EXPLORING THE COMPLEXITY OF THE INTEGER IMAGE PROBLEM IN THE MAX-ALGEBRA*

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Abstract. We investigate the complexity of the problem of finding an integer vector in the max-algebraic column span of a matrix, which we call the integer image problem. We show some cases where we can determine in polynomial time whether such an integer vector exists, and find such an integer vector if it does exist. On the other hand we also describe a group of related problems each of which we prove to be NP-hard. Our main results demonstrate that the integer image problem is equivalent to finding a special type of integer image of a matrix satisfying a property we call *column typical*. For a subclass of matrices this problem is polynomially solvable but if we remove the column typical assumption then it becomes NP-hard.

Key words. max-algebra, integer vector, column span, image space, complexity

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1. Introduction. This paper deals with the task of finding integer vectors in the max-algebraic column span of a matrix. For $a, b \in \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$ we define $a \oplus b = \max(a, b)$, $a \otimes b = a + b$ and extend the pair $(\oplus, \otimes)$ to matrices and vectors in the same way as in linear algebra, that is (assuming compatibility of sizes)

$$\begin{aligned}(\alpha \otimes A)_{ij} &= \alpha \otimes a_{ij},\\ (A \oplus B)_{ij} &= a_{ij} \oplus b_{ij}, \text{ and}\\ (A \otimes B)_{ij} &= \bigoplus_k a_{ik} \otimes b_{kj}.\end{aligned}$$

All multiplications in this paper are in max-algebra and we will usually omit the $\otimes$ symbol. Note that α^{-1} stands for $-\alpha$, and we will use ε to denote $-\infty$ as well as any vector or matrix whose every entry is $-\infty$. The zero vector is denoted by $\mathbf{0}$. A vector/matrix whose every entry belongs to $\mathbb{R}$ is called *finite*. If a matrix has no ε rows (columns) then it is called *row* (*column*) $\mathbb{R}$*-astic* and it is called *doubly* $\mathbb{R}$*-astic* if it is both row and column $\mathbb{R}$-astic.

Max-algebra (also called tropical algebra) is a rapidly expanding area of idempotent mathematics, linear algebra and applied discrete mathematics. One key advantage is that problems from areas such as operational research, science and engineering which are non-linear in the conventional algebra, can be modeled as linear problems within the max-algebraic setting [1, 7, 9, 10]. Applications of max-algebra are both theoretical and practical; in [10] the Dutch railway system is modeled using max-algebra.

The *integer image problem* (IIm) is the problem of determining whether there is an integer vector in the column span (called here the image space) of a matrix $A \in \overline{\mathbb{R}}^{m \times n}$. The set of integer images is

$$IIm(A) := \{z \in \mathbb{Z}^m : (\exists x \in \overline{\mathbb{R}}^n) Ax = z\}.$$

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We also define $X(A)$ to be the set of vectors x for which Ax belongs to the set of integer images, that is

$$X(A) := \{x \in \overline{\mathbb{R}}^n : Ax \in IIm(A)\}.$$

A related question is whether $X(A) \cap \overline{\mathbb{Z}}^n$ is nonempty, we define

$$IIm^*(A) := \{z \in \mathbb{Z}^m : (\exists x \in \overline{\mathbb{Z}}^n)Ax = z\}.$$

One application of the integer image problem is as follows [4]. Suppose machines $M_1, ..., M_n$ produce components for products $P_1, ..., P_m$. Let x_j denote the starting time of M_j and a_{ij} be the time taken for M_j to complete its component for P_i. Then all components for product P_i are ready at completion time

$$c_i = \max(a_{i1} + x_1, ..., a_{in} + x_n) \ i = 1, ..., m.$$

Equivalently this can be written as $Ax = c$. In this context the integer image problem asks whether there exists a set of starting times for which the completion times are integer (this can easily be extended to ask for any discrete set of values). If we additionally require that the starting times are integer/discrete values then we want to find $c \in IIm^*(A)$.

Further it is known [5] that the *max-algebraic integer eigenspace*, defined as

$$\{x \in \mathbb{Z}^n : Ax = \lambda x, x \neq \varepsilon\}$$

for a fixed eigenvalue $\lambda \in \mathbb{R}$, is equal to the integer image space of a matrix B obtained from A. Currently is it not known whether it is possible to find an integer eigenvector in polynomial time. The eigenproblem in max-algebra can be used to analyse stability in production systems [2, 7]: Assume machines $M_1', ..., M_n'$ work interactively and in stages. In each stage all the machines produce components for the other machines to use in the next stage. Let $x_i(r)$ denote the starting time of the r^{th} stage on machine M_i' and a_{ij} denote the time taken for M_i' to complete its component for machine M_j'. Then

$$x_i(r+1) = \max(x_1(r) + a_{i1}, ..., x_n(r) + a_{in}) \ i = 1, ..., n, r = 0, 1, ...$$

This can be written as $x(r+1) = Ax(r)$. A *steady regime* is reached if this process moves forwards in regular steps, i.e. if $x(r+1) = \lambda x(r)$ for all $r \geq 0$. Clearly this occurs if and only if $x(0)$ is an eigenvector of A corresponding to some eigenvalue $\lambda \in \mathbb{R}$. It is natural to look for integer starting times, and therefore we aim to solve the integer eigenproblem. A solution to the corresponding integer image problem would achieve this.

An algorithm for testing whether $IIm(A) \neq \emptyset$ and finding an integer image if it exists was described in [5]. This algorithm always terminates in a finite number of steps and is pseudopolynomial if the input matrix is finite. We investigate whether the problem could in fact be in P, the class of polynomially solvable problems.

In searching for integer solutions to the integer image problem one helpful tool is being able to identify potential active positions. Given vectors x, z such that $Ax = z$ we say that a position (i, j) is *active* with respect to x or z if $a_{ij} + x_j = z_i$, and *inactive* otherwise. It will be useful in this paper to talk about the entries of the matrix corresponding to active positions and therefore we say that an element a_{ij} of

