline{x}_2) = (-1, 1)$. Then, the next value is $\lambda_2 = (2 + \underline{x}_1) \vee (-4 + \underline{x}_2) = 1$. Now $\phi(\lambda_2) = 0.5$ and $\sigma(1) = 2, \sigma(2) = 2, \sigma(3) = 2, \sigma(4) = 1, \sigma(5) = 3, \sigma(6) = 3, \sigma(7) = 3$ and $\sigma(8) = 3$ is the optimal strategy for player Max. For the next Newton iteration, we find the minimal solution of the system

$$\begin{aligned} 0 \leq x_2, \quad 0 \leq x_2 - 1, \quad 0 \leq x_2 - 2, \quad (x_2 - 3) \vee 0 \leq x_1 + 2, \\ x_2 - 4 \leq 0, \quad x_2 - 5 \leq 0, \quad x_2 - 6 \leq 0, \end{aligned}$$

which is $(\underline{x}_1, \underline{x}_2) = (-2, 2)$. This gives $\lambda_3 = (2 + \underline{x}_1) \vee (-4 + \underline{x}_2) = 0$, which is the optimal value λ^* . The optimality of $\lambda^* = 0$ can be certified applying [Theorem 12](#), see [Example 2](#) above.

The vectors $(2, -\infty)$, $(-1, 1)$ and $(-2, 2)$ found by the Newton iterations are indicated on the left-hand side of [Fig. 5](#) as “1”, “2” and “3”.

The right-hand side of [Fig. 5](#) displays the graph of $\phi(\lambda)$, together with the Newton iterations. The graphs of partial spectral functions $\phi^\sigma(\lambda)$ are given by red dashed lines.

4.2. Maximization

Consider the maximization problem of [Example 1](#). We next apply the positive Newton method to the homogeneous version (29) of the equivalent minimization problem.

With this aim, firstly observe that in this case $n + 1 = 3 \in \text{supp}(u) = \{3\}$, and so Condition (33) is always satisfied. Therefore, as explained in [Section 3.5](#), at each iteration of the positive Newton method applied to this problem, $\lambda_k + v_l$ can be computed by minimizing y_3 subject to the system obtained by setting $y_l = 0$ in $Cy \leq D^\sigma y$, which corresponds to Problem (35) in the case of this example.

Let us take $\lambda_0 = 3$. Then, we obtain that $\sigma(1) = 3, \sigma(2) = 3, \sigma(3) = 3, \sigma(4) = 3$ and $\sigma(5) = 2$ is an optimal strategy for player Max at λ_0 . To perform the Newton iteration we first notice that $l = \sigma(m + 1) = \sigma(5) = 2$, which means that $\lambda_1 + v_2 = \lambda_1 + 3$ is the minimum of y_3 subject

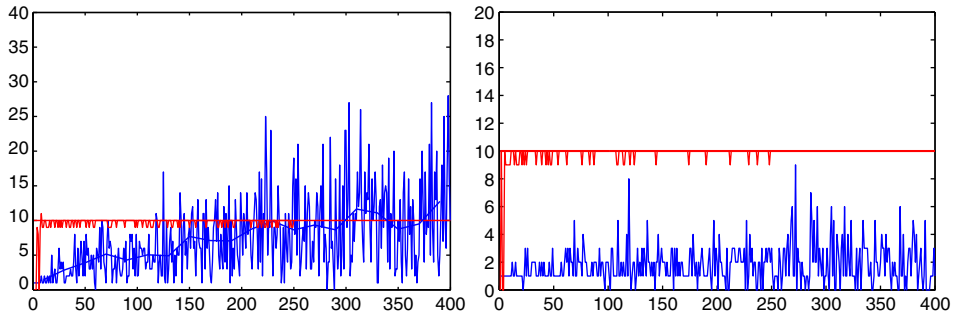


Fig. 7. Number of iterations of the Newton method (thin blue line) and the bisection method (thin red line) in the cases of minimization (left) and maximization (right). Thick blue line: average number of iterations of the Newton method for each interval of 20 dimensions. Thick red line: level $\log(2M) \approx 10$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

to the system obtained by setting $y_2 = 0$ in $Cy \leq D^\sigma y$, as explained above. Thus, we have to find the minimal solution of the following system:

$$x_3 \geq -1, \quad x_3 \geq (-2 + x_1) \vee (-2), \quad x_3 \geq -1 + x_1, \quad x_3 \geq x_1,$$

