On scaling to an integer matrix and graphs with integer weighted cycles

M MacCaig

School of Mathematics, University of Birmingham, Edgbaston, Birmingham B15 2TT, UK

Abstract

Between 1970 and 1982 Hans Schneider and co-authors produced a number of results regarding matrix scalings. They demonstrated that a matrix has a diagonal similarity scaling to any matrix with entries in the subgroup generated by the cycle weights of the associated digraph, and that a matrix has an equivalent scaling to any matrix with entries related to the weights of cycles in an associated bipartite graph. Further, given matrices A and B, they produced a description of all diagonal X such that $X^{-1}AX = B$. In 2005 Butkovič and Schneider used max-algebra to give a simple and efficient description of this set of scalings. In this paper we focus on the additive group of integers, and work in the max-algebra to give a full description of all scalings of a real matrix A to any integer matrix. We do this for four types of scalings; beginning with the familiar $X^{-1}AX$, XAY and XAX scalings and finishing with a new scaling which we call a signed similarity scaling. This is a scaling of the form XAY where we specify for each row i, either $x_i = y_i$ or $x_i = -y_i$. In all of our results we use necessary and sufficient conditions for existence which are based on integer weighted cycles in the associated digraph, or associated bipartite graph, of the matrix.

Keywords: max-algebra, diagonal similarity scaling, direct similarity scaling, symmetric scaling, integer matrices, integer cycles
AMS classification: 15A24, 15A15, 15A80

1. Introduction

In [6, 7, 9] the authors study matrix scalings of the form $X^{-1}AX$ and XAY. Given an irreducible matrix $A \in G^{n\times n}$ for some group G, they considered in [6] the problem of determining whether there exists a diagonal similarity scaling of A to a matrix with entries in a subgroup of G. Specifically they showed that if H is the subgroup of G generated by the weights of cycles in A, then there

Email address: m.maccaig.maths@outlook.com (M MacCaig)
[1]Website: https://sites.google.com/site/mariemaccaig/home
Supported by EPSRC grant EP/J00829X/1.

exists X such that $X^{-1}AX \in H^{n\times n}$. Further they described how to find one such X. They extend these results to scalings of the form XAY. In [9] they gave necessary and sufficient conditions for two matrices $A, B \in G^{m\times n}$ to be diagonally similar (for $m = n$) or directly similar (called diagonally equivalent in that paper). Additionally, for additive groups G, they demonstrated that diagonal similarity can be characterised in terms of flows on the corresponding graph of the matrices, and direct similarity in terms of flows on the bipartite graph. In [7], given A and B, they gave a full description of all X such that $X^{-1}AX = B$. The authors in [3] demonstrated that max-algebra could be used to describe the set of all such scalings satisfying $XAX = B$ in a simple and efficient manner. Symmetric scalings, that is scalings of the form XAX, have also been extensively studied, often in relation to specified row and column sums, or row and column maxima, see eg [8].

In this paper we focus on the additive group of integers. We introduce a new scaling problem for square matrices which we call a signed similarity scaling. This is a scaling of the form XAY where we specify for each row i, whether $x_i = y_i$ or $x_i = -y_i$, thus it contains both $X^{-1}AX$ and XAX scalings as subproblems. We give necessary and sufficient conditions for a matrix to have a signed similarity scaling to an integer matrix, and additionally describe all possible scalings to an integer matrix for finite (real) matrices. We work in the max-algebra as it allows us a simple way of describing the set of all solutions to the problems we consider. In order to state our main results we first consider the problems of determining whether a matrix A has a diagonal similarity, direct similarity or symmetric scaling to an integer matrix. Sections 3 and 4 contains the results on diagonal similarity scalings and direct similarity scalings respectively. In these sections we state necessary and sufficient conditions for a matrix to have a scaling to an integer matrix. Although these conditions follow from the results in [6, 9] we state them using simple terminology that ties together all the problems we consider. Additionally we are able to give a complete and simple description of all scalings of A to an integer matrix. In Section 6 we perform the same procedure for symmetric scalings. Finally in Section 7 we bring together all our previous results to give necessary and sufficient conditions and a full description of all signed similarity scalings of any matrix A to an integer matrix.

2. Background and Preliminary Results

In max-algebra, for $a, b \in \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$, we define $a \oplus b = \max(a, b)$, $a \otimes b = a + b$ and extend the pair $(\oplus, \otimes)$ to matrices and vectors in the same way as in linear algebra, that is (assuming compatibility of sizes)

$$\begin{aligned}(A \oplus B)_{ij} &= a_{ij} \oplus b_{ij},\\ (A \otimes B)_{ij} &= \bigoplus_k a_{ik} \otimes b_{kj} \text{ and}\\ (\alpha \otimes A)_{ij} &= \alpha \otimes a_{ij}.\end{aligned}$$

We denote the set $\mathbb{Z} \cup \{-\infty\}$ by $\overline{\mathbb{Z}}$.

In this paper X, Y, W, Z will always denote diagonal matrices, described by vectors x, y, w, z, thus $X = diag(x)$, $Y = diag(y)$ and so on. The size of these matrices will always be apparent from the problem. Note that it is well known that the only invertible matrices in the max-algebra are generalised permutation matrices [2, 5].

For positive integers m, n we denote $M = \{1, ..., m\}$ and $N = \{1, ..., n\}$.

If $A = (a_{ij}) \in \overline{\mathbb{R}}^{n\times n}$ then $A_{\bullet j}$ stands for the j^{th} column of A, and $A_{j\bullet}$ stands for the j^{th} row. When discussing columns of A, we may sometimes simply write A_j for the j^{th} column. The notation $A[I, J]$ stands for the submatrix of A with row indices from $I \subseteq N$ and column indices from $J \subseteq N$.

Note that α^{-1} stands for $-\alpha$, and we will use ϵ to denote $-\infty$ as well as any vector or matrix whose every entry is $-\infty$. Further $A^{(-1)}$ denotes $-A$.

A vector/matrix whose every entry belongs to $\mathbb{R}$ is called *finite*. If a matrix has no ϵ rows (columns) then it is called *row* (*column*) $\mathbb{R}$-*astic* and it is called *doubly* $\mathbb{R}$-*astic* if it is both row and column $\mathbb{R}$-astic.

Given $a \in \mathbb{R}$ we define $fr(a)$ to be the fractional part of a, that is $fr(a) = a - \lfloor a \rfloor$, additionally we define $fr(\epsilon) = \epsilon$. Further given a matrix $A \in \overline{\mathbb{R}}^{m\times n}$ we denote by $fr(A)$ the matrix with entries $fr(a_{ij})$. We say that A is *fractionally antisymmetric* if, for all $i \in M, j \in N$ either $fr(a_{ij}) = fr(-a_{ji})$ or $a_{ij} = \epsilon = a_{ji}$.

The following lemma describes simple properties of $fr(\cdot)$ which we will use throughout.

Lemma 2.1. *Let* $a, b, c \in \mathbb{R}$.

(i) $fr(-a) = \begin{cases} fr(a) & \text{if } a \in \mathbb{Z} \\ 1 - fr(a) & \text{otherwise} \end{cases}$

(ii)

$$fr(a + b) = fr(a + fr(b)) = fr(fr(a) + fr(b)) \text{ and}$$
$$fr(a - b) = fr(a - fr(b)) = fr(fr(a) - fr(b))$$

(iii) $a + b \in \mathbb{Z} \Leftrightarrow fr(a) = fr(-b)$

(iv) If $a + b \in \mathbb{Z}$, *then* $fr(a + c) = fr(-b + c)$.

(v) $fr(a) = fr(b + c) \Leftrightarrow fr(b) = fr(a - c)$.

(vi)

$$fr(a) = fr(b) \Leftrightarrow fr(a + c) = fr(b + c) \text{ and}$$
$$fr(a) = b \Rightarrow fr(a + c) = fr(b + c)$$

(vii) If $a + b \in \mathbb{Z}$ *and* $fr(b) = fr(c)$ *then* $a + c \in \mathbb{Z}$.

(viii) $fr(a) = fr(b) \Leftrightarrow fr(-a) = fr(-b)$.

Definition 2.2. *A vector* $x \in \mathbb{R}^n$ *is called* uni-fractional *if*

$$(\exists f \in [0, 1))(\forall i \in N) fr(x_i) = f.$$

Equivalently if $(\exists f \in \mathbb{R})(\exists z \in \mathbb{Z}^n) x = f \otimes z$.

*Given $f \in [0,1)$, an f-*fractional *vector is a uni-fractional vector $y \in \mathbb{R}^n$ with $fr(y_j) = f, \forall j \in N$.*

Definition 2.3. *A matrix $A \in \overline{\mathbb{R}}^{m\times n}$ is called a* direct sum *if it has the form*

$$\begin{pmatrix} A^{(1)} & & & \\ & A^{(2)} & & \\ & & \ddots & \\ & & & A^{(s)} \end{pmatrix}$$

where the blank inputs represent ϵ entries. Let $M_1, ..., M_s$ and $N_1, ..., N_s$ be the partitions of M and N respectively such that $A^{(p)} = A[M_p, N_p]$. For all $p \in \{1, ..., s\}$ define $m_p := |M_p|$ and $n_p := |N_p|$.

Definition 2.4. *A matrix $A \in \overline{\mathbb{R}}^{n\times n}$ is said to be in Frobenius Normal Form (FNF) if it has the form*

$$A = \begin{pmatrix} A_{11} & \epsilon & \dots & \epsilon \\ A_{21} & A_{22} & \dots & \epsilon \\ \vdots & \vdots & & \vdots \\ A_{r1} & A_{r2} & \dots & A_{rr} \end{pmatrix}$$

where A_{ii} are irreducible.