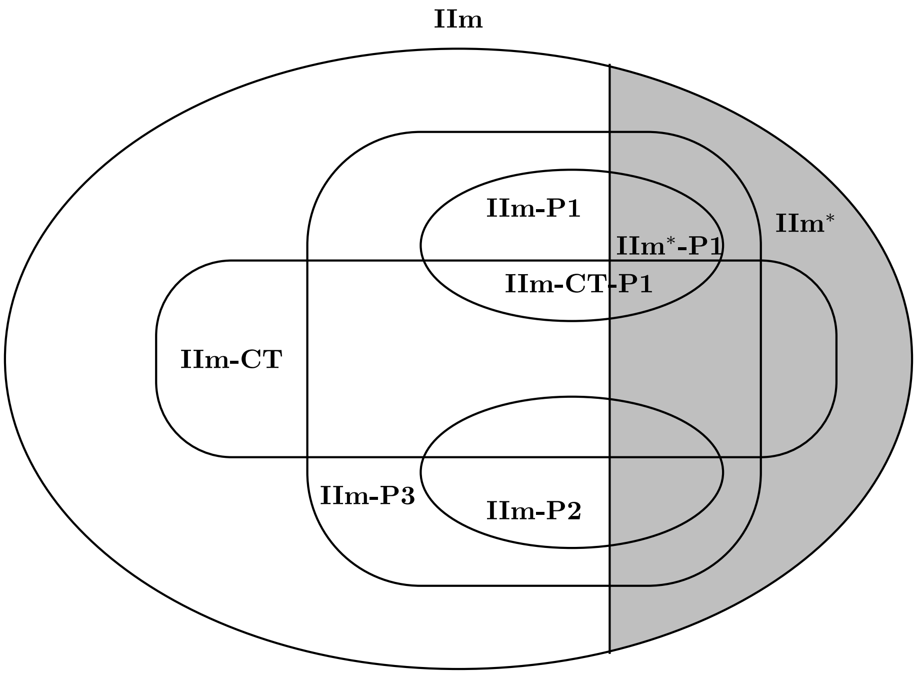

FIG. 1.1. *Simple relations between the different versions of the integer image problem considered in this paper, excluding IIm-P4.*

A is *active* if and only if the position (i, j) is active. In the same way we call a column A_j *active* if it contains an active position.

We define a *column typical* matrix to be a matrix $A \in \overline{\mathbb{R}}^{m\times n}$ such that for each j we have $fr(a_{ij}) \neq fr(a_{kj})$ for any i and k, $i \neq k$ such that $a_{ij}, a_{kj} \in \mathbb{R}$. Note that $fr(\cdot)$ denotes the fractional part and will be defined in Section 2.

In this paper we will consider a number of integer image problems, each with an additional requirement on the set of integer images. These are detailed in the definition below. Figure 1 outlines the relations between these problems.

DEFINITION 1.1. *Given $A \in \overline{\mathbb{R}}^{m\times n}$ we consider the following related problems to IIm.*
(IIm-CT) If A is column typical does there exist $x \in \overline{\mathbb{R}}^n$ such that $Ax \in \mathbb{Z}^m$?
(IIm-CT-P1) If A is column typical does there exist $x \in \overline{\mathbb{R}}^n$ such that $Ax \in \mathbb{Z}^m$ with exactly one active entry per row with respect to x?
(IIm-P1) Does there exist $x \in \overline{\mathbb{R}}^n$ such that $Ax \in \mathbb{Z}^m$ with exactly one active entry per row with respect to x?
(IIm-P2) Does there exist $x \in \overline{\mathbb{R}}^n$ such that $Ax \in \mathbb{Z}^m$ with exactly two active entries per row with respect to x?
(IIm-P3) Does there exist $x \in \overline{\mathbb{R}}^n$ such that $Ax \in \mathbb{Z}^m$ with at most two active entries per row with respect to x?
(IIm-P4) Given $t \in \mathbb{N}$ does there exist $x \in \overline{\mathbb{R}}^n$ such that $Ax \in \mathbb{Z}^m$ with at most t active columns of A with respect to x?
(IIm) Does there exist $x \in \overline{\mathbb{Z}}^n$ such that $Ax \in \mathbb{Z}^m$?*
(IIm-P1) Does there exist $x \in \overline{\mathbb{Z}}^n$ such that $Ax \in \mathbb{Z}^m$ with exactly one active entry per row with respect to x?*

In Section 2 we summarise the existing theory necessary for the presentation of our results and describe some simple cases for which we can determine whether $IIm(A) \neq \emptyset$ in polynomial time, and find an integer image if it exists. These cases include IIm-CT. In Section 3 we give two different transformations of a general matrix $A \in \overline{\mathbb{R}}^{m\times n}$ to a matrix $B \in \overline{\mathbb{R}}^{m\times mn}$ for which $IIm(A) = IIm(B)$ and give reasons why we suspect $IIm(B)$ will be easier to describe than $IIm(A)$. In particular determining whether $IIm(B) \neq \emptyset$ reduces to checking whether B is an instance of IIm-CT or IIm* and for both these problems we can find an integer image in a special case. However in general B fails to satisfy the requirements of this special case so this does not solve the integer image problem, but it does lend support to the idea that the integer image problem could be solvable in polynomial time. In Section 4 we show that IIm is polynomially equivalent to IIm-CT-P1 and IIm-CT. Section 5 contains proofs that IIm-P1, IIm-P2, IIm-P3, IIm-P4 and IIm*-P1 are NP-hard. Since the only difference between IIm-P1 and IIm-CT-P1 is the assumption that the matrix is column typical this raises the question of whether the integer image problem may in fact be NP-hard. Section 6 contains the proof that the transformation described in Section 3 is valid and can be achieved in polynomial time.

What this paper aims to demonstrate is that on the one hand the integer image problem for general matrices is closely related to the integer image problem for column typical matrices $A \in \overline{\mathbb{R}}^{m\times n}$, which is polynomially solvable if either $m \geq n$ or we fix the value of m. On the other hand IIm-CT and IIm-CT-P1 are polynomially equivalent and if we remove the assumption that the matrix is column typical IIm-P1 is NP-hard. So we are in essence approaching the integer image problem from two sides, one a set of problems in P and the other a set of problems that are NP-hard.

2. Preliminaries and simple cases. We denote by P the class of all problems which are solvable in polynomial time. The class NP and the definition of an NP-hard problem can be found, for example, in [8]. For general problems P1 and P2 we write $P1 \leq_p P2$ to mean that P1 can be reduced to P2 in polynomial time. It is known that if $P1 \leq_p P2$ and $P1$ is NP-hard then $P2$ is NP-hard, if instead $P2 \in P$ then $P1 \in P$.

We will use the following standard notation. For positive integers m, n we denote $M = \{1, ..., m\}$ and $N = \{1, ..., n\}$. If $A = (a_{ij}) \in \overline{\mathbb{R}}^{m\times n}$ then A_j stands for the j^{th} column of A.

Given a square matrix, the *maximum cycle mean*, $\lambda(A)$ is,

$$\lambda(A) = \max\left\{\frac{a_{i_1 i_2} + ... + a_{i_k i_1}}{k} : i_1, ... i_k \in N, k = 1, ..., n\right\}.$$

It is known that $\lambda(A)$ can be calculated in $\mathcal{O}(n^3)$ time [3, 11]. If $\lambda(A) = 0$ then we say that A is *definite*. If moreover $a_{ii} = 0$ for all $i \in N$ then A is called *strongly definite*. Given a definite matrix we define

$$A^* = I \oplus A \oplus A^2 \oplus ... \oplus A^n.$$

A matrix is called *diagonal* if its diagonal entries are finite and its off diagonal entries are ε. The *identity* matrix is a diagonal matrix in which all diagonal entries are equal to 0. A matrix Q is called a *generalised permutation matrix* if it can be obtained from a diagonal matrix by permuting the rows and/or columns.