which is $(x_1, x_3) = (-\infty, -1)$. The full vector $y = (-\infty, 0, -1)$ is a translate of $y + 1 = (x_1, x_2, 0)$, where $(x_1, x_2) = (-\infty, 1)$ is marked as “1” at the left of Fig. 6. Meanwhile we obtain $\lambda_1 = x_3 - 3 = -4$, and $\sigma(1) = 1, \sigma(2) = 3, \sigma(3) = 2, \sigma(4) = 2$ and $\sigma(5) = 2$ is now a new optimal strategy for player Max. Again, here $l = \sigma(m + 1) = \sigma(5) = 2$ and so the second columns of C and D^σ are the free terms in (35). We have to find the minimal solution of the following system:

$$x_1 \geq -1, \quad x_3 \geq (-2 + x_1) \vee (-2), \quad 0 \geq -1 + x_1, \quad 2 \geq x_1,$$

which is $(x_1, x_3) = (-1, -2)$. We obtain $\lambda_2 = x_3 - 3 = -5$, which is the optimal value, and so the value of the original maximization problem is 5. The full vector $y = (-1, 0, -2)$ is a translate of $y + 2 = (x_1, x_2, 0)$, where $(x_1, x_2) = (1, 2)$ is marked as “2” at the left of Fig. 6.

As in the case of Fig. 5, the right-hand side of Fig. 6 displays the graph of $\phi(\lambda)$, together with the Newton iterations. The graphs of partial spectral functions $\phi^\sigma(\lambda)$ are given by red dashed lines.

4.3. Numerical experiments

A preliminary implementation of the bisection and Newton methods for tropical linear-fractional programming was developed in MATLAB. We next present some graphs showing how they behave on randomly generated instances of tropical linear-fractional programming problems, in which the entries of matrices and vectors range from -500 to 500 . The matrices A and B in (28) and (26) are square, with dimensions ranging from 1 to 400.

Fig. 7 displays the cases of the tropical linear programming (28), in which all the entries are finite. Here the certificates of unboundedness reduce to the solvability of a two-sided tropical system of inequalities. When a feasible and bounded problem is generated, it is solved by the bisection and Newton methods.

For the bisection method, we use the lower initial values λ_0 of (Butkovič and Aminu, 2008), see also Remark 10. Following (Butkovič and Aminu, 2008), the upper initial values $\bar{\lambda}_0$ for the bisection method come from a solution of $Ax \vee c \leq Bx \vee d$. To find this solution, we use the policy iteration of (Dhingra and Gaubert, 2006) instead of the alternating method of (Cunningham-Green and Butkovič, 2003). Shown by the thin red line (up to $m = n = 250$), the bisection method worked similarly in the case of minimization and maximization. In our experiments, the interval between lower and upper initial values never exceeded $2M = 1000$, with the number of iterations quickly approaching a constant level of 9 or 10 iterations ($\log M \approx 9$ or $\log 2M \approx 10$).

The thin blue line represents the run of the Newton method, and the thick blue line represents their average number calculated for each interval of 20 dimensions. For the sake of fair comparison with

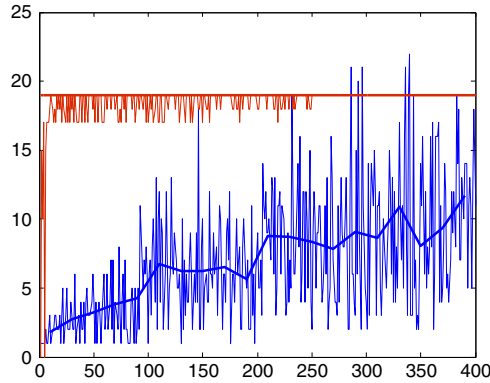


Fig. 8. Number of iterations of the Newton method (thin blue line) and the bisection method (thin red line) in the case of minimization with $M = 500000$. Thick blue line: average number of iterations of the Newton method for each interval of 20 dimensions. Thick red line: level $\log(2M) - 1 \approx 19$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

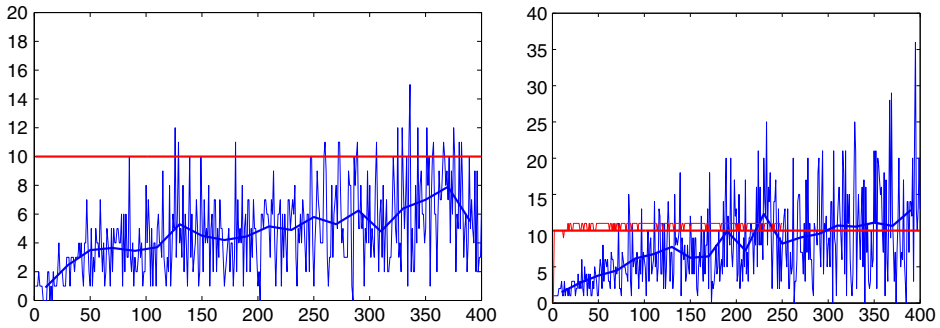


Fig. 9. Number of iterations of the Newton method (thin blue line) and the bisection method (thin red line) in the cases of linear-fractional programming with finite entries (right) and when the average proportion of $-\infty$ entries is 0.7 (left). Thick blue line: average number of iterations of the Newton method for each interval of 20 dimensions. Red line on the left: level $\log(2M) \approx 10$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