It is known [1] that any matrix can be transformed to FNF by a simultaneous permutation of the rows and columns in $\mathcal{O}(n^3)$ time.

In this paper we work with two different graphs obtained from a matrix $A \in \overline{\mathbb{R}}^{m\times n}$.

If $m = n$ then $D(A)$ is defined to be the directed graph with vertex set N, and the edge (i,j) exists if and only if $a_{ij} > \epsilon$. The weight of the edge (i,j) is a_{ij}.

By paths and cycles in $D(A)$ we mean elementary paths and cycles.

We define $BG(A)$ to be the bipartite graph with vertex sets M and N, and edges $(i,j) \in M \times N$ exist if and only if $a_{ij} > \epsilon$. The weight of the edge (i,j) is defined to be a_{ij}.

A matrix $A \in \overline{\mathbb{R}}^{n\times n}$ is called irreducible if $D(A)$ is strongly connected (if there is an $i - j$ path for all $i, j \in N$) or if $n = 1$. If $A \in \overline{\mathbb{R}}^{n\times n}$ is interpreted as a matrix of direct-distances in $D(A)$ then A^k (where k is a positive integer) is the matrix of the weights of heaviest paths with k arcs. For square matrices, $A^+ := A \oplus A^2 \oplus A^3 \oplus ... \oplus A^n$. Note that A^+ is the matrix of the weights of heaviest paths. Following this observation it is not difficult to deduce:

Lemma 2.5. *[2] Let $A \in \overline{\mathbb{R}}^{n\times n}$, $n \geq 2$. If A is irreducible then A^+_λ, is finite.*

Lemma 2.6 below is well known, see e.g. [2].

Lemma 2.6. *Let* $A \in \overline{\mathbb{R}}^{n\times n}, B \in \overline{\mathbb{R}}^{n\times n}$ *with* $B = X^{-1} \otimes A \otimes X$ *for some diagonal matrix* X*. Then* $w(\sigma, A) = w(\sigma, B)$ *for all cycles* σ*.*

The following is easily seen to be true.

Lemma 2.7. *Let* $A, B \in \overline{\mathbb{R}}^{m\times n}$ *and suppose* $fr(A) = fr(B)$*.*

(i) If $m = n$ *then, for all diagonal* X*,*

$$X^{-1} \otimes A \otimes X \in \overline{\mathbb{Z}}^{n\times n} \Leftrightarrow X^{-1} \otimes B \otimes X \in \overline{\mathbb{Z}}^{n\times n}.$$

(ii) For all diagonal X, Y*,*

$$X \otimes A \otimes Y \in \overline{\mathbb{Z}}^{m\times n} \Leftrightarrow X \otimes B \otimes Y \in \overline{\mathbb{Z}}^{m\times n}.$$

3. Diagonal Similarity Scaling

Given $A \in \overline{\mathbb{R}}^{n\times n}$ we say that A has a *diagonal similarity scaling to an integer matrix* if there exists X such that $X^{-1} \otimes A \otimes X \in \overline{\mathbb{Z}}^{n\times n}$.

3.1. A Necessary Condition

Definition 3.1. *A digraph* D *satisfies the* integer cycle property *(ICP) if for all cycles* σ *in* D*,* $w(\sigma, D) \in \mathbb{Z}$*.*

A matrix $A \in \overline{\mathbb{R}}^{n\times n}$ *satisfies the ICP if and only if* $D(A)$ *satisfies the ICP.*

Remark 1. Note that if D satisfies the ICP then every closed walk also has integer weight, this is because the cycles removed have fractional part zero.

The following result can be deduced from [6], and is also an immediate corollary of Lemma 2.6.

Proposition 3.2. *Let* $A \in \overline{\mathbb{R}}^{n\times n}$*. If there exists a diagonal similarity scaling of* A *to an integer matrix then* A *satisfies the ICP.*

3.2. The Finite Case

We show here that, for finite A, the ICP is necessary and sufficient (which follows from [6]) and additionally describe all diagonal similarity scalings of A to an integer matrix.

Theorem 3.3. *Let* $A \in \mathbb{R}^{n\times n}$ *satisfy the ICP. Then* A *has a diagonal similarity scaling to an integer matrix. Further* $X = diag(x)$ *satisfies* $X^{-1}\otimes A\otimes X \in \mathbb{Z}^{n\times n}$ *if and only if, for any fixed* $t \in N$*,*

$$x \in \{diag(A_t) \otimes u : u \text{ is uni-fractional}\}.$$

Proof. Fix $t \in N$. Let $u \in \mathbb{R}^n$ be uni-fractional with $fr(u_i) = f, \forall i \in N$.

[$\Leftarrow$]

Let $x = (x_i) \in \mathbb{R}^n$ where $x_i = a_{it} + u_i$. Then, using Lemma 2.1(vi), for any $i, j \in N$,

$$\begin{aligned} fr(-x_i + a_{ij} + x_j) &= fr(-(a_{it} + u_i) + a_{ij} + a_{jt} + u_j) \\ &= fr(-a_{it} - f + a_{ij} + a_{jt} + f) \\ &= fr(-a_{it} + a_{ij} + a_{jt}). \end{aligned} \tag{3.1}$$

Observe that $fr(-a_{it}) = fr(a_{ti})$ by the ICP and Lemma 2.1(iii). Therefore Equation (3.1) becomes

$$fr(-x_i + a_{ij} + x_j) = fr(a_{ti} + a_{ij} + a_{jt}) = 0,$$

again due to the ICP. Hence $X^{-1} \otimes A \otimes X \in \mathbb{Z}^{n \times n}$.

[$\Rightarrow$]

Assume that $X = diag(x)$ satisfies $X^{-1} \otimes A \otimes X \in \mathbb{Z}^{n \times n}$.

Let $f = fr(x_t)$, We show $(\forall i \in N)(\exists z \in \mathbb{Z})x_i = a_{it} + z + f$. Indeed, for all $i \in N$,

$$-x_i + a_{it} + x_t \in \mathbb{Z} \Rightarrow (\exists z' \in \mathbb{Z})x_i = a_{it} + (z' + \lfloor x_t \rfloor) + f.$$

□

From Proposition 3.2 and Theorem 3.3 we conclude.

Corollary 3.4. *Let $A \in \mathbb{R}^{n \times n}$. Then A has a diagonal similarity scaling to an integer matrix if and only if A satisfies the ICP.*

3.3. Extension to General Matrices

3.3.1. Irreducible matrices

We first show that any irreducible matrix satisfying the ICP can be completed to a finite matrix also satisfying the ICP, with the same set of diagonal similarity scalings to integer matrices. Recall from Lemma 2.5 that if $A \in \overline{\mathbb{R}}^{n \times n}$ is irreducible then A^+ is finite.

Proposition 3.5. *Let D be strongly connected. If D satisfies the ICP then, for any $i, j \in V(D)$, there exists $f \in [0, 1)$ such that all $i - j$ paths P have $fr(w(P)) = f$.*

Proof. Let P and P' be two $i - j$ paths in D. Let Q be any $j - i$ path in D. Since D satisfies the ICP,

$$w(P) + w(Q) \in \mathbb{Z} \text{ and } w(P') + w(Q) \in \mathbb{Z}.$$

Hence $fr(w(P)) = fr(-w(Q)) = fr(w(P'))$ as required. □

The next two corollaries follow immediately from Proposition 3.5.

Corollary 3.6. *Suppose A is irreducible and satisfies the ICP.*
(i) For all $i, j \in N$ such that a_{ij} is finite,

$$fr(a_{ij}) = fr(a_{ij}^k) \ \forall k \in \mathbb{N}.$$

(ii) A^k satisfies the ICP for all $k \in \mathbb{N}$, (note that it will follow from Corollary 3.9 that this implies $(\exists X) X^{-1} \otimes A^k \otimes X \in \overline{\mathbb{Z}}^{n\times n}$),
(iii) For every diagonal matrix X we have,

$$X^{-1} \otimes A \otimes X \in \overline{\mathbb{Z}}^{n\times n} \Leftrightarrow X^{-1} \otimes A^k \otimes X \in \overline{\mathbb{Z}}^{n\times n}.$$

Corollary 3.7. *Let $A \in \overline{\mathbb{R}}^{n\times n}$. Then A satisfies the ICP if and only if A^+ satisfies the ICP.*

Proof. Since $(\forall i, j \in N) a_{ij}^+ = w(P_{ij})$ any cycle in $D(A^+)$ also represents a closed walk in $D(A)$ with the same fractional part, the sufficient direction follows. For the necessary direction note that for all finite a_{ij}, we have $fr(a_{ij}) = fr(a_{ij}^+)$. □

Proposition 3.8. *Let $A \in \overline{\mathbb{R}}^{n\times n}$ be irreducible and satisfy the ICP. Then*

$$X^{-1} \otimes A \otimes X \in \overline{\mathbb{Z}}^{n\times n} \Leftrightarrow X^{-1} \otimes A^+ \otimes X \in \mathbb{Z}^{n\times n}.$$

Proof. [⇐]
For any finite a_{ij}, by Proposition 3.5, $fr(a_{ij}) = fr(a_{ij}^+)$ since they are both weights of $i - j$ paths in $D(A)$. Clearly then, from Lemma 2.1(vi),

$$fr(-x_i + a_{ij} + x_j) = fr(-x_i + a_{ij}^+ + x_j) = 0.$$

[⇒]

$$X^{-1} \otimes A \otimes X \in \overline{\mathbb{Z}}^{n\times n} \Rightarrow X^{-1} \otimes A \otimes X \otimes X^{-1} \otimes A \otimes X \in \overline{\mathbb{Z}}^{n\times n}.$$

It follows that, for all $r \in \mathbb{N}$, $X^{-1} \otimes A^r \otimes X \in \overline{\mathbb{Z}}^{n\times n}$ and hence also $X^{-1} \otimes A^+ \otimes X \in \mathbb{Z}^{n\times n}$. □

From Proposition 3.2, Corollary 3.7 and Proposition 3.8 we conclude.

