*Feasibility and Reachability in Max-linear Systems (EPSRC Project EP/F000480/1)*
*Final report*

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A Dissemination

Results of this research appear in a research monograph [8] and 28 research papers. Of these 15 have been published or accepted for publication in peer-reviewed academic journals, 2 in peer-reviewed proceedings and 3 have been submitted for publication. The list of references at the end of this report contains bibliographic details of all papers prepared as part of this research (including departmental working papers) and three other publications ([3], [12] and [16]).

Results of this research have also been reported by the members of the research team at more than 30 conferences and seminars. Some of them are listed below:

- 2008: Workshop on Nonnegative Matrices, American Institute of Mathematics in Palo Alto, California (two invited plenary talks); Seminar CICADA, Manchester (two talks); Workshop on Discrete Event Systems, Göteborg, Sweden; Minisymposium on max-algebra at ILAS symposium, Cancun (three talks); IWOTA08, Williamsburg (invited plenary talk).
- 2009: 16th International Conference on Mathematical Methods in Economy and Industry České Budějovice, Czech Rep; Workshop on Idempotent and Tropical Mathematics, Montreal; Geometry in Astrakhan, Astrakhan, Russia; Algorithms of Approximation VI, Ambleside; 14th Haifa Matrix Theory Conference; Colloquium, Technical University Vienna.
- 2010: Workshop of the German Society for Applied and Industrial Mathematics, GAMM (invited plenary talk); International conference Applied Linear Algebra, Novi Sad, Serbia (two talks, one of them invited plenary talk); Minisymposium on max-algebra at ILAS symposium, Pisa; Coimbra Meeting on 0-1 Matrix Theory and Related Topics, Coimbra, Portugal; Applied Analysis Workshop, University Chemnitz, Germany (three talks); Applied Linear Algebra Meeting, Northern Illinois University.
- 2011: MOPNET4 Manchester (invited plenary talk); SIAM Conference on Control and Its Applications, Baltimore; Workshop on Tropical Mathematics, University of Warwick; Workshop on Tropical Mathematics, University of Birmingham; Colloquium, University of Kent, Canterbury; Summer School INKOV, University Hr.Králové, Czech Rep; Colloquium, Department of Statistics, University Chicago.

All papers and produced software have been placed on the PI's website and can be freely downloaded. Software enables users to solve the max-linear programming problem for both maximisation and minimisation (Task 3) and also provides a possibility of solving two-sided homogeneous systems with separated variables or not, as well as non-homogeneous systems.

A consequence of this project is the creation of the Joint Research Group in Tropical Mathematics supported by the LMS. The PI is the chair of this group, which consists of researchers from Birmingham, Manchester and Warwick. Workshops of this group (organised three times a year) provide another platform for the dissemination of the results of the project.

Some of the results were achieved and published jointly with the following external collaborators: R. A. Cuninghame-Green (Birmingham), S. Gaubert (Paris), R. Katz (Rosario, Argentina), V. Nitica (West Chester, Pennsylvania), E. Wagneur (Montreal). Two postgraduate students of the PI (A. Aminu and K. P. Tam, both privately funded) were also involved; the first of them prepared the software.

B Most significant findings (all formulations are max-algebraic):

B1 Two fundamental questions of the reachability problem have been fully resolved. One of them is the case when the orbit of the matrix reaches an eigenvector with any non-trivial starting vector (strongly stable or robust matrices - Task 6), the other one is when the orbit never reaches an eigenvector unless it starts with an eigenvector (weakly stable matrices - this is a new concept not known at the time of the grant application). These cases are now fully and efficiently characterised for both irreducible and reducible matrices. See C1 and D4.

B2 CSR representation of matrix powers (for details see D2). This result confirms the intuition that the dynamics of max-algebraic matrix powers is essentially controlled by the powers of certain Boolean matrices.

B3 Remarkable spectral properties of (any number of) commuting matrices, in particular the existence of common eigenvalues, and the impact of these properties on commuting Boolean matrices. See D3.