We use P_n to denote the set of permutations on N. For $A \in \overline{\mathbb{R}}^{n\times n}$ the *max-algebraic permanent* is given by

$$maper(A) = \bigoplus_{\pi \in P_n} \bigotimes_{i \in N} a_{i,\pi(i)}.$$

The *weight* of a permutation π with respect to A is

$$w(\pi, A) = \bigotimes_{i \in N} a_{i,\pi(i)}$$

and the set of permutations whose weight is maximum is

$$ap(A) = \{\pi \in P_n : w(\pi, A) = maper(A)\}.$$

For $a \in \mathbb{R}$ the *fractional part* of a is $fr(a) := a - \lfloor a \rfloor$. For a matrix $A \in \overline{\mathbb{R}}^{m\times n}$ we use $\lfloor A \rfloor$ ($\lceil A \rceil$) to denote the matrix with (i, j) entry equal to $\lfloor a_{ij} \rfloor$ ($\lceil a_{ij} \rceil$) and similarly for vectors. We set $fr(\varepsilon) = \varepsilon = \lfloor \varepsilon \rfloor = \lceil \varepsilon \rceil$. We outline a number of simple properties of $fr(\cdot)$ below.

Lemma 2.1. *Let $a, b, c \in \mathbb{R}$ and $x \in \mathbb{Z}$. Then*

(1) $fr(a) \geq 0$ so $a \geq 0 \Leftrightarrow fr(a) \leq a$

(1) $fr(-a) = 1 - fr(a)$

(2) $fr(a + b) = fr(fr(a) + fr(b))$

(3) $fr(a - b) = fr(fr(a) - fr(b))$

In fact,

$$fr(a) > fr(b) \Rightarrow fr(a - b) = fr(a) - fr(b),$$
$$fr(a) < fr(b) \Rightarrow fr(a - b) = 1 - fr(b) + fr(a),$$

(4) $\lfloor a + b \rfloor > \lceil a \rceil \Rightarrow b > 1 - fr(a)$

(5) $fr(x + a) = fr(a)$

(6) $fr(x - a) = 1 - fr(a)$

(7)

$$\lfloor -a \rfloor = \begin{cases} -a & \text{if } a \in \mathbb{Z} \\ -1 - \lfloor a \rfloor & \text{otherwise} \end{cases}$$

(8)

$$\lceil -a \rceil = \begin{cases} -a & \text{if } a \in \mathbb{Z} \\ 1 - \lceil a \rceil & \text{otherwise} \end{cases}$$

(9) If $b + c \in \mathbb{Z}$ then

$$fr(a + c) = fr(a - b)$$

Proof. Many follow directly from definition. We give details for a few.

(4) $a + b > \lceil a \rceil$ therefore $fr(a) + fr(b) > 1$

(9) Since $b + c \in \mathbb{Z}$ we have $fr(c) = 1 - fr(b)$. Then

$$fr(a + c) = fr(fr(a) + fr(c)) = fr(fr(a) + 1 - fr(b)) = fr(a - b)$$

using (2) and (3). □

We detail here two max-linear systems of equations that will be useful in this paper. For $A \in \overline{\mathbb{R}}^{n\times n}$ we define the set of *integer subeigenvectors* with respect to $\lambda \in \mathbb{R}$ as

$$IV^*(A,\lambda) = \{x \in \mathbb{Z}^n : Ax \leq \lambda x, x \neq \varepsilon\}.$$

The set of all integer subeigenvectors can be described in $\mathcal{O}(n^3)$ time due to the following result.

THEOREM 2.2. *[5] Let $A \in \overline{\mathbb{R}}^{n\times n}$, $\lambda \in \mathbb{R}$.*
(i) $IV^*(A,\lambda) \neq \emptyset$ if and only if

$$\lambda(\lceil \lambda^{-1} A\rceil) \leq 0.$$

(ii) If $IV^*(A,\lambda) \neq \emptyset$ then

$$IV^*(A,\lambda) = \{\lceil \lambda^{-1} A\rceil^* z : z \in \mathbb{Z}^n\}.$$

Given $A \in \overline{\mathbb{R}}^{m\times n}$ and $B \in \overline{\mathbb{R}}^{m\times k}$ the *two-sided system with separated variables* is $Ax = By$ for vectors $x \in \overline{\mathbb{R}}^{n}, y \in \overline{\mathbb{R}}^{k}$. The problem of finding integer vectors x, y such that $Ax = By$ was studied in [6] and we outline below the results we need.

The pair (A, B) is said to satisfy *Property OneFP* if for each $i \in M$ there is exactly one pair of indices $(r(i), r'(i))$ such that $fr(a_{ir(i)}) = fr(b_{ir'(i)})$ and $a_{ir(i)}, b_{ir'(i)} \in \mathbb{R}$. Without loss of generality these fractional parts are zero (this can be assumed since a constant can be subtracted from each row of the matrix equation without changing the set of integer solutions).

THEOREM 2.3. *[6] Let $A \in \overline{\mathbb{R}}^{m\times n}$, $B \in \overline{\mathbb{R}}^{m\times k}$ satisfy Property OneFP. For all $i, j \in M$ let*

$$w_{ij} = \lceil a_{j,r(i)}\rceil a^{-1}_{i,r(i)} \oplus \lceil b_{j,r'(i)}\rceil b^{-1}_{i,r'(i)}.$$

Then an integer solution to $Ax = By$ exists if and only if $\lambda(W) \leq 0$. If this is the case then $Ax = By = z^{-1}$ where $z \in IV^*(W, 0)$.

COROLLARY 2.4. *[6] For $A \in \overline{\mathbb{R}}^{m\times n}$, $B \in \overline{\mathbb{R}}^{m\times k}$ it is possible to decide whether an integer solution to $Ax = By$ exists in $\mathcal{O}(m^3 + n + k)$ time.*

REMARK 2.5. *Once z from Theorem 2.3 is determined we can find the integer solution (x, y) to the two sided system by solving each of the equations $Ax = z^{-1}$ and $By = z^{-1}$. These can be solved in $\mathcal{O}(mn)$ and $\mathcal{O}(mk)$ time respectively [5].*

The first treatment of the integer image problem appeared in [5]. We briefly note the important results from that paper here, the main one being that for column typical matrices $A \in \overline{\mathbb{R}}^{n\times n}$ IIm can be solved in $\mathcal{O}(n^3)$ time by Theorem 2.6 below.