Corollary 3.9. *Let $A \in \overline{\mathbb{R}}^{n\times n}$ be irreducible. Then A has a diagonal similarity scaling to an integer matrix if and only if A satisfies the ICP. Further the set of all scalings can be described using Theorem 3.3 applied to A^+.*

3.3.2. Reducible matrices

Corollary 3.10. *Let $A \in \overline{\mathbb{R}}^{n\times n}$ be a direct sum (see Definition 2.3) with square, irreducible blocks $A^{(1)}, A^{(2)}, ..., A^{(s)}$. Then there exists a diagonal similarity scaling of A to an integer matrix if and only if A satisfies the ICP. Further X satisfies $X^{-1}\otimes A\otimes X \in \overline{\mathbb{Z}}^{n\times n}$ if and only if $X[N_p, N_p]^{-1}\otimes A^{(p)}\otimes X[N_p, N_p] \in \overline{\mathbb{Z}}^{n_p\times n_p}$ for all $p = 1, ..., s$. Hence all scalings of A can be described by combining the solutions for each irreducible submatrix, for which we use Corollary 3.9.*

Let $A \in \overline{\mathbb{R}}^{n\times n}$ be reducible and in FNF (see Definition 2.4). This can be assumed without loss of generality since a simultaneous permutation of the rows and columns will only change the labelling of the vertices in $D(A)$, and the order of the diagonal elements in X.

Given a matrix $B \in \overline{\mathbb{R}}^{n\times n}$ we define $B^{(-)} = (b^{(-)}_{ij})$ by

$$b^{(-)}_{ij} = \begin{cases} b^{-1}_{ij}, \text{if } b_{ij} > \epsilon \\ b_{ij}, \text{if } b_{ij} = \epsilon. \end{cases}$$

Proposition 3.11. *Let $A \in \overline{\mathbb{R}}^{n\times n}$ be in FNF. Let*

$$\bar{A} = \begin{pmatrix} A_{11} & (A^{(-)}_{21})^T & \dots & (A^{(-)}_{r1})^T \\ A_{21} & A_{22} & \dots & (A^{(-)}_{r2})^T \\ \vdots & \vdots & & \vdots \\ A_{r1} & A_{r2} & \dots & A_{rr} \end{pmatrix}.$$

Then

$$X^{-1} \otimes A \otimes X \in \overline{\mathbb{Z}}^{n\times n} \Leftrightarrow X^{-1} \otimes \bar{A} \otimes X \in \overline{\mathbb{Z}}^{n\times n}.$$

Proof. [$\Leftarrow$] Trivial since $(\forall i \geq j) a_{ij} = \bar{a}_{ij}$.

[$\Rightarrow$]

Assume $X^{-1}\otimes A\otimes X \in \overline{\mathbb{Z}}^{n\times n}$. For $i \geq j$ the result holds trivally. So assume $i < j$, then

$$-x_j + a_{ji} + x_i \in \mathbb{Z} \Rightarrow -x_i + \bar{a}_{ij} + x_j = -(-x_j + a_{ji} + x_i) \in \mathbb{Z}.$$

□

Corollary 3.12. *Let $A \in \overline{\mathbb{R}}^{n\times n}$. Then A has a diagonal similarity scaling to an integer matrix if and only if $\bar{A}$ satisfies the ICP.*

If such a scaling exists then, since $\bar{A}$ is a direct sum (with s irreducible diagonal blocks, $1 \leq s \leq r$), all diagonal similarity scalings can be described using Corollary 3.10.

Theorem 3.13. *Given a digraph D on n vertices, it is possible to decide whether D satisfies the ICP in $\mathcal{O}(n^4)$ time.*

Proof. Let A be the adjacency matrix of D. Then D satisfies the ICP if and only if each strongly connected component of A satisfies the ICP. Hence, after writing A in FNF, which requires $\mathcal{O}(n^3)$ operations, we need only check whether each of the (at most n) irreducible blocks A_{ss} satisfies the ICP.

By Corollary 3.9 we can decide whether A_{ss} satisfies the ICP by checking whether there exists a diagonal X such that $X^{-1} \otimes A_{ss} \otimes X \in \overline{\mathbb{Z}}^{n\times n}$. Since we know the description of all possible vectors x with $diag(x) = X$, it suffices to choose one and check. The most computationally expensive part of this is computing $X^{-1} \otimes A_{ss} \otimes X$, and possibly computing A_{ss}^{+}, both of which are $\mathcal{O}(n^3)$. □

3.4. A Note on Eigenvectors and Visualisation

This subsection is self contained, and notes briefly some observations about the links between diagonal scaling to an integer matrix and eigenvectors in the max algebra.

The following lemma is trivial to prove.

Lemma 3.14. *Let $A \in \overline{\mathbb{R}}^{n\times n}$ and $\gamma \in \mathbb{Z}$. Then*

$$X^{-1} \otimes A \otimes X \in \overline{\mathbb{Z}}^{n\times n} \Leftrightarrow X^{-1} \otimes (\gamma^{-1}A) \otimes X \in \overline{\mathbb{Z}}^{n\times n}$$

We say that a matrix $A \in \overline{\mathbb{R}}^{n\times n}$ is *definite* if,

$$\bigoplus_{\text{cycle } \sigma} w(\sigma, A) = 0.$$

Observe that the above lemma means that we could assume, without loss of generality, that the matrix we are considering is definite.

An *eigenvector* with respect to eigenvalue 0 is $x \in \overline{\mathbb{R}}^{n}$ such that $A \otimes x = x$. The following result is well known.

Proposition 3.15. *[5]Let $A \in \overline{\mathbb{R}}^{n\times n}$ be definite. Then any eigenvector is a max-combination of the columns of A^{+} with $a_{jj} = 0$.*

An eigenvector x is called *fundamental* if there exists $j \in N$ such that $x = A_{j}^{+}$ and $a_{jj}^{+} = 0$.

From Proposition 3.15 and Corollary 3.9 we conclude the following.

Corollary 3.16. *Let A be irreducible and definite. Then any diagonal X such that $X^{-1} \otimes A \otimes X \in \overline{\mathbb{Z}}^{n\times n}$ is generated by*

$$x \in \{diag(u) \otimes y : y \text{ is a fundamental eigenvector of } A, u \text{ is uni-fractional}\}.$$

Example 3.1. *Note that this does not hold for <u>any</u> eigenvector. Let*

$$A = \begin{pmatrix} 0 & -0.6 \\ -1.4 & 0 \end{pmatrix}.$$

Then $(0, 0)^T$ is an eigenvector but cannot be used to scale A to an integer matrix.

We end this subsection with a note on visualizing to an integer matrix. A matrix $A \in \overline{\mathbb{R}}^{n\times n}$ is *visualised* if

$$a_{ij} \leq \lambda(A) \forall i, j \in N$$

and

$$a_{ij} = \lambda(A) \forall (i,j) \in E_c(A).$$

Proposition 3.17. *[] Let $A \in \overline{\mathbb{R}}^{n\times n}$ be definite. If $A \otimes x \leq x$ then $X^{-1} \otimes A \otimes X$ is visualised for $X = diag(x)$.*

We ask whether there exists X such that $X^{-1} \otimes A \otimes X$ is visualised and integer. Obviously a necessary condition is that A satisfies the ICP.

Proposition 3.18. *Let $A \in \overline{\mathbb{R}}^{n\times n}$ be irreducible and definite. If A satisfies the ICP then $X^{-1} \otimes A \otimes X \in \overline{\mathbb{Z}}^{n\times n}$ and is visualised for*

$$x \in \{y : y \text{ is a fundamental eigenvector of } A\}.$$

Proof. Follows from Proposition 3.17 and Proposition 3.8. □

3.5. A note on even cycles in digraphs

This is another self contained section, and can be skipped without affecting readability of the rest of the paper.

The following results can easily be seen to be true.

Proposition 3.19. *Let $A \in \{\epsilon, 0\}^{n\times n}$. Then every cycle in $D(A)$ has even length if and only if A satisfies the ICP.*

Proposition 3.20. *Let $A \in \mathbb{Z}^{n\times n}$. Then every cycle in $D(A)$ has even weight if and only if $\frac{1}{2} \times A$ satisfies the ICP.*

Corollary 3.21. *Let $A \in \mathbb{Z}^{n\times n}$. Then every cycle in $D(A)$ has $w(\sigma) \equiv 0$* mod k *if and only if $\frac{1}{k} \times A$ satisfies the ICP.*

4. Direct Similarity Scaling

Given $A \in \overline{\mathbb{R}}^{m\times n}$ we say that A has a *direct similarity scaling to an integer matrix* if there exist X, Y such that $X \otimes A \otimes Y \in \overline{\mathbb{Z}}^{m\times n}$.