B4 Full description of visualisation scaling, which emerges as a new and powerful tool in max-algebra and highlights the role of subeigenvectors and Kleene stars (see C2). Visualisation scaling was used as an essential tool for solving Task 5 (see C3) and in the analysis of commuting matrices (see D3).

C Results addressing the questions set in the original research proposal (Tasks 1-7):

C1 Full and efficient characterisation of robust reducible matrices (Task 6 in the project proposal) has been proved using the max-algebraic spectral theory of reducible matrices [10]. This result shows a remarkable similarity between max-algebra and non-negative linear algebra by using the Frobenius normal form.

C2 The concept of diagonal similarity scaling of matrices existed for many years. Shortly before the start of the present project the idea of strict visualisation scaling of a matrix was formulated in [12]. This idea was substantially developed within this project using subeigenvectors and Kleene stars [31]. Together with the theory of cyclic classes applied to critical digraphs it was then used to describe periodic behaviour of powers of irreducible matrices and matrix orbits [25]. An $O\left(n^3 \log n\right)$ method was derived for finding the period of a given irreducible matrix, the period of the matrix orbit starting from a given vector and for finding any periodic power. This enables to decide whether the eigenspace of a given power of an irreducible matrix is reached by the orbit of this matrix from a given starting vector. As a special case, it decides whether the eigenspace of an irreducible matrix is reached by its orbit, which is Task 4.

C3 The idea of strict visualisation and cyclic classes was also instrumental for the description of attraction spaces (that is subspaces of starting vectors from which the orbit reaches an eigenspace) for irreducible matrices, using max-linear systems of equations [28], which solves Task 5 for irreducible matrices. This motivated to investigate circulant symmetries of periodic powers. It has been shown that the dimension of the problem for an $n \times n$ matrix can be effectively reduced from $n$ to $\widetilde{c} + \overline{c}$, where $\widetilde{c}$ is the number of cyclic classes of the critical digraph and $\overline{c}$ is the number of non-critical nodes. It has also been shown that the equations corresponding to non-critical nodes are not needed for the description of attraction spaces. The remaining critical part breaks into several chains of equations, each of which corresponds to one strongly connected component of the critical digraph. The obtained chains of equations can be simplified using the max-algebraic cancellation rule. The coefficients of the reduced system form a matrix called the core matrix. The main result then is a description of the attraction space using the core matrix and cyclic classes of the critical digraph.

C4 Theory and algorithms for solving max-linear programs subject to one or two-sided max-linear constraints (both minimisation and maximisation) have been developed [9], [1] for programs with finite entries. This was based on the Alternating Method for solving two-sided systems [16] and includes criteria for the objective function to be bounded, and the proof that the finite bounds are always attained, if they exist. Also, bisection methods for localizing the optimal value with a given precision have been compiled. For programs with integer entries these methods turn out to be exact, of pseudopolynomial computational complexity. This solves Task 3. Since the announcement of the bisection method [9] a number of studies on max-linear programming have been undertaken and so in a sense our paper plays a role of the generic work in this area. A method for matrices with non-finite entries has then also been developed [18].

C5 A number of results have been obtained giving answers to Task 2.

i. A polynomial method for narrowing the search for generalised eigenvalues using symmetrised semirings has been developed ([8], Sec. 9.4) for problems with finite entries. Since no polynomial method for finding a generalised eigenvalue seems to exist, this is a substantial contribution to Task 2, as in some cases it identifies all generalised eigenvalues, even if there are an infinite number of them.

ii. A number of special cases have been resolved [17].

iii. It has been proved [27] that the union of any finite set of closed intervals can be the set of generalised eigenvalues for a suitably taken pair of $2 \times n$ matrices.