Let A be a square matrix. Consider a generalised permutation matrix Q. It is easily seen that $IIm(A) = IIm(A \otimes Q)$. Further, from [3] we know that for every matrix A with $maper(A) > \varepsilon$ there exists a generalised permutation matrix Q such that $A \otimes Q$ is strongly definite and Q can be found in $\mathcal{O}(n^3)$ time.

Observe that if $A \in \overline{\mathbb{R}}^{m\times n}$ has an ε row then $IIm(A) = \emptyset$, and if A has an ε column then $IIm(A)$=$IIm(A')$ where A' is obtained from A by removing the ε column. Hence it is sufficient to only consider doubly $\mathbb{R}$-astic matrices for the entirety of this paper.

THEOREM 2.6. *[5] Let $A \in \overline{\mathbb{R}}^{n\times n}$ be a column typical matrix.*
(a) If $maper(A) = \varepsilon$ then A has no integer image.
(b) If $maper(A) > \varepsilon$ and $|ap(A)| > 1$ then A has no integer image.
(c) If $maper(A) > \varepsilon$ and $|ap(A)| = 1$ let Q be the unique generalised permutation matrix such that $A \otimes Q$ is strongly definite. Then

$$IIm(A) = IIm(A \otimes Q) = IV^*(A \otimes Q, 0).$$

Observe that if $A \in \overline{\mathbb{R}}^{m\times n}$ is column typical with $m \leq n$ then

$$(\exists x)Ax = z \Leftrightarrow (\exists j_1, ..., j_m \in N)(\exists x')A'x' = z$$

where $A' \in \overline{\mathbb{R}}^{m\times m}$ is the matrix formed of columns $A_{j_1}, ..., A_{j_m}$. Therefore if A is column typical with $m \leq n$ then we could simply check each of the $\binom{n}{m}$ square submatrices of A to see if they have an integer image. Checking each submatrix can be achieved in $\mathcal{O}(m^3)$ time by Theorems 2.2 and 2.6.

COROLLARY 2.7. *For fixed m the integer image problem is solvable in polynomial time.*

It is also shown in [5] that the integer image problem can be easily solved if either $m = 2$ or $n = 2$.

3. Transformations which preserve the set of integer images. We present two transformations which allow us to assume some structure on the matrix for which we are seeking an integer image. In both cases the transformation can be achieved in polynomial time and we expect that the added structure will help in finding integer images. Indeed for each type of structure described we find a small class of matrices for which we can solve the integer image problem efficiently.

3.1. Transformation to matrices with integer active entries. Here we describe a transformation $A \to B$ such that $IIm(A) = IIm^*(B)$. Further we show that if a general matrix C has at most one integer entry in each row then we can decide in $\mathcal{O}(m^3 + n)$ time whether $C \in$IIm*.

Given a matrix $A \in \overline{\mathbb{R}}^{m\times n}$ let A^{int} be constructed from A by replacing each column $A_j, j \in N$ with m columns,

$$fr(a_{1j})^{-1}A_j, fr(a_{2j})^{-1}A_j, ..., fr(a_{mj})^{-1}A_j.$$

EXAMPLE 3.1.

$$A = \begin{pmatrix} 0 & 1.1 \\ 0.5 & -2.3 \\ -0.6 & -0.9 \end{pmatrix}, A^{int} = \begin{pmatrix} 0 & -0.5 & -0.4 & 1 & 0.4 & 1 \\ 0.5 & 0 & 0.1 & -2.4 & -3 & -2.4 \\ -0.6 & -1.1 & -1 & -1 & -1.6 & -1 \end{pmatrix}$$

Note that each column takes at least one entry of the matrix and makes it integer.

Observe that for any $z \in IIm^*(A^{int})$, if the position (i, j) is active then it is necessary that $a_{ij} \in \mathbb{Z}$ since $a_{ij} + x_j = z_i$ where by definition x_j and z_i are integer. Therefore the following result tells us that when considering the integer image problem we can assume without loss of generality that only integer entries can be active.

THEOREM 3.2. *$IIm(A) = IIm(A^{int}) = IIm^*(A^{int})$.*

Proof. Let A^{int} have columns $A_{j(i)}$ where $A_{j(i)} = fr(a_{ij})^{-1}A_j$. We first show that $IIm(A) = IIm(A^{int})$.

Suppose $\exists x \in \overline{\mathbb{R}}^n$ such that $Ax = z \in IIm(A)$. Then

$$z = \bigoplus_{j \in N} A_j x_j = \bigoplus_{j \in N} \bigoplus_{i \in M} A_j fr(a_{ij}) fr(a_{ij})^{-1} x_j$$
$$= \bigoplus_{j \in N} \bigoplus_{i \in M} A^{int}_{j(i)} \left(fr(a_{ij}) x_j \right) \in IIm(A^{int}).$$

For the other inclusion assume that

$$y = (y_{1(1)}, ..., y_{1(m)}, y_{2(1)}, ..., y_{2(m)}, ..., y_{n(1)}, ..., y_{n(m)}) \in \overline{\mathbb{R}}^{mn}$$

satisfies $Ay = z \in IIm(A)$. Then

$$z = \bigoplus_{j \in N} \bigoplus_{i \in M} A_{j(i)} y_{j(i)} = \bigoplus_{j \in N} \bigoplus_{i \in M} A_j fr(a_{ij})^{-1} y_{j(i)}$$
$$= \bigoplus_{j \in N} A_j \left(\bigoplus_{i \in M} fr(a_{ij})^{-1} y_{j(i)} \right) \in IIm(A).$$

Further it is clear that $IIm^*(A^{int}) \subseteq IIm(A^{int})$. This together with $IIm(A) = IIm(A^{int})$ implies $IIm^*(A^{int}) \subseteq IIm(A)$. It remains to show $IIm(A) \subseteq IIm^*(A^{int})$.

Clearly

$$(\exists x \in \overline{\mathbb{R}}^n) Ax = z \in \overline{\mathbb{Z}}^m \Rightarrow (\exists y \in \overline{\mathbb{Z}}^{mn}) A^{int} y = z \in \overline{\mathbb{Z}}^m$$

since if A_j is active with respect to x then $(\exists i \in M) fr(a_{ij}) = 1 - fr(x_j)$ and a_{ij} is active, therefore

$$a_{ij} + x_j = \lfloor a_{ij} \rfloor + fr(a_{ij}) + \lfloor x_j \rfloor + fr(x_j) = \lfloor a_{ij} \rfloor + 1 - fr(x_j) + \lfloor x_j \rfloor + fr(x_j) = a^{int}_{it} + \lceil x_j \rceil$$

for some $t \in \{1, .., mn\}$. This means that $x_j A_j = \lceil x_j \rceil (fr(a_{ij})^{-1} A_j) = y_t A^{int}_t$ where $y_t = \lceil x_j \rceil$. Hence the result. $\square$

This transformation is expected to be helpful in solving the integer image problem since it allows us to look for integer images of the matrix for which active positions are (i, j) where $a_{ij} \in \mathbb{Z}$. While it remains unknown whether IIm* is in P or not we can describe one class of matrices for which it is solvable in $\mathcal{O}(m^3 + n)$ time.