Using the known techniques for transforming the problem of finding a direct similarity scaling to the problem of finding a diagonal similarity scaling, see eg [7, 6, 9], we immediately get the following result.

Proposition 4.1. *Let $A \in \overline{\mathbb{R}}^{m\times n}$. Then*

$$(\exists Y, Z) Z \otimes A \otimes Y \in \overline{\mathbb{Z}}^{m\times n} \Leftrightarrow (\exists X) X^{-1} \otimes \begin{pmatrix} \epsilon & A \\ \epsilon & \epsilon \end{pmatrix} \otimes X \in \overline{\mathbb{Z}}^{(m+n)\times(m+n)}.$$

Therefore we could use our results on diagonal similarity scalings of this $(m + n) \times (m + n)$ matrix to solve the problem of finding a direct similarity scaling of $A \in \overline{\mathbb{R}}^{m\times n}$. However, to give the simplest possible description of all scalings, and to allow us to use our methods to solve the signed scaling problem later, we develop separate methods which do not increase the size of the matrix we are studying.

4.1. A Necessary Condition

Note that cycles in $BG(A)$ are

$$(i_1, j_1), (i_2, j_1), (i_2, j_2), ..., (i_t, j_t), (i_1, j_t).$$

This cycle has length $2t$. We define the *alternating sign weight* of a cycle σ in $BG(A)$ to be

$$asw(\sigma) := \sum_{p=1}^{t} a_{i_p,j_p} - \sum_{p=1}^{t} a_{i_{p+1},j_p}$$

where indices in the summation are calculated modulo t.

Definition 4.2. *A bipartite graph G satisfies the* bipartite integer cycles property *(BICP) if*

$$\forall \text{ cycles } \sigma : asw(\sigma, G) \in \mathbb{Z}.$$

A matrix $A \in \overline{\mathbb{R}}^{m\times n}$ satisfies the BICP if $BG(A)$ does.

Proposition 4.3. *Let $A \in \overline{\mathbb{R}}^{m\times n}$. If A has a direct similarity scaling to a $\overline{\mathbb{Z}}$ matrix then $BG(A)$ satisfies the BICP.*

Proof. Suppose $X \otimes A \otimes Y \in \overline{\mathbb{Z}}^{m\times n}$ and A satisfies the BICP. Take any cycle σ in $BG(A)$, say

$$(i_1, j_1), (i_2, j_1), (i_2, j_2), ..., (i_t, j_t), (i_1, j_t).$$

Then

$$\sum_{p=1}^{t}(x_{i_p} + a_{i_p j_p} + y_{j_p}) - \sum_{p=1}^{t}(x_{i_{p+1}} + a_{i_{p+1} j_p} + y_{j_p}) \in \mathbb{Z}$$

since every term in both summations is integer. It follows that $asw(\sigma, A) \in \mathbb{Z}$ as required. □

4.2. The Finite Case

Following the same structure as in Section 3 we first show that the BICP is necessary and sufficient for a finite matrix A to have a direct similarity scaling to an integer matrix (again this result can be deduced from [6]). We further describe all such scalings in the case when they exist.

Theorem 4.4. *Let $A \in \mathbb{R}^{m\times n}$ satisfy the BICP. Then there exist diagonal X, Y such that $X \otimes A \otimes Y \in \mathbb{Z}^{m\times n}$. Further, $X \otimes A \otimes Y \in \mathbb{Z}^{m\times n}$ if and only if, for a fixed $t \in M \cap N$,*

$$y \in \left\{ diag(u) \otimes A_{t\bullet}^{(-1)} : u \text{ is uni-fractional} \right\} := \mathcal{Y}$$

and

$$x = x(y) \in \left\{ diag(v) \otimes A_{\bullet t}^{(-1)} : v \text{ is } fr(-y_t)\text{-fractional} \right\} := \mathcal{X}(y).$$

Proof. Observe that $(x(y), y) \in \mathcal{X}(y) \times \mathcal{Y}$ if and only if there exists $f \in [0, 1)$ such that

$$(\forall j \in N) fr(y_j) = fr(f - a_{tj}) \text{ and } (\forall i \in M) fr(x_i) = fr(-f + a_{tt} - a_{it}).$$

[⇐] If $(x(y), y) \in \mathcal{X}(y) \times \mathcal{Y}$ then, for all $i \in M, j \in N$,

$$\begin{aligned} fr(x_i + a_{ij} + y_j) &= fr(-f + a_{tt} - a_{it} + a_{ij} + f - a_{tj}) \\ &= fr(a_{tt} - a_{it} + a_{ij} - a_{tj}) = 0 \end{aligned}$$

where the final equality is either by the BICP or, if either $i = t$ or $j = t$, by cancellation.

[⇒]Assume that $X \otimes A \otimes Y \in \mathbb{Z}^{m\times n}$. Let $f = fr(y_t + a_{tt})$.

Note that, for all $i \in M$,

$$x_i + a_{it} + y_t \in \mathbb{Z} \Rightarrow (\exists z \in \mathbb{Z}) x_i = z - y_t - a_{it} \text{ so } fr(x_i) = fr(-y_t - a_{it}).$$

Thus $x \in \mathcal{X}(y)$.

Similarly, for all $j \in N$,

$$fr(y_j) = fr(-x_t - a_{tj}) = fr(y_t + a_{tt} - a_{tj}) = fr(f - a_{tj})$$

where the second equality uses Lemma 2.1 and the substitution $fr(x_t) = fr(-y_t - a_{tt})$ from above. We conclude $y \in \mathcal{Y}$ to complete the proof. □

From Proposition 4.3 and Theorem 4.4 we conclude.

Corollary 4.5. *Let $A \in \mathbb{R}^{m\times n}$. Then A has a direct similarity scaling to an integer matrix if and only if A satisfies the BICP.*

4.3. Extension to General Matrices

4.3.1. The Connected Case

We now consider the case when $BG(A)$ is connected, and show that any such matrix satisfying the BICP can be completed to a finite matrix also satisfying the BICP, and which has the same set of direct similarity scalings to an integer matrix. Note that we can again assume without loss of generality that $A \in \overline{\mathbb{R}}^{m\times n}$ is doubly $\mathbb{R}$-astic.

Algorithm 4.6. *Input: Connected, bipartite B satisfying BICP.*
Output: $B^{(C)}$, complete, satisfying BICP.
1: For all $i \in M$;
For all $j \in N$;
If $(i,j) \notin E(B)$ then $B := B \cup \{(i,j)\}$. Let $P = (i = i_1, j_1, i_2, j_2, ..., i_t, j_t = j)$ be a path in B, then

$$w_{ij} := fr(w_{i_1 j_1} - w_{i_2 j_1} + w_{i_2 j_2} - ... + w_{i_t j_t}).$$

2: $B^{(C)} = B$.

Proposition 4.7. *Algorithm 4.6 is correct and runs in $\mathcal{O}((mn)^2)$ time.*

Proof. We show that, if B is connected and satisfies the BICP and we add one edge according to Step 1 then the graph obtained, B' say, also satisfies the BICP.

Let $\{(i,j)\} = E(B') - E(B)$. Let

$$P = (i = i_1, j_1, i_2, j_2, ..., i_t, j_t = j)$$

be the path in B used to define w_{ij}. Clearly any cycle in B' not containing (i,j) will have integer alternating sign weight. Further the cycle obtained by adding (i,j) to P also has integer alternating sign weight be construction. So assume that P' is an elementary path in B' between i and j different from P and (i,j) (if no such P' exists then we are done).

Suppose $P' = (i = p_1, q_1, p_2, q_2, ..., p_s, q_s = j)$. Then $P' \cup P$ is a cycle (closed walk) in B, and hence

$$w_{i_1 j_1} - w_{i_2 j_2} + ... - w_{i_t j_{t-1}} + w_{i_t, j_t} - w_{p_s q_s} + w_{p_s q_{s-1}} - ... + w_{p_2 q_1} - w_{p_1 q_1} \in \mathbb{Z}.$$

Therefore

$$fr(w_{i_1 j_1} - w_{i_2 j_2} + w_{i_2 j_2} - ... + w_{i_t j_t}) = fr(w_{p_s q_s} - w_{p_s q_{s-1}} + ... - w_{p_2 q_1} + w_{p_1 q_1})$$

by Lemma 2.1(iii) which implies

$$fr(asw(P \cup (i,j))) = fr(asw(P' \cup (i,j)))$$

and hence all cycles in B' have integer alternating sign weight.

For the complexity note that for each of at most mn edges we find a path from one endpoint to the other, and its weight, this requires $\mathcal{O}(mn)$ operations. □

Given $A \in \overline{\mathbb{R}}^{m \times n}$ satisfying the BICP, input $BG(A)$ into Algorithm 4.6 and define $A^{(C)}$ to be the matrix corresponding to $BG(A)^{(C)}$.

Proposition 4.8. *Let $A \in \overline{\mathbb{R}}^{m \times n}$ satisfy the BICP and suppose $BG(A)$ is connected. Then*

$$X \otimes A \otimes Y \in \overline{\mathbb{Z}}^{m \times n} \Leftrightarrow X \otimes A^{(C)} \otimes Y \in \mathbb{Z}^{m \times n}.$$

Proof. [$\Leftarrow$] Obvious.

[$\Rightarrow$] Fix $i \in M, j \in N$. If $a_{ij} > \epsilon$ then $fr(x_i + a_{ij} + y_j) = 0$ since $a_{ij} = a_{ij}^{(C)}$. So assume not.