iv. The approach of [20] is based on the use of a min-max function (that is, a function involving min, max and + operations), associated with a two-sided system. This function is a composition of non-linear projectors on certain max-plus spaces. The key idea is to investigate the spectral radius of the min-max function associated with the generalised eigenproblem. One of its variables is the parameter $\lambda$ of the generalised eigenproblem. The spectrum of the two-sided eigenproblem then coincides with the values of $\lambda$ for which the spectral radius of the min-max function is zero. On one hand, the spectral function (that expresses the dependence of the spectral radius on $\lambda$), which is piecewise-linear and Lipschitz, can be fully reconstructed. This is done by a pseudopolynomial number of calls to an oracle, which evaluates the spectral function and its slope at a given value of $\lambda$. On the other hand, it has a natural geometric interpretation as the least Chebyshev distance between $A \otimes x$ and $\lambda \otimes B \otimes x$. Thus, [20] provides an approximate method (exact in the integer case) for finding the whole spectrum for a two-sided eigenproblem and other parametric problems of this kind.

C6 The main results related to Task 7 are: the use of the Frobenius normal form of reducible matrices for the analysis of robust matrices [10], the use of the Frobenius trace down method for solving $Z-$ equations in max-algebra in the same way as in nonnegative linear algebra [14] and the use of the Frobenius normal forms of commuting matrices, particularly when the Perron roots of the components are distinct [21].

C7 The original plan for Task 1 was modified. The team has concentrated on investigating the properties of the pseudopolynomial Alternating Method [16], see also [24]. First, it has been verified that this method is non-polynomial by constructing an example (by S.Sergeev), for which the method needs a non-polynomial number of iterations [8]. The Alternating Method was further studied in [26]. The method was generalised there to the case of homogeneous multi-sided systems and it was proved, using the cellular decomposition, that the Alternating Method converges in a finite number of iterations to a finite solution of a multi-sided system with real entries, if such a solution exists. This extends the previously known result for two-sided systems with integer entries [16]. The paper also presents new bounds on the number of iterations of the Alternating Method. They are expressed in terms of projective matrix norms and the cyclic projective distance associated with several cones, being similar to the Renegar condition number [3]. A more general purpose of [26] is to present a survey on certain methods of max-algebraic convexity. These methods include the description of extremals in terms of a multiorder, the theory of non-linear projectors with application to separation theorems, and the role of Kleene stars as building blocks in the cellular decomposition. New results on these topics include a representation of non-linear projectors in terms of minimal elements, new algebraic identities involving max-algebraic pseudoinverses of matrices and a new, multi-cone generalization of the max-algebraic Minkowski's theorem about extremals.

D A number of results beyond the original research plan have been achieved. They are all related to the originally proposed research.

D1 A book on max-linear systems [8] has been prepared and published. The aim of this book is to present max-algebra as a modern modelling and solution tool. It provides both an introduction to max-algebra and results on advanced topics, in particular on reachability and feasibility. Most of the results achieved in this project appear in the book. The book is intended for a wide-ranging readership, from undergraduate and postgraduate students to researchers and mathematicians working in industry, commerce or management. No prior knowledge of max-algebra is assumed. The text in the first five chapters is self-contained and may be used as a support for undergraduate or postgraduate courses. The theory is illustrated by numerical examples and complemented by exercises at the end of every chapter. A number of practical and theoretical applications and a list of open problems are included. The book contains a number of new, never published results which would otherwise be published in about three journal papers.