Indeed suppose $C \in \overline{\mathbb{R}}^{m \times n}$ has at most one integer entry in each row. Then either the matrix does not satisfy the necessary condition that every row has an active entry, in which case $IIm^*(C) = \emptyset$, or C has exactly one integer in each row. In this case let I be the identity matrix with dimension m. Then the pair (A, I) satisfies Property OneFP and

$$IIm^*(A) \neq \emptyset \Leftrightarrow (\exists x \in \mathbb{Z}^n, y \in \mathbb{Z}^m) Ax = Iy.$$

Therefore in this case we can determine whether $IIm^*(A) \neq \emptyset$ in $\mathcal{O}(m^3 + n)$ by Theorem 2.3.

Generally however A^{int} will not be square, nor will it have a small number of integer entries in each row. We finish this subsection by detailing a few observations about $IIm^*(C)$ for an arbitrary matrix C.

Proposition 3.3. *Suppose* $C \in \overline{\mathbb{R}}^{m \times n}$.

(i) $IIm^*(C) \subseteq IIm(\lceil C \rceil)$
(ii) $IIm^*(C) \subseteq IIm(\lfloor C \rfloor)$
Proof. If $Cx = z \in \mathbb{Z}^m$ where $x \in \overline{\mathbb{Z}}^n$ then

$$\max_j (c_{ij} + x_j) = z_i \Rightarrow \max_j (\lceil c_{ij} \rceil + x_j) = z_i.$$

The other result is also trivial to prove. □

For each $i \in M$ let

$$d_i(C) = \max_{j \in N} \lceil c_{ij} \rceil - \max_{j : c_{ij} \in \mathbb{Z}} c_{ij}.$$

Clearly $d_i(C) \geq 0$ for all $i \in M$. Using this we obtain a simple sufficient condition for $IIm^*(C) \neq \emptyset$.

PROPOSITION 3.4. *Let $C \in \overline{\mathbb{R}}^{m \times n}$ have at least one integer in each row. If $(\forall i \in M) d_i(C) = 0$ then $C \otimes \mathbf{0} \in IIm^*(C)$.*

Proof.

$$(\forall i) d_i(C) = 0 \Rightarrow (\forall i \in M)(\exists j(i) \in N) c_{ij(i)} \in \mathbb{Z} \text{ and } c_{ij(i)} = \bigoplus_{t \in N} c_{it}.$$

$$\therefore \begin{pmatrix} c_{1j(1)} \\ c_{2j(2)} \\ \vdots \\ c_{mj(m)} \end{pmatrix} = C \otimes \mathbf{0}.$$

This belongs to $IIm(C)$ since the left hand side is an integer vector. □

3.2. Transforming to column typical matrices. Here we show that, for the problem of determining if $IIm(A) \neq \emptyset$, we may assume without loss of generality that A is column typical with $m \leq n$. It follows that in order to solve the problem of whether or not a matrix has an integer vector in its column span it is sufficient to find a method for column typical matrices only.

First observe that if $A \in \overline{\mathbb{R}}^{m \times n}$ is column typical and $Ax \in \mathbb{Z}^m$ then each column A_j contains at most one active entry with respect to x. Since every row contains an active entry it is necessary that at least m columns are active in this equation. We conclude:

OBSERVATION 3.5. *Suppose $A \in \overline{\mathbb{R}}^{m \times n}$ is column typical with $m > n$. Then $IIm(A) = \emptyset$.*

Suppose without loss of generality in this subsection that $A \in \overline{\mathbb{R}}^{m \times n}$ is doubly $\mathbb{R}$-astic and no two columns in A are the same. Let

$$J^{ct}(A) = \{j \in N : A_j \text{ is column typical}\}.$$

If $j \in N - J^{ct}(A)$ then define

$$I_j^{ct} = \{i \in M : \exists t \in M, t \neq i \text{ such that } fr(a_{ij}) = fr(a_{tj})\}$$

otherwise set $I_j^{ct} = \{\emptyset\}$. The *column typical counterpart*, A^{ct}, of A is the

$$m \times (\sum_{j \in N} |I_j^{ct}|)$$

matrix obtained from A by replacing each column A_j with $|I_j^{ct}|$ columns as follows:

If $I_j^{ct} = \{\emptyset\}$ then add one copy of A_j. Otherwise for each $i \in I_j^{ct}$ add a column with entries

$$\begin{cases} a_{tj} - \delta_t & \text{if } t \in I_j^{ct} - \{i\} \\ a_{tj} & \text{otherwise.} \end{cases} \tag{3.1}$$

The values $\delta_i, i \in M$ will satisfy $0 < \delta_i < 1$ and be chosen to ensure that each new column has entries with different fractional parts.

EXAMPLE 3.6. *The columns*

$$\begin{pmatrix} 0 \\ 0 \\ 0.5 \\ 0 \\ 0.5 \\ 0.2 \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} 0 \\ 0.1 \\ 0.2 \\ 0.3 \\ 0.4 \\ 0.5 \end{pmatrix}$$

would be replaced by

$$\begin{pmatrix} 0 & 0-\delta_1 & 0-\delta_1 & 0-\delta_1 & 0-\delta_1 \\ 0-\delta_2 & 0 & 0-\delta_2 & 0-\delta_2 & 0-\delta_2 \\ 0.5-\delta_3 & 0.5-\delta_3 & 0.5 & 0.5-\delta_3 & 0.5-\delta_3 \\ 0-\delta_4 & 0-\delta_4 & 0-\delta_4 & 0 & 0-\delta_4 \\ 0.5-\delta_5 & 0.5-\delta_5 & 0.5-\delta_5 & 0.5-\delta_5 & 0.5 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} 0 \\ 0.1 \\ 0.2 \\ 0.3 \\ 0.4 \\ 0.5 \end{pmatrix}$$

where $0 < \delta_s < 1$ are such that the new matrix is column typical.

The columns in A^{ct} which replace A_j will be called the *counterparts* of A_j. For now we suppose that $\delta_i \in (0,1), i \in M$ satisfy the following four assumptions:

(A1) δ_i are distinct;

(A2) A^{ct} is column typical;

(A3i) $(\forall j \in N)(\forall i,t \in M) fr(a_{ij}) \neq fr(a_{tj})$ & $a_{ij}, a_{tj} > \varepsilon \Rightarrow fr(a_{ij} - a_{tj}) > \delta_i, \delta_t$;

(A3ii) $(\forall i \in M)(\forall j,p \in M) fr(a_{ij}) \neq fr(a_{ip})$ & $a_{ij}, a_{ip} > \varepsilon \Rightarrow fr(a_{ij} - a_{ip}) > \delta_i$;

(A4) $(\forall i \in M)(\forall j \in N) fr(a_{ij}) \neq 0$ & $a_{ij} > \varepsilon \Rightarrow \delta_i < \min(fr(a_{ij}), 1 - fr(a_{ij}))$.