Since $BG(A)$ is connected there exists a path

$$P = (i = i_1, j_1, i_2, j_2, ..., i_r, j_r = j)$$

between i and j in $BG(A)$. Now, since $X \otimes A \otimes Y \in \overline{\mathbb{Z}}^{m\times n}$,

$$(x_{i_1}+a_{i_1j_1}+y_{j_1})-(x_{i_2}+a_{i_2j_1}+y_{j_1})+...-(x_{i_r}+a_{i_rj_{r-1}}+y_{j_{r-1}})+(x_{i_r}+a_{i_rj_r}+y_{j_r}) \in \mathbb{Z}.$$

After cancelling, and using $x_{i_1} = x_i, y_{j_r} = y_j$,

$$x_i + a_{i_1j_1} - a_{i_2j_1} + ... - a_{i_rj_{r-1}} + a_{i_rj_r} + y_j \in \mathbb{Z}. \tag{4.1}$$

Now take any path P' in $BG(A)^{(C)}$, suppose $P' = (i = i_1', j_1', i_2', j_2', ..., i_s', j_s' = j)$. Since $BG(A)^{(C)}$ satisfies the BICP by Proposition 4.7 we have

$$fr(a_{ij}^{(C)}) = fr(a_{i_1'j_1'} - a_{i_2'j_1'} + ... - a_{i_s'j_{s-1}'} + a_{i_s'j_s'}). \tag{4.2}$$

Also $P \cup P'$ is a cycle in $BG(A)^{(C)}$ so has integer alternating sign weight, hence,

$$fr(a_{i_1j_1} - a_{i_2j_1} + ... - a_{i_rj_{r-1}} + a_{i_rj_r}) = fr(a_{i_1'j_1'} - a_{i_2'j_1'} + ... - a_{i_s'j_{s-1}'} + a_{i_s'j_s'}). \tag{4.3}$$

Therefore, by (4.2) and (4.3),

$$fr(a_{ij}^{(C)}) = fr(a_{i_1j_1} - a_{i_2j_1} + ... - a_{i_rj_{r-1}} + a_{i_rj_r}). \tag{4.4}$$

Finally, using (4.1), (4.4) and Lemma 2.1(vi), we get

$$x_i + a_{ij}^{(C)} + y_j \in \mathbb{Z}$$

as required. $\square$

From Propositions 4.3 and 4.8 we conclude the following.

Corollary 4.9. *Let $A \in \overline{\mathbb{R}}^{m\times n}$ and suppose $BG(A)$ is connected. Then A has a direct similarity scaling to an integer matrix if and only if A satisfies the BICP. Further the set of all scalings can be described using Theorem 4.4 applied to $A^{(C)}$.*

4.3.2. The Non-Connected Case

The following key points are easy to prove.

Lemma 4.10. *Given $A \in \overline{\mathbb{R}}^{m\times n}$, we can assume without loss of generality that A is a direct sum (see Definition 2.3) with rectangular blocks $A^{(1)}, A^{(2)}, ..., A^{(s)}$, each satisfying that $BG(A^{(p)})$ is connected.*

Further, for all X, Y, it is clear that

$$X \otimes A \otimes Y \in \overline{\mathbb{Z}}^{m\times n} \Leftrightarrow (\forall p) X[M_p, M_p]^{-1} \otimes A^{(p)} \otimes Y[N_p, N_p] \in \overline{\mathbb{Z}}^{m_p \times n_p}.$$

Additionally A satisfies the BICP if and only if, for all p, $A^{(p)}$ satisfies the BICP.

Definition 4.11. *Given $A \in \overline{\mathbb{R}}^{m\times n}$. Let*

$$\bar{A}^{(C)} = \begin{pmatrix} A^{(C1)} & & & \\ & A^{(C2)} & & \\ & & \ddots & \\ & & & A^{(Cs)} \end{pmatrix}$$

where $A^{(Ci)} = (A^{(i)})^{(C)}$ for all $i = 1, ..., s$.

Using Lemma 4.10 the following result is immediate from Proposition 4.8.

Corollary 4.12. *Let $A \in \overline{\mathbb{R}}^{m\times n}$ satisfy the BICP and $\bar{A}^{(C)}$ be as defined in Definition 4.11. Then*

$$X \otimes A \otimes Y \in \overline{\mathbb{Z}}^{m\times n} \Leftrightarrow X \otimes \bar{A}^{(C)} \otimes Y \in \overline{\mathbb{Z}}^{m\times n}.$$

The next two corollaries follow from the results in this section.

Corollary 4.13. *Let $A \in \overline{\mathbb{R}}^{m\times n}$. Then there exists a direct similarity scaling of A to an integer matrix if and only if A satisfies the BICP. Further the set of all such scalings can be described as the set of all solutions obtained by combining the solutions for each block $A^{(Cp)}$ of $\bar{A}^{(C)}$ (see Definition 4.11), for which we use Theorem 4.4.*

Corollary 4.14. *Given a bipartite graph B with vertex sets of size m and n, we can decide in $\mathcal{O}((mn)^2)$ time whether B satisfies the BICP.*

5. The ICP and the BICP: Links and Observations

Here we begin by showing that, in the finite case, the ICP and the BICP can be verified using only short cycles. Later we investigate when the two properties are equivalent.

5.1. Short Cycles

Here we consider only finite matrices.

Proposition 5.1. *Let $A \in \mathbb{R}^{n\times n}$. Then A satisfies the ICP if and only if the weight of cycle with length 1, 2 or 3 is integer.*

Proof. We prove that if σ is a cycle in $D(A)$ then $w(\sigma, A) \in \mathbb{Z}$ by induction on $l(\sigma)$ (the length of the cycle). Obviously the statement holds for $l(\sigma) = 1, 2, 3$. So suppose $l(\sigma) = t \geq 4$ and the statement holds for any cycle of smaller length.

Assume $\sigma = (i_1, i_2, ..., i_t, i_1)$. Then, since 2-cycles have integer weight,

$$\begin{aligned} fr(w(\sigma, A)) &= fr(w(\sigma, A) + a_{i_1 i_3} + a_{i_3 i_1}) \\ &= fr(w(\tau, A) + w(\tau', A)) \end{aligned}$$

where $\tau = (i_1, i_2, i_3, i_1)$ and $\tau' = (i_1, i_3, i_4, ..., i_t, i_1)$ are cycles in $D(A)$ with $l(\tau) = 3, l(\tau') = t - 1$. Since τ and τ' both have integer weight by induction, it follows that σ has integer weight. □

Observe that the finiteness assumption is necessary, this can be seen by considering, for example, a matrix for which $D(A)$ is a directed cycle of length 4.

Proposition 5.2. *Let $A \in \mathbb{R}^{m\times n}$. Then*

$$(\forall \text{ cycles } \sigma \text{ of length } 4) asw(\sigma, A) \in \mathbb{Z} \Leftrightarrow A \text{ satisfies BICP.}$$

Proof. [$\Leftarrow$] Obvious

[$\Rightarrow$] We prove any cycle of length $2t$, $4 \leq t \in \mathbb{N}$ has integer alternating sign weight by induction. Obviously it is true for $t = 2$.

Consider any cycle $\sigma = (i_1, j_t)(i_1, j_1)(i_2, j_1), ..., (i_t, j_t)$ of length $2t$, $t \geq 3$. Then

$$\begin{aligned} asw(\sigma, A) &= -a_{i_1 j_t} + a_{i_1 j_1} - a_{i_2 j_1} + a_{i_2 j_2} - a_{i_3 j_2} + ... + a_{i_t j_t} \\ &= -a_{i_1 j_t} + a_{i_1 j_1} - a_{i_2 j_1} + (a_{i_1 j_2} - a_{i_1 j_2}) + a_{i_2 j_2} - a_{i_3 j_2} + ... + a_{i_t j_t} \in \mathbb{Z} \end{aligned}$$

since both cycles $(i_1, j_1)(i_2, j_1)(i_2, j_2)(i_1, j_2)$ and $(i_1, j_2)(i_3, j_2)(i_3, j_3)...(i_t, j_t)(i_1, j_t)$ have integer weight by induction hypothesis. □

Again, the finiteness assumption is necessary, consider for example a matrix for which $BG(A)$ is a single cycle of length 6.

5.2. When the ICP and the BICP are Equivalent

Proposition 5.3. *Let $A \in \mathbb{R}^{n\times n}$ have $a_{ii} \in \mathbb{Z}, \forall i \in N$. Then A satisfies the ICP if and only if A satisfies the BICP.*

Proof. Suppose first that A satisfies the ICP, so $fr(a_{ij}) = fr(-a_{ji})$ for all $i, j \in N$. Consider any cycle $\sigma = (i_1, j_1)(i_2, j_1)(i_2, j_2)(i_1, j_2)$ of length 4 in $BG(A)$. Now,

$$\begin{aligned} fr(asw(\sigma, A)) &= fr(a_{i_1 j_1} - a_{i_2 j_1} + a_{i_2 j_2} - a_{i_1 j_2}) \\ &= fr(a_{i_1 j_1} + a_{j_1 i_2} + a_{i_2 j_2} + a_{j_2 i_1}) = 0 \end{aligned}$$

where the last equality is due to the ICP.

By Proposition 5.2 we conclude that A satisfies the BICP.