the bisection method, the initial value λ_0 coincides with the upper initial value $\bar{\lambda}_0$ for the bisection method. This value comes from a solution of $Ax \vee c \leq Bx \vee d$, instead of the theoretical value $2M(n+1)$, which depends on n and may be much greater. In the case of minimization, the average number of Newton iterations slowly grows, being smaller than 10 before $n \approx 250$, but exceeding 10 at larger dimensions. Naturally, the number of iterations for the same dimension may be very different, depending on the configuration and complexity of the tropical polytopes (i.e., the solution sets of $Ax \vee c \leq Bx \vee d$). In the case of maximization, the number of iterations is usually below 5. Note that maximization is resolved immediately if we find the greatest point of the solution set, which suggests that the maximization problem may be simpler. We also remark that there is no correlation between the number of iterations of the bisection and Newton methods. In particular, it is easy to construct instances with large integers in which the number of bisection iterations becomes arbitrarily large, whereas the number of Newton iterations remains bounded. This agrees with Fig. 8, where in comparison to the graph on the left-hand side of Fig. 7, M is equal to 500 000 instead of 500.

Fig. 9 displays the cases of tropical linear-fractional programming (26) with all entries finite (right) and with a 0.7 frequency of $-\infty$ entries (left). In the case of $-\infty$ entries, as required by Assumptions 1 and 2, we ensure that the set of constraints contains neither $-\infty$ rows on the right-hand side nor $-\infty$ columns on the left-hand side. The case of tropical linear-fractional programming with finite entries

shows almost the same picture as in the case of minimization above. The case when $-\infty$ appears with a regular frequency is even more favorable for the Newton method, due to the sparsity of β .

5. Conclusion

In this paper, we developed an algorithm to solve tropical linear-fractional programming problems. This is motivated by the works (Allamigeon et al., 2008; Allamigeon, 2009), in which disjunctive invariants of programs are computed by tropical methods: tropical linear-fractional programming problems are needed to tropicalize the method of templates introduced by Sankaranarayanan et al. (2005, 2006).

The main technical ingredient, which combines ideas appearing in Akian et al. (2009); Allamigeon et al. (2011b); Gaubert and Sergeev (2010) is to introduce a *parametric* zero-sum two-player game (in which the payments depend on a scalar variable), in such a way that the value of the initial tropical linear-fractional programming problem coincides with the smallest value of the variable for which the game is winning for one of the players. The value of the parametric game, which we call the *spectral function*, is a piecewise affine function of the variable. Then, the problem is reduced to finding the smallest zero of the spectral function, which we do by a Newton-type algorithm, in which at each iteration, we solve a one-player auxiliary game.

Using this game-theoretic connection, we present concise certificates expressed in terms of the strategies of both players, allowing one to check whether a given feasible solution is optimal, or whether the tropical linear-fractional programming problem is unbounded. This is inspired by Allamigeon et al. (2011b), in which certificates of the same nature were given for the simpler problem of certifying whether a tropical linear inequality is a logical consequence of a finite family of such inequalities.

We also develop a generalization of the bisection method of Butkovič and Aminu (2008). The latter, as well as the Newton method, are shown to be pseudo-polynomial. Note that at each iteration, both methods call an oracle solving a mean payoff game problem (for which the existence of a polynomial time algorithm is an open question—only pseudo-polynomial algorithms are known). The pseudo-polynomial bound that we give for the Newton method is worse than the one concerning the bisection method, however, for the former we also give a non pseudo-polynomial bound, involving the number of strategies, which is better than the pseudo-polynomial bound if the integers of the instance are very large. This is confirmed by experiments, with a preliminary implementation, which indicates that the Newton method scales better as the size of the integers grows. In addition, it has the advantage of maintaining feasibility, and there are significant special instances in which it converges in very few iterations.

The Newton method of this paper appears as a natural product of the game-theoretic connection and the spectral function approach. We further concentrate on the implementation of each Newton step by reduction to optimal path algorithms, and on the proof of pseudo-polynomiality. This method could be also considered in the framework of more abstract Newton methods in a generalized domain, which also means making decent comparison with other Newton schemes, like (Esparza et al., 2008). The comparison of Newton and bisection methods, as well as possible alternative approaches, also remain to be further examined.

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