THEOREM 3.7. *Let $A \in \overline{\mathbb{R}}^{m\times n}$ be doubly $\mathbb{R}$-astic and A^{ct} be the column typical counterpart of A where $\delta_i, i \in M$ satisfy A1-A4. Then*

$$IIm(A) = IIm(A^{ct}).$$

This will be proved in Section 6, where we also prove that $\delta_i, i \in M$ satisfying A1-A4 can be found efficiently and hence A^{ct} can be constructed in $\mathcal{O}((mn)^2)$ time.

3.3. Combining the transformations. So far in this section we have described two transformations which create some structure in the input matrix of the integer image problem. If the matrix A is column typical then we know that each column will have at most one active position with respect to any $z \in IIm(A)$. Further IIm is equivalent to IIm* and in this case we know that the only candidates for active positions are (i,j) such that $a_{ij} \in \mathbb{Z}$.

Both structures lead to subcases where the problem is polynomially solvable hence it seems helpful to know as much about the locations of the possible active positions as we can as well as knowing that there are not too many.

Given a matrix $A \in \overline{\mathbb{R}}^{m\times n}$ we could first construct $B = A^{ct} \in \overline{\mathbb{R}}^{m\times mn}$ and then $C = B^{int} \in \overline{\mathbb{R}}^{m\times m^2 n}$. Then $IIm(A) = IIm^*(C)$ and further the candidates for active position of C are (i,j) such that $c_{ij} \in \mathbb{Z}$ of which there is exactly one per column since C is column typical (it inherits the property from B). We conclude that $A \in IIm$ if and only if $C \in IIm^*$-P1. We will prove in Section 5 that IIm*-P1 is NP-hard. Of course, this does not resolve the complexity of IIm, C is also column typical and there is evidence to suggest that column typical matrices add structure which would help in finding a polynomial solution method to the integer image problem, if one exists.

4. Problems that are polynomially equivalent to IIm. We show that IIm, IIm-CT and IIm-CT-P1 are polynomially equivalent, and therefore belong in the same complexity class. Recall, from Definition 1.1 that IIm-CT is the class of column typical matrices with an integer image, and IIm-CT-P1$\subseteq$ IIm-CT which also requires that an integer image exists with exactly one active position in each row of the matrix.

THEOREM 4.1. *IIm-CT-P1$\in P \Leftrightarrow$ IIm-CT$\in P \Leftrightarrow$ IIm$\in P$, i.e.*

(i) IIm-CT-P1 $\leq_p$ IIm-CT.

(ii) IIm-CT $\leq_p$ IIm-CT-P1.

(iii) IIm-CT $\leq_p$ IIm.

(iv) IIm $\leq_p$IIm-CT.

Proof.

(i) and (ii): We show that if A is column typical then A has an integer image if and only if A has an integer image in which there is exactly one active entry per row.

The sufficient direction is clear. So assume that A has an integer image z. Then $\exists x \in \overline{\mathbb{R}}^n$ such that $Ax = z$. If there exist $k, j \in N$ such that a_{ij} and a_{ik} are both active for some $i \in M$ then the vector x' obtained from x by replacing x_k by ε also satisfies $Ax' = z$. This is because A is column typical meaning there is at most one active entry in every column and so removing A_k from the system will not affect active entries in any other row. In this way we can construct a vector x'' such that $Ax'' = z$ and A has exactly one active entry per row.

(iii) Obvious.

(iv) Let $A \in \overline{\mathbb{R}}^{m\times n}$. Let $A^{ct} \in \overline{\mathbb{R}}^{m\times k}, k \leq mn$ be the column typical counterpart of A as defined in Section 3.

We have $IIm(A) \neq \emptyset \Leftrightarrow IIm(A^{ct}) \neq \emptyset$.

Therefore A is an instance of IIm-CT if and only if A^{ct} is an instance of IIm-CT-P1 and A^{ct} can be constructed in $\mathcal{O}((mn)^2)$ time. ∎

COROLLARY 4.2. *To show that $IIm \in NPC$ or $IIm \in P$ it is enough to consider either IIm-CT-P1 or IIm-CT.*

We know from Theorem 2.6 that checking whether a square matrix is in IIm-CT is achievable in polynomial time. But this does not imply that IIm for square matrices is polynomially solvable since in transforming a matrix to its column typical counterpart we increase the number of columns.

5. Related NP-hard problems. In this section we consider modifications of the integer image problem which we can prove to be NP-hard. The hardness of these related problems does not imply hardness of IIm, that question remains open. The related problems we consider are IIm-P1, IIm-P2, IIm-P3, IIm-P4 and IIm*-P1 as described in Definition 1.1. Recall that each problem class requires the following additional conditions on the matrix.

IIm-P1: Integer image with exactly one active position per row.

IIm-P2: Integer image with exactly two active position per row.

IIm-P3: Integer image with at most two active position per row.
IIm-P4: Integer image with at most t active columns.
IIm*-P1: Integer image with exactly one active position per row and integer active entries.

Each proof will use the following key result.

PROPOSITION 5.1. *Fix $\alpha \in (0,1)$. Let $A \in \{0, \alpha^{-1}\}^{m\times n}$ be a matrix in which each column has at least one zero entry. If $z \in IIm(A)$ then*

(i) for any $x \in X(A)$ such that $Ax = z$ all active entries of A are integer (zero),

(ii) z is a constant vector, and

(iii) if $A_j, j \in N$ contains an active position then (i,j) is active for all $i \in M$ such that $a_{ij} = 0$.

Proof. Assume $(\exists z \in \mathbb{Z}^m)(\exists x \in \overline{\mathbb{R}}^n)Ax = z$.

(i) Suppose $a_{ij} = \alpha^{-1}$ is active, so $x_j = z_i\alpha \notin \mathbb{Z}$. By assumption there exists a zero entry in every column so let t be an index such that $a_{tj} = 0$. Then $a_{tj}x_j \notin \mathbb{Z}$, so a_{tj} is inactive and there exists l such that a_{tl} is active. Hence we have

$$\begin{aligned} \alpha^{-1}x_j =& z_i, \\ 0x_j <& z_t, \\ a_{il}x_l \le& z_i \text{ and} \\ a_{tl}x_l =& z_t. \end{aligned}$$

From the first two equations we obtain $z_i = \alpha^{-1}x_j < x_j < z_t$ and therefore $z_i 1 \le z_t$. Using this and the last two equations we get

$$a_{tl}x_l = z_t \ge z_i 1 \ge a_{il}x_l 1.$$

This implies that $a_{tl} \ge a_{il}1$, a contradiction with $|a_{il}a_{tl}^{-1}| \le \alpha < 1$.