For the other direction assume that A satisfies the BICP and has $a_{ii} \in \mathbb{Z}, \forall i \in N$. Then A is fractionally antisymmetric since we can take the cycle

$$(i_1, j_1)(i_2, j_1)(i_2, j_2)(i_1, j_2)$$

with $p = i_1 = j_1, q = i_2 = j_2$ which has integer alternating sign weight, implying

$$fr(a_{pq} + a_{qp}) = fr(a_{pp} + a_{qq}) = 0.$$

Thus clearly the weight of any cycle in $D(A)$ of length 1 or 2 is integer. By Proposition 5.1 it is enough to show that the weight of any cycle in $D(A)$ of length 3 is integer. Let $\sigma = (i_1, i_2, i_3, i_1)$. Consider the cycle $\tau = (i_1, i_2)(i_2, i_2)(i_2, i_3)(i_1, i_3)$ in $BG(A)$, which has integer alternating sign weight. Then

$$\begin{aligned} 0 &= fr(a_{i_1 i_2} - a_{i_2 i_2} + a_{i_2 i_3} - a_{i_1 i_3}) = fr(a_{i_1 i_2} + a_{i_2 i_3} - a_{i_1 i_3}) \\ &= fr(a_{i_1 i_2} + a_{i_2 i_3} + a_{i_3 i_1}) = w(\sigma, A). \end{aligned}$$

□

Corollary 5.4. *Let $A \in \overline{\mathbb{R}}^{n \times n}$ be irreducible with $a_{ii} \in \mathbb{Z}, \forall i \in N$. Then A satisfies the ICP if and only if A satisfies the BICP.*

Proof. Note that, since $D(A)$ is strongly connected and contains all loops, $BG(A)$ is connected. Thus we can complete A in two ways to obtain the finite matrices A^+ and $A^{(C)}$. Now, for all i, j such that $a_{ij} > \epsilon$, $fr(a_{ij}) = fr(a^+_{ij})$, hence,

$$\begin{aligned} A \text{ satisfies the ICP } &\Rightarrow A^+ \text{ satisfies the ICP } \Rightarrow A^+ \text{ satisfies the BICP} \\ &\Rightarrow A \text{ satisfies the BICP.} \end{aligned}$$

Additionally, using that for all i, j such that $a_{ij} > \epsilon$, $a_{ij} = a^{(C)}_{ij}$,

$$\begin{aligned} A \text{ satisfies the BICP } &\Rightarrow A^{(C)} \text{ satisfies the BICP } \Rightarrow A^{(C)} \text{ satisfies the ICP} \\ &\Rightarrow A \text{ satisfies the ICP.} \end{aligned}$$

□

In the general case the result is only true in one direction.

Corollary 5.5. *Let $A \in \overline{\mathbb{R}}^{n\times n}$ have $a_{ii} \in \mathbb{Z}, \forall i \in N$. If A satisfies the BICP then A satisfies the ICP.*

Proof. Let $\bar{A}^{(C)}$ be as described in Corollary 4.12. So $\bar{A}^{(C)}$ is a direct sum of matrices $\bar{A}^{(C)}_{ss}, s = 1, ..., r'$, where each $\bar{A}^{(C)}_{ss}$ describes a connected bipartite graph and has with integer diagonal. Most importantly, if $a_{ij} > \epsilon$ then $a_{ij} = \bar{a}^{(C)}_{ij}$.

Then

$$\begin{aligned} A \text{ satisfies the BICP } &\Rightarrow \bar{A}^{(C)} \text{ satisfies the BICP } \Rightarrow (\forall s)\bar{A}^{(C)}_{ss} \text{ satisfies the BICP} \\ &\Rightarrow (\forall s)\bar{A}^{(C)}_{ss} \text{ satisfies the ICP } \Rightarrow A \text{ satisfies the ICP.} \end{aligned}$$

□

Example 5.1. *Consider*

$$A = \begin{pmatrix} 0 & 0.1 \\ 0.2 & \epsilon \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & \epsilon & \epsilon \\ 0.1 & 0 & \epsilon \\ 0.4 & 0.2 & 0 \end{pmatrix}.$$

Note that A satisfies the BICP but does not satisfy the ICP.

Observe that the condition that the matrix has an integer diagonal in the above results cannot be reduced to $a_{ii} \in \overline{\mathbb{Z}}$, for example B satisfies the BICP trivially, but does not satisfy the ICP.

The following corollary of Proposition 5.3 will be useful when considering signed scalings.

Corollary 5.6. *Let $A \in \mathbb{R}^{n\times n}$. If A satisfies the ICP, then A satisfies the BICP and additionally, A is fractionally antisymmetric.*

Proof. If A satisfies the ICP then it satisfies the BICP by Proposition 5.3. Further, from the proof of that proposition, A is fractionally antisymmetric.

□

6. Symmetric Scaling

Let $A \in \overline{\mathbb{R}}^{n\times n}$. Here we consider the problem of determining whether there exists X such that $X \otimes A \otimes X \in \overline{\mathbb{Z}}^{n\times n}$.

Proposition 6.1. *Let $A \in \mathbb{R}^{n\times n}$. Then there exists X such that $X \otimes A \otimes X \in \mathbb{Z}^{n\times n}$ if and only if A satisfies the BICP and $fr(A)$ is symmetric.*

Proof. [⇒] Obviously $X \otimes A \otimes X$ is an example of a direct similarity scaling and hence the BICP is necessary from Corollary 4.3. To see that we also require $fr(A)$ symmetric, note that $fr(x_i + a_{ij} + x_j) = 0 = fr(x_j + a_{ji} + x_i)$ and cancel using Lemma 2.1(vi).

[⇐] Let $x = \frac{fr(a_{11})}{2}(a_{11}^{-1}, a_{12}^{-1}, ..., a_{in}^{-1})^T$ and $X = diag(x)$. Then, for all $i, j \in N$,

$$\begin{aligned} fr(x_i + a_{ij} + x_j) &= fr\left(\frac{fr(a_{11})}{2} - a_{i1} + a_{ij} + \frac{fr(a_{11})}{2} - a_{1j}\right) \\ &= fr(a_{11} - a_{1i} + a_{ij} - a_{1j}) = 0 \end{aligned}$$

where the final equality holds as follows. If $i, j \neq 1$ and $i \neq j$ then it holds by the BICP. If $i \neq j$ and either $i = 1$ or $j = 1$ then it holds by cancellation. Finally if $i = j$ then $fr(a_{11} - a_{1i} + a_{ii} - a_{1i}) = fr(a_{11} - a_{1i} + a_{ii} - a_{i1})$ since $fr(a_{i1}) = fr(a_{1i})$ and we can again apply the BICP. Hence $X \otimes A \otimes X \in \mathbb{Z}^{n\times n}$. □

As before, we can also describe all signed similarity scalings of A to an integer matrix.

Theorem 6.2. *Let $A \in \mathbb{R}^{n\times n}$ satisfy the BICP and suppose $fr(A)$ is symmetric. Then $X \otimes A \otimes X \in \mathbb{Z}^{n\times n}$ if and only if $X = diag(x)$ where, for some fixed $t \in N$*

$$x \in \left\{ diag(u) \otimes A_t^{-1} : \ u \text{ is } \frac{fr(a_{tt})}{2}\text{-fractional or } \frac{1 + fr(a_{tt})}{2}\text{-fractional.} \right\} := \mathbb{X}$$

Proof. Let $\mathcal{X}(x), \mathcal{Y}$ be as defined in Proposition 4.4. We show that $x \in \mathbb{X} \Leftrightarrow x \in \mathcal{X}(x) \cap \mathcal{Y}$. Recall that $x \in \mathcal{X}(x) \cap \mathcal{Y}$ if and only if, for some fixed $t \in N$,

$$(\forall j \in N) fr(x_j) = fr(f - a_{tj}) = fr(-f + a_{tt} - a_{jt})$$

for some $f \in [0, 1)$.

Suppose $x \in \mathbb{X}$. Then, since $fr(a_{jt}) = fr(a_{tj})$ for all $j \in N$, clearly $x \in \mathcal{Y}$. Further, for $f = \frac{1+fr(a_{tt})}{2}$, we have, for all $j \in N$,

$$\begin{aligned} fr(-f + a_{tt} - a_{jt}) &= fr\left(- \frac{1 + fr(a_{tt})}{2} + \lfloor a_{tt} \rfloor + fr(a_{tt}) - a_{jt} \right) \\ &= fr\left(- \frac{1}{2} + \frac{fr(a_{tt})}{2} - a_{jt} \right) = fr\left(\frac{1}{2} + \frac{fr(a_{tt})}{2} - a_{jt} \right) \\ &= fr\left(\frac{fr(1 + a_{tt})}{2} - a_{jt} \right) = fr(f - a_{tj}), \end{aligned}$$

where we have used that $fr(\frac{1}{2}) = fr(-\frac{1}{2})$ and $fr(a_{tj}) = fr(a_{jt})$. Therefore $\mathcal{X}(x) = \mathcal{Y}$ for this value of f. We can argue similarly for $f = \frac{fr(a_{tt})}{2}$. Hence $x \in \mathcal{X}(x) \cap \mathcal{Y}$

If instead $x \in \mathcal{X}(x) \cap \mathcal{Y}$, which is non-empty (using for example $f = \frac{fr(a_{tt})}{2}$), then

$$fr(x_1) = fr(f - a_{t1}) = fr(-f + a_{tt} - a_{1t}) = fr(-f + a_{tt} - a_{t1})$$

which implies that $fr(f) = fr(-f + a_{tt})$ and therefore $fr(f + f) = fr(a_{tt})$. This holds for $f = \frac{fr(a_{tt})}{2}$ or $f = \frac{1+fr(a_{tt})}{2}$. Thus $x \in \mathbb{X}$. □

6.2. Symmetric Scaling: Extension to General Matrices

When $BG(A)$ is connected the result follows immediately from Proposition 4.8 and Proposition 6.1.