D2 The theory of so-called CSR representations of matrix powers has been developed: In [30] it has been proved that for any (reducible) matrix $A$ all powers $A^r, r \geq 3n^2$, can be expressed as a max-algebraic sum of terms of the form $C \otimes S^r \otimes R$, called $CSR$ products. All these terms can be found in $O\left(n^4 \log n\right)$ time. Here $C$ and $R$ are extracted from the columns and rows of a certain Kleene star (the same for both) and $C \otimes R$ is the spectral projector of $A$ if $A$ is irreducible. The matrix $S$ is diagonally similar to the Boolean incidence matrix of a certain critical digraph. It is shown that the powers have a well-defined ultimate behaviour, where certain terms are totally or partially suppressed, thus leading to ultimate $C \otimes S^r \otimes R$ terms and the corresponding ultimate expansion. This generalizes both the Cyclicity Theorem to reducible matrices and the concept of robustness for ultimate periodicity (see Task 6). The expansion is then used to derive an $O\left(n^4 \log n\right)$ method for solving the question whether the orbit of a reducible matrix is ultimately periodic with any starting vector. This result is of fundamental importance as it shows that the dynamics of max-algebraic powers can be described using Boolean matrices. It is expected that this will yield strong results for instance for the little understood problem of roots of matrices and ultimately for other max-algebraic functions of matrices.

D3 Little had been known about commuting matrices in max-algebra in the past. Paper [21] is an attempt to begin substantial research in this area. As a fundamental result and a starting point for new research it has been proved that two commuting matrices have a common eigenvector. This result is then generalised to any finite number of pairwise commuting matrices, which implies a number of max-algebraic analogues of classical results. The paper analyses Frobenius normal form of commuting matrices and shows that the set of common eigenvectors can be described as the max-algebraic eigenspace of a matrix which can be computed in polynomial time. Visualisation scaling has been used to deduce links to Boolean matrices. These enabled us then to prove that two commuting matrices have a common critical node. The paper also offers analogues of some of these results for nonnegative matrices in conventional linear algebra.

The proof of existence of a common eigenvector works for both max-algebra and nonnegative linear settings and uses elements of the simultaneous triangularization of commuting matrices. Following the results of Frobenius in the conventional linear algebra, this allows for characterising eigenvalues of polynomials of commuting matrices, and leads to inequalities involving Perron roots. In the case when all Perron roots of irreducible blocks are different it is proved that the matrices have the same spectral classes and the same closure of condensation digraph, which leads to a precise enumeration of all eigenvalues of matrix polynomials.

D4 In contrast to robust (or strongly stable) matrices whose investigation has been proposed and successfully undertaken (see C1), a new type of matrices has emerged during this project, namely matrices whose orbit with any starting vector does not reach an eigenvector unless the starting vector itself is an eigenvector. Since multiprocessor interactive systems with such matrices will not reach a stable regime unless they start in such a regime, we call these matrices weakly stable. Full characterisation of weakly stable matrices has been proved: an irreducible matrix is weakly stable if and only if its critical graph is a Hamiltonian cycle in the associated graph; this criterion has been extended to reducible matrices using the Frobenius normal form. Both criteria can be checked in polynomial time. The proof of the irreducible case required the introduction of the concept of supereigenvectors, that is vectors satisfying $Ax \geq \lambda x$, in contrast to well known subeigenvectors (satisfying $Ax \leq \lambda x$) and seeing the eigenspace as the intersection of these two sets. It also emerged that despite the apparent symmetry there is a profound difference in hardness of the description of these two sets.

D5 In [18] the initial max-linear programming formulation was extended to fractional max-linear programming: minimize$\frac{px \oplus r}{qx \oplus s}$ subject to $A \otimes x \oplus c \leq B \otimes x \oplus d$. This is motivated by geometric applications: finding the minimum value of the parameter of a max-algebraic halfspace. It appears that the fractional max-linear programming, and hence the initial max-linear programming, are equivalent to finding minimal value $\lambda$ at which a certain function $\varphi(\lambda)$ is nonnegative. Another ingredient is the recently discovered equivalence between max-algebraic two-sided systems and mean-payoff games which enables to solve a max-algebraic two-sided system by calling a mean-payoff game oracle. This oracle gives a value of $\varphi(\lambda)$ at a given point, or just answers whether it is nonnegative. This approach led to a Newton iteration scheme for fractional max-linear programming, see [18], which depends on the "configuration" or "complexity" of the solution set rather than on the range of entries, and finds exact solution of the problem in general case, not just for the integer input. The new approach also allowed to extend the bisection algorithm of [9] to fractional max-linear programming with possibly infinite entries (see [29]).