(ii) Suppose there exists $x \in \overline{\mathbb{R}}^n$ such that $Ax = z \in \mathbb{Z}^m$ where $(\exists i, t \in M)z_i \ne z_t$. Without loss of generality assume that $z_i > z_t$, in fact $z_i \ge z_t 1$. Let a_{ij}, a_{tl} be active entries in rows i and t respectively. Note that by (i), $a_{ij} = 0 = a_{tl}$, meaning $x_j = z_i$ and $x_l = z_t$. But then

$$A_j x_j = A_j z_i \ge A_j(z_t 1) = (1A_j)z_t > A_l z_t = A_l x_l$$

which implies that A_l is inactive. This is a contradiction since a_{tl} is active.

(iii) Denote $S = \{j :$ There exists an active entry in $A_j\}$. Fix $j \in S$ and suppose $a_{ij}x_j = z_i$. Then by (i), $x_j = z_i$ and hence

$$A_j x_j = A_j z_i \le \begin{pmatrix} z_1 \\ \vdots \\ z_m \end{pmatrix} = \begin{pmatrix} z_i \\ \vdots \\ z_i \end{pmatrix}$$

where the final equality is obtained using (ii). Finally for all $t \in M$ such that $a_{tj} = 0$ we have $a_{tj}x_j = z_i$, therefore every integer (zero) entry in A_j is active. □

It is important to observe that any matrix $A \in \{0, \alpha^{-1}\}^{m\times n}$ with at least one zero entry in each column has an integer image if and only if there is a zero in every row, which occurs if and only if $\mathbf{0} \in IIm(A)$. In fact

$$IIm(A) \ne \emptyset \Leftrightarrow IIm(A) = \{\gamma\mathbf{0} : \gamma \in \mathbb{Z}\}.$$

Thus when we consider whether the matrix has an integer image that also satisfies some specified property (number of active entries per row, column etc) this property will be determined by the vector x such that $Ax = \mathbf{0}$. Note that it can be assumed that $x \in \{0, \varepsilon\}^n$ where the finite components correspond to active columns of A. These ideas will be used throughout the proofs of NP-hardness in this section.

We will use reductions from the following NP-hard problems.

(Monotone 1-in-3 SAT): 1-in-3 SAT is a modification of the SAT problem in which each clause has 3 literals and we ask whether there exists a satisfying assignment such that exactly one literal in each clause is TRUE. The monotone version of the problem satisfies the additional condition that each clause contains only unnegated variables. Note that without loss of generality each literal appears in at least one clause.
(Monotone NAE-3-SAT) Not all equal 3-SAT is a modification of the SAT problem in which every clause contains 3 literals and we ask whether there exists a satisfying assignment for which no clause contains only TRUE literals. This means that every clause will contain at least one TRUE and at least one FALSE literal (no clause will contain only FALSE literals with respect to a satisfying assignment). The monotone version of the problem satisfies the additional condition that each clause contains only unnegated variables.
(MCCP) The minimum cardinality cover problem: Given a universe U, a family S of finite subsets of U and a positive integer t, does there exist a subfamily $C \subseteq S$, $|C| \leq t$ such that C is a cover of U?

Remark 5.2.

(i) 1-in-3 SAT is problem L04 in [8], where it is noted that it remains NP complete even if no clause contains a negated literal. The result follows from the classification of NP-hard satisfiability problems in [13].

(ii) Monotone NAE-3-SAT is also NP complete, as noted in [12] and is again due to the results in [13].

(iii) MCCP is problem SP5 in [8].

Theorem 5.3.

(i) Monotone 1-in-3 SAT$\leq_p$ IIm-P1.

(ii) Monotone 1-in-3 SAT$\leq_p$ IIm-P2.

Proof. Let $F = C_1 \wedge ... \wedge C_m$ where every clause contains 3 unnegated literals from $\{y_1, ..., y_n\}$.

Construct an $m \times n$ matrix $A = (a_{ij})$ as follows: For some $\alpha \in (0, 1)$,

$$a_{ij} = \begin{cases} 0 & \text{if } y_j \in C_i \\ \alpha^{-1} & \text{otherwise.} \end{cases}$$

Note that A can be constructed in polynomial time.

Example 5.4. *For $F = (y_1 \vee y_2 \vee y_3) \wedge (y_1 \vee y_3 \vee y_4)$ we obtain*

$$A = \begin{pmatrix} 0 & 0 & 0 & \alpha^{-1} \\ 0 & \alpha^{-1} & 0 & 0 \end{pmatrix}.$$

Now assume there exists $z \in \mathbb{Z}^m$ such that $(\exists x \in \overline{\mathbb{R}}^n)\ Ax = z$.

Since A satisfies the conditions of Proposition 5.1 we know that there exists $\gamma \in \mathbb{Z}$ such that $z = \gamma\mathbf{0}$ and active entries are integer, thus $x \in \mathbb{Z}^n$. Further for all $j \in S$,

$$a_{ij} = 0 \Rightarrow a_{ij} \text{ is active}$$

where $S = \{j :$ There exists an active entry in $A_j\}$.

(i) If $Ax = z$ with exactly one active entry per row then $y = (y_1, ..., y_n)^T$ is a satisfying assignment of F with exactly one TRUE literal per clause where

$$y_j = \begin{cases} 1 & \text{if } j \in S \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand if F has a satisfying assignment y in which exactly one literal in each clause is satisfied then for all $j \in N$ let

$$x_j = \begin{cases} 0 & \text{if } y_j = 1 \\ \varepsilon & \text{else.} \end{cases}$$

The vector $x = (x_1, ..., x_n) \in \overline{\mathbb{R}}^n$ is such that $Ax = \mathbf{0}$ and there is exactly one active entry per row.

Therefore F has a satisfying assignment with exactly one TRUE literal per clause if and only if A has an integer image with exactly one active entry per row.

(ii) If $Ax = z$ with exactly two active literals per clause then $y = (y_1, ..., y_n)^T$ is a satisfying assignment of F with exactly two TRUE literals per clause where

$$y_j = \begin{cases} 1 & \text{if } j \in S \\ 0 & \text{otherwise.} \end{cases}$$

Hence $\bar{y}$ is a satisfying assignment of F with exactly one TRUE literal per clause.

Finally if F has a satisfying assignment y in which exactly one literal in each clause are satisfied then for all $j \in N$ let

$$x_j = \begin{cases} 0 & \text{if } \bar{y}_j = 1 \\ \varepsilon & \text{else.} \end{cases}$$

The vector $x = (x_1, ..., x_n) \in \overline{\mathbb{R}}^n$ is such that $Ax = \mathbf{0}$ and there are exactly two active entries per row.