Corollary 6.3. *Let $A \in \overline{\mathbb{R}}^{n\times n}$ and suppose $BG(A)$ is connected. There exists X such that $XAX \in \overline{\mathbb{Z}}^{n\times n}$ if and only if $A^{(C)}$ satisfies the BICP and additionally, $fr(A^{(C)})$ is symmetric. Further, the set of all such scalings (if they exist) can be described using Theorem 6.2 applied to $A^{(C)}$.*

Finally we consider the case when $BG(A)$ is not connected. Obviously A could be written as the direct sum of rectangular matrices $A_{11}, ..., A_{rr}, r \geq 2$, which describe the connected components of $BG(A)$. We would like to use Corollary 4.12 to conclude that A has the desired scaling if and only if each block $A_{11}^{(C)}$ satisfies some condition. However, we can only apply Corollary 6.3 to square blocks with the same index set for rows and columns. This may not be the case. To fix this problem, we introduce the *symmetrisation* of A, this is $A^{sym} := A \oplus A^T$.

Observe that a necessary condition for X such that $X \otimes A \otimes X \in \overline{\mathbb{Z}}^{n\times n}$ to exist is

$$(\forall i, j \in N)\Big(a_{ij}, a_{ji} > \epsilon \Rightarrow fr(a_{ij}) = fr(a_{ji})\Big). \tag{6.1}$$

Under this condition the following result is trivial.

Proposition 6.4. *Let $A \in \overline{\mathbb{R}}^{n\times n}$ satisfy* (6.1). *Then*

$$X \otimes A \otimes X \in \overline{\mathbb{Z}}^{n\times n} \Leftrightarrow X \otimes A^{sym} \otimes X \in \overline{\mathbb{Z}}^{n\times n}.$$

Consider any connected component of $BG(A^{sym})$, call it B and let its vertex sets be $U \subseteq N$ and $V \subseteq N$. Then, due to symmetry, either $U = V$ or B is a tree with $U \cap V = \emptyset$. In this case there exists another connected component of $BG(A^{sym})$, B' say, with vertex sets $U' = V$ and $V' = U$, satisfying

$$(i, j) \in E(B) \Leftrightarrow (j, i) \in E(B') \text{ and } E(B) \cap E(B') = \emptyset.$$

Proposition 6.5. *Let $A \in \overline{\mathbb{R}}^{n\times n}$ satisfy* (6.1)*. Then there exists a partition $N_1, ..., N_t$ of N such that A^{sym} is a direct sum with blocks*

$$A^{sym}[N_p, N_p] := B_{pp}, p = 1, ..., t$$

each satisfying exactly one of the conditions C1, C2 below.

C1: $BG(B_{pp})$ is connected.

C2: $BG(B_{pp})$ has two components, which are trees, T and T', satisfying,

$$(\forall i \in N_p)(\exists i \neq j \in N_p)(i,j) \in E(T) \cup E(T')$$

and

$$(\forall (i,j) \in E(BG(B_{pp})))\ (i,j) \in T \Leftrightarrow (j,i) \in T'.$$

Corollary 6.6. *Let $A \in \overline{\mathbb{R}}^{n\times n}$ satisfy* (6.1) *and $N_1, ..., N_t$ be a partition of N as found in Proposition 6.5. Then*

$$X \otimes A \otimes X \in \overline{\mathbb{Z}}^{n\times n} \Leftrightarrow (\forall p \in \{1, ..., t\}) X[N_p] \otimes A[N_p, N_p] \otimes X[N_p] \in \overline{\mathbb{Z}}^{n_p \times n_p}.$$

Clearly, for B_{pp} satisfying C1, the existence (and description) of Y for which $Y \otimes B_{pp} \otimes Y \in \mathbb{Z}^{n_p\times n_p}$ follows immediately from Proposition 6.3. So it remains to consider B_{pp} satisfying $C2$.

Proposition 6.7. *Suppose $B := B_{pp}$ is symmetric and satisfies C2. Then there exists Y such that $Y \otimes B \otimes Y \in \overline{\mathbb{Z}}^{n_p\times n_p}$.*

Further, for fixed $j \in N$ and $\alpha \in [0,1)$, let $y(\alpha)$ be the unique vector satisfying $y(\alpha)_j = \alpha$ and $diag(y(\alpha)) = Y$ for Y such that $YBY \in \overline{\mathbb{Z}}^{n\times n}$. Then

$$Y \otimes B \otimes Y \in \overline{\mathbb{Z}}^{n\times n} \Leftrightarrow y \in \{diag(u) \otimes y(\alpha) : \alpha \in [0,1), u \in \mathbb{Z}^{n_p}.\}$$

Proof. Suppose $B \in \overline{\mathbb{R}}^{n'\times n'}$ describes $BG(B_{pp})$ having two components, which are trees, T and T', satisfying,

$$(\forall i \in N_p)(\exists i \neq j \in N_p)(i,j) \in E(T) \cup E(T')$$

and

$$(\forall (i,j) \in E(BG(B_{pp})))\ (i,j) \in T \Leftrightarrow (j,i) \in T'.$$

Note that each edge belongs to exactly one tree, and, for all $i \in N_p$, $(i,i) \notin BG(B_{pp})$.

To prove the result we observe that, if we fix any $j \in N_p$, then, after setting $x_j = \alpha \in \mathbb{R}$, the fractional parts of all other values $x_i, i \neq j$ are predetermined, and describe a vector which works. This is since the graph is acyclic, and so we never encounter any conflict with previous values. □

Corollary 6.8. *Let $A \in \overline{\mathbb{R}}^{n\times n}$ satisfy (6.1). Then there exists X such that $X \otimes A \otimes X \in \overline{\mathbb{Z}}^{n\times n}$ if and only if, for all $p \in \{1, ..., t\}$, $B^{(C)}_{pp}$ satisfies the BICP and $fr(B^{(C)}_{pp})$ is symmetric. Further, a full description of all such scalings can be obtained by combining the solutions from each sub-block.*

7. Signed Similarity Scalings

Let $A \in \overline{\mathbb{R}}^{n\times n}$. For a specified partition J^+, J^- of N we say that A has a *signed similarity scaling to an integer matrix* if there exist diagonal X, Y such that $X \otimes A \otimes Y \in \overline{\mathbb{Z}}^{n\times n}$ where

$$Y_{jj} = \begin{cases} X_{jj}; & \text{if } j \in J^+, \\ X_{jj}^{-1}; & \text{if } j \in J^-. \end{cases}$$

Note that for $J^+ = \emptyset, J^- = N$ this is exactly the problem of determining whether there exists a diagonal similarity scaling of A^T to an integer matrix. When $J^+ = N, J^- = \emptyset$ this is exactly the symmetric scaling.

Observe that, given $A \in \overline{\mathbb{R}}^{n\times n}$, $X \otimes A \otimes Y \in \overline{\mathbb{Z}}^{n\times n}$ we can, without loss of generality, perform simultaneous permutations of the rows and columns so that

$$A = \begin{pmatrix} A[J^+, J^+] & A[J^+, J^-] \\ A[J^-, J^+] & A[J^-, J^-] \end{pmatrix}$$

without changing the answer to the question of whether A has a signed similarity scaling to an integer matrix. Thus we wish to determine whether there exist diagonal W, Z such that

$$X \otimes A \otimes Y = \begin{pmatrix} W & \epsilon \\ \epsilon & Z^{-1} \end{pmatrix} \otimes \begin{pmatrix} A[J^+, J^+] & A[J^+, J^-] \\ A[J^-, J^+] & A[J^-, J^-] \end{pmatrix} \otimes \begin{pmatrix} W & \epsilon \\ \epsilon & Z \end{pmatrix} \in \overline{\mathbb{Z}}^{n\times n}.$$

7.1. Signed Similarity Scaling: The Finite Case

We will prove here that the assumptions listed below are necessary and sufficient for a finite matrix to have a signed similarity scaling to an integer matrix.

Assumption 7.1. *Let $A \in \mathbb{R}^{n\times n}$ and J^+, J^- be a partition of N with $J^+, J^- \neq \emptyset$.*

A1: A satisfies the BICP,
A2: $fr(A[J^+, J^+])$ is symmetric,
A3: $A[J^-, J^-]$ is fractionally antisymmetric.

Proposition 7.2. *Let $A \in \mathbb{R}^{n\times n}$ and J^+, J^- be a partition of N with $J^+, J^- \neq \emptyset$. If there exist diagonal X, Y with $X[J^+] = Y[J^+]$ and $X[J^-] = Y^{-1}[J^-]$ such that $X \otimes A \otimes Y \in \mathbb{Z}^{n\times n}$ then A satisfies A1, A2 and A3.*

Proof. Suppose

$$\begin{pmatrix} W & \epsilon \\ \epsilon & Z^{-1} \end{pmatrix} \otimes \begin{pmatrix} A[J^+,J^+] & A[J^+,J^-] \\ A[J^-,J^+] & A[J^-,J^-] \end{pmatrix} \otimes \begin{pmatrix} W & \epsilon \\ \epsilon & Z \end{pmatrix} \in \overline{\mathbb{Z}}^{n\times n}.$$

Then clearly A has a direct similarity scaling to an integer matrix, and so A satisfies A1 by Corollary 4.13.