D6 In [29] the notion of parametric two-sided systems $A(\lambda)\, x \leq B(\lambda)\, x$ has been introduced, which generalizes both two-sided eigenproblem and max-linear programming. Here each entry of $A(\lambda)$ and $B(\lambda)$ is a piecewise linear function with bounded slopes and offsets of linear pieces. The problems of max-linear and fractional max-linear programming (possibly with infinite entries) and reconstructing spectral function are generalized to this setting. All these problems are shown to be pseudopolynomial, based on the equivalence between max-algebraic two-sided systems and mean-payoff games, for which pseudopolynomial and subexponential algorithms are known. The paper also contains particular results on the two-sided eigenproblem, namely 1) an explicit formula for the unique eigenvalue of $Ax = \lambda Bx$ (under natural assumptions which assure such uniqueness) and 2) evaluation of spectral function in the case when $B = I$ (the one-matrix eigenproblem). For the fractional max-linear (and in particular, max-linear) programming, bisection and Newton iteration schemes are studied and numerical experiments comparing these methods are described.

D7 Paper [4] demonstrates one particular aspect of max-algebra that may be useful in other areas of mathematics. It is the potential to efficiently describe all solutions to a problem in contrast to other methods that usually find one or a few solutions. This may provide a possibility of finding a solution with additional properties or a solution optimal with respect to a given criterion. The problem on which this aspect is demonstrated is the system of dual network inequalities (SDNI), for which the Kleene star efficiently describes all solutions. This is then used to derive a pseudopolynomial algorithm that either finds a bounded mixed-integer solution to the SDNI or decides that no such exists. An explicit mixed-integer solution is derived for the SDNI provided that all entries are integer.

D8 Another type of applications of max-algebra is presented in [15], where max-algebraic techniques are used to efficiently find solutions to various tasks relating starting, completion and loading times in multi-machine interactive processes.

D9 In [32] the multiorder principle has been used to explicitly describe all extremal solutions to the system $A \otimes x \leq B \otimes x$ with two inequalities.

D10 In [20] a separation theorem for several max cones was proved and the notion of cyclic projections was introduced. As a consequence, it was possible to prove a max-algebraic analogue of Helly's theorem.

D11 The following two max-algebraic combinatorial problems have been shown to be $NP$-complete [5]: (a) Given a matrix $A$ and a vector $b$, is it possible to permute the components of $b$ so that for the obtained vector $b'$ the system $A \otimes x = b'$ has a solution? (b) Given a matrix $A$ and a vector $x$, is it possible to permute the components of $x$ so that the obtained vector $x'$ is an eigenvector of $A$? The linear algebraic versions of these two problems have also been shown to be $NP$-complete [7]. A similar problem for nonnegative matrices remains open.

D12 Basic properties of max-algebraic tensors are investigated in [11]. It is shown that these tensors are useful for solving max-algebraic matrix equations.

D13 The concept of separation by hyperplanes and halfspaces is fundamental for convex geometry and its max-algebraic analogue. However, similar separation results in max-min convex geometry are based on semispaces. Paper [23] answers the question, which semispaces are hyperplanes and when it is possible to separate sets by hyperplanes in max-min convex geometry. In [22] the separation of a closed box from a max-min convex set by max-min semispaces is studied. This can be regarded as an interval extension of the known separation results. A constructive proof is given in the case when the box satisfies a certain condition, and it is also shown that the separation is never possible when this condition is not satisfied. The paper also studies the separation of two max-min convex sets by a box and by a box and a semispace.

**Remark 1** *The original duration of this project (Feb 2008 - Jan 2011) has been extended by the EPSRC on the PI's request by 3 months in January 2011. This was necessary due to an accident of the VR's wife in October 2010. Following this accident the VR had to cancel his planned visit to Birmingham. No additional funding was required and the visit took place in March/April 2011.*

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