Therefore F has a satisfying assignment with exactly one TRUE literal per clause if and only if A has an integer image with exactly two active entries per row. ∎

THEOREM 5.5. *Monotone NAE-3-SAT$\leq_p$ IIm-P3.*

Proof. Let $F = C_1 \wedge ... \wedge C_m$ where every clause contains 3 unnegated literals from $\{y_1, ..., y_n\}$.

Construct a matrix $A = (a_{ij})$ exactly as before: For some $\alpha \in (0, 1)$,

$$a_{ij} = \begin{cases} 0 & \text{if } y_j \in C_i \\ \alpha^{-1} & \text{otherwise.} \end{cases}$$

Now assume there exists $z \in \mathbb{Z}^m$ such that $(\exists x \in \overline{\mathbb{R}}^n)$ $Ax = z$ with at most two active entries per row.

By Proposition 5.1 there exists $\gamma \in \mathbb{Z}$ such that $z = \gamma\mathbf{0}$ and if A_j contains an active entry then all integer (zero) entries in A_j will be active. Observe that by construction A has at most three integer (zero) entries in each row, so the condition that there are at most two active entries in each row means that not all zero entries are active.

Now define a Boolean vector $y = (y_1, ..., y_n)^T$ where

$$y_j = \begin{cases} 1 & \text{if } A_j \text{ is active} \\ 0 & \text{otherwise.} \end{cases}$$

Then y is a satisfying assignment of F with no clause containing only TRUE literals.

Finally if F has a satisfying assignment y in which no clause contains only TRUE literals then for all $j \in N$ let

$$x_j = \begin{cases} 0 & \text{if } y_j = 1 \\ \varepsilon & \text{else.} \end{cases}$$

The vector $x \in \overline{\mathbb{R}}^n$ is such that $Ax = \mathbf{0}$ and there are at most two active entries per row.

Therefore A can be constructed in polynomial time and F has a satisfying assignment with no clause containing only TRUE literals if and only if A has an integer image with at most two active entries per row. □

THEOREM 5.6. *MCCP$\leq_p$ IIm-P4.*

Proof. Let $U = \{u_1, ..., u_m\}$, $S = \{S_1, ..., S_n\}, S_j \subseteq U$ and $t \in \mathbb{N}$ be an instance of the MCCP.

Fix $\alpha \in (0, 1)$ and define $A = (a_{ij}) \in (0, \alpha^{-1})^{m \times n}$ by

$$a_{ij} = \begin{cases} 0, & \text{if } u_i \in S_j; \\ \alpha^{-1}, & \text{otherwise.} \end{cases}$$

Note that since S contains no duplicate sets the columns of A are distinct. Clearly active entries will be the integer (zero) entries of A and any $z \in IIm(A)$ is a constant vector by Proposition 5.1.

Now assume there exists $z \in \mathbb{Z}^m$ such that $(\exists x \in \overline{\mathbb{R}}^n)\ Ax = z$ with at most t active columns. For all $j \in N$ if A_j is active then every integer (zero) entry in A_j is active and so we place S_j into C. It follows that C is a cover of U and $|C| \leq t$.

Finally if C is a cover of U with $|C| \leq t$ then define $x = (x_1, ..., x_n) \in \overline{\mathbb{R}}^n$ by

$$x_j = \begin{cases} 0 & \text{if } S_j \in C \\ \varepsilon & \text{otherwise.} \end{cases}$$

Now $Ax = \mathbf{0}$ and the number of active columns is at most the number of j such that $x_j > \varepsilon$, which is t.

Therefore A has an integer image for which at most t columns are active if and only if there exists a cover of U of size at most t. Note that A can be constructed in polynomial time. □

COROLLARY 5.7. *The following problems are NP-hard.*

(i) IIm-P1,

(ii) IIm-P2,

(iii) IIm-P3,

(iv) IIm-P4 and

(v) IIm*-P1.

Proof. (i)-(iv) follow immediately from Theorems 5.3, 5.5 and 5.6 and Remark 5.2.

(v) holds for the exact same reasons as (i), an identical proof can be used to show Monotone 1-in-3 SAT$\leq_p$ IIm*-P1. □

6. Proving the validity of column typical counterparts. We prove the results from Subsection 3.2 which we repeat below.

Theorem 3.7 states: Let $A \in \overline{\mathbb{R}}^{m\times n}$ be doubly $\mathbb{R}$-astic and A^{ct} be the column typical counterpart of A where $\delta_i, i \in M$ satisfy A1-A4. Then

$$IIm(A) = IIm(A^{ct}).$$

Further $\delta_i, i \in M$ satisfying A1-A4 can be found efficiently and A^{ct} constructed in $\mathcal{O}((mn)^2)$ time where assumptions A1-A4 are as as follows.

(A1) δ_i are distinct;

(A2) A^{ct} is column typical;

(A3i) $(\forall j \in N)(\forall i, t \in M) fr(a_{ij}) \neq fr(a_{tj})$ & $a_{ij}, a_{tj} > \varepsilon \Rightarrow fr(a_{ij} - a_{tj}) > \delta_i, \delta_t$;

(A3ii) $(\forall i \in M)(\forall j, p \in M) fr(a_{ij}) \neq fr(a_{ip})$ & $a_{ij}, a_{ip} > \varepsilon \Rightarrow fr(a_{ij} - a_{ip}) > \delta_i$;

(A4) $(\forall i \in M)(\forall j \in N) fr(a_{ij}) \neq 0$ & $a_{ij} > \varepsilon \Rightarrow \delta_i < \min(fr(a_{ij}), 1 - fr(a_{ij}))$.

6.1. Proof of Theorem 3.7. Recall (from Subsection 3.2) that we assume $A \in \overline{\mathbb{R}}^{m\times n}$ has no two identical columns.

PROPOSITION 6.1. *If A has no two identical columns then all columns of A^{ct} are different when $\delta_i, i \in M$ are chosen according to assumptions A1-A4 .*

Proof. If $A^{ct}_{j_1}$ and $A^{ct}_{j_2}$ are both counterparts to A_j and $j_1 \neq j_2$ then by definition $A^{ct}_{j_1} \neq A^{ct}_{j_2}$.

Assume then that $A^{ct}_{c(j)}$ and $A^{ct}_{c(p)}$ are counterparts of A_j and A_p respectively, $j \neq p$. Since $A_j \neq A_p$ there exists i such that $a_{ij} \neq a_{ip}$. We prove that $a^{ct}_{ic(j)} \neq a^{ct}_{ic(p)}$by showing

$$\begin{aligned} a_{ij} - \delta_i &\neq a_{ip} - \delta_i, \\ a_{ij} - \delta_i &\neq a_{ip} \text{ and} \\ a_{ij} &\neq a_{ip} - \delta_i. \end{aligned}$$

The first is immediate, the