Further $W \otimes A[J^+,J^+] \otimes W$ is integer and hence $fr(A[J^+,J^+])$ is symmetric by Corollary 6.1.

Finally $Z^{-1} \otimes A[J^-,J^-] \otimes Z$ is integer, therefore $A[J^-,J^-]$ is fractionally antisymmetric since $A[J^-,J^-]$ satisfies the ICP by Corollary 3.12. □

We aim to prove that A1-A3 are also sufficient.

Theorem 7.3. *Let $A \in \mathbb{R}^{n\times n}$ and J^+, J^- be a partition of N with $J^+, J^- \neq \emptyset$. Suppose A satisfies A1, A2 and A3. Then there exist diagonal matrices X, Y with $X[J^+] = Y[J^+]$ and $X[J^-] = Y^{-1}[J^-]$ such that $X \otimes A \otimes Y \in \mathbb{Z}^{n\times n}$.*

Further, for any fixed $t \in J^+$, all possible scalings are given by

$$y \in \left\{ diag(u) \otimes A_{t\bullet}^{(-1)} : u \text{ is } \frac{fr(a_{tt})}{2}\text{-fractional or } \frac{1+fr(a_{tt})}{2}\text{-fractional} \right\}.$$

Proof. Fix $t \in J^+$. Recall from Proposition 4.4 that $X \otimes A \otimes Y \in \mathbb{Z}^{n\times n}$ if and only if there exists $f \in [0,1)$ such that

$$(\forall j \in N) fr(y_j) = fr(f - a_{tj}) \text{ and } fr(x_j) = fr(-f + a_{tt} - a_{jt}).$$

Let

$$y \in \left\{ diag(u) A_{t\bullet}^{(-1)} : u \text{ is } \frac{fr(a_{tt})}{2}\text{-fractional or } \frac{1+fr(a_{tt})}{2}\text{-fractional} \right\}.$$

Recall that $(\forall j \in N) fr(y_j) = fr(f - a_{tj})$ for $f \in \{\frac{fr(a_{tt})}{2}, \frac{1+fr(a_{tt})}{2}\}$. Hence it suffices to show that $x = (x_j)$ given by

$$x_j = \begin{cases} y_j & \text{if } j \in J^+ \\ -y_j & \text{if } j \in J^- \end{cases}$$

satisfies $(\forall r \in N) fr(x_r) = fr(-f + a_{tt} - a_{rt})$.

First assume $r \in J^+$. Then $fr(x_r) = fr(y_r) = fr(f - a_{tr})$.

Case 1: $f = \frac{fr(a_{tt})}{2}$

Here

$$fr(x_r) = fr\left(\frac{fr(a_{tt})}{2} - a_{tr}\right) = fr\left(fr(a_{tt}) - \frac{fr(a_{tt})}{2} - a_{tr}\right) = fr(a_{tt} - f - a_{tr}).$$

Observe that $r, t \in J^+$, so, since $A[J^+,J^+]$ is symmetric, $fr(a_{tr}) = fr(a_{rt})$. This implies $fr(x_r) = fr(a_{tt} - f + a_{tr})$ as required.

Case 2: $f = \frac{1+fr(a_{tt})}{2}$

Similarly to Case 1, and using that $fr(\frac{1}{2}) = fr(-\frac{1}{2})$, we obtain

$$\begin{aligned} fr(x_r) &= fr\left(\frac{1+fr(a_{tt})}{2} - a_{tr}\right) \\ &= fr\left(\frac{1}{2} + fr(a_{tt}) - \frac{fr(a_{tt})}{2} - a_{tr}\right) \\ &= fr(-f + a_{tt} - a_{tr}). \end{aligned}$$

Further, for all $r \in J^-$,

$$\begin{aligned} fr(-y_r) &= fr(-f + a_{tr}) \\ &= fr(-f + a_{rr} - a_{rt} + a_{tt}) && (\because a_{rr} - a_{rt} + a_{tt} - a_{tr} \in \mathbb{Z}) \\ &= fr(-f - a_{rt} + a_{tt}). && (\because a_{rr} \in \mathbb{Z} \text{ by ICP}) \end{aligned}$$

Hence any y of this form does describe a signed similarity scaling of A to an integer matrix.

Finally we show there are no other possible scaling vectors. Indeed, from Proposition 4.4, we require

$$y \in \left\{ diag(u) \otimes A_{j\bullet}^{(-1)} : u \text{ is uni-fractional} \right\}.$$

So suppose u is f-fractional where $f \notin \{\frac{fr(a_{jj})}{2}, \frac{1+fr(a_{jj})}{2}\}$. Then, by Proposition 6.2, and using that $fr(A[J^+,J^+]_{\bullet j}) = fr(A[J^+,J^+]_{j\bullet})$,

$$Y[J^+] \otimes A[J^+,J^+] \otimes Y[J^+]$$

is not integer, a contradiction. □

Corollary 7.4. *Let $A \in \mathbb{R}^{n\times n}$ and J^+, J^- be a partition of N with $J^+, J^- \neq \emptyset$. Then A has a signed similarity scaling to an integer matrix if and only if A satisfies A1, A2 and A3.*

7.2. Signed Similarity Scaling: Extension to General Matrices

The case when $BG(A)$ is connected follows immediately from Proposition 4.8.

Corollary 7.5. *Let $A \in \overline{\mathbb{R}}^{n\times n}$ be such that $BG(A)$ is connected. Let J^+, J^- be a partition of N with $J^+, J^- \neq \emptyset$. Then there exist diagonal X, Y with $X[J^+] = Y[J^+]$ and $X[J^-] = Y^{-1}[J^-]$ such that $X \otimes A \otimes Y \in \overline{\mathbb{Z}}^{n\times n}$ if and only if $A^{(C)}$ satisfies A1, A2 and A3. Further the set of all such scalings (if any exist) can be described using Theorem 7.3 applied to $A^{(C)}$.*

When $BG(A)$ is not connected we can use the techniques from all previous sections as follows. Let

$$\bar{\bar{A}} := \begin{pmatrix} A[J^+, J^+]^{sym} & A[J^+, J^-] \\ A[J^-, J^+] & \overline{A[J^-, J^-]} \end{pmatrix}$$

where $\bar{B}$ is as defined in Proposition 3.11. Observe that $X \otimes A \otimes X \in \overline{\mathbb{Z}}^{n\times n} \Leftrightarrow X \otimes \bar{\bar{A}} \otimes Y \in \overline{\mathbb{Z}}^{n\times n}$. Further, $\overline{A[J^-, J^-]}$ can be written as a direct sum of irreducible matrices, and $A[J^+, J^+]^{sym}$ can be written as a direct sum of square blocks which either describe a connected bipartite graph, or two symmetric trees.

Proposition 7.6. *Let $A \in \overline{\mathbb{R}}^{n\times n}$. Let J^+, J^- be a partition of N with $J^+, J^- \neq \emptyset$. Let $J_1, ..., J_t$ be a partition of J^+ such that $A[J^+, J^+]^{sym}$ is a direct sum with t blocks each satisfying exactly one of the conditions C1, C2 from Proposition 6.5.*

Then there exist diagonal X, Y with $X[J^+] = Y[J^+]$ and $X[J^-] = Y^{-1}[J^-]$ such that $X \otimes A \otimes Y \in \overline{\mathbb{Z}}^{n\times n}$ if and only if

(i) $\bar{\bar{A}}$ satisfies the BICP,

(ii) for each block of $A[J^+, J^+]^{sym}$, $fr((A[J^+, J^+]^{sym})_{pp}^{(C)})$ is symmetric,

(iii) for each irreducible block of $\overline{A[J^-, J^-]}$, we have that $\overline{A[J^-, J^-]}_{pp}^{+}$ is fractionally antisymmetric.

8. Acknowledgements

Thanks to Professor Peter Butkovič for numerous valuable discussions on this subject, as well as guidance on finding previous work in the area.

References

[1] R. A. Brualdi, H. Ryser, *Combinatorial matrix theory*, Cambridge, Cambridge University Press, 1991.

[2] P. Butkovič, *Max-linear Systems: Theory and Algorithms*, Springer-Verlag, London, 2010.

[3] P. Butkovič, H. Schneider, *Applications of max-algebra to diagonal scaling of matrices*, Electronic Journal of Linear Algebra, Vol. 13, 262-273, 2005.

[4] P. Butkovič, M. MacCaig, *On integer eigenvectors and subeigenvectors in the max-plus algebra*, University of Birmingham, School of Mathematics, 2012/09.

[5] R. A. Cuninghame-Green, *Minimax algebra*, Lecture notes in economics and math systems, Vol. 166, Springer, Berlin, 1979.

[6] G. M. Engel, Owega, H. Schneider, *Diagonal similarity and equivalence for matrices over groups with 0*, Czechoslovak Mathematical Journal, 25 (100) 1975, Praha.

[7] G. M. Engel, H. Schneider, *Algorithms for testing the diagonal similarity of matrices and related problems*, Siam J. Alg. Disc. Meth. Vol. 3, No. 4, 1982.

[8] U. G. Rothblum, H. Schneider, M. Schneider, *Scaling matrices to prescribed row and column maxima*, Siam J. Matrix Anal. Appl. Vol. 15, No. 1, 1-14, 1994.

[9] B. D. Saunders, H. Schneider, *Flows on graphs applied to diagonal similarity and diagonal equivalence for matrices*, Discrete Mathematics 24, 205-220, 1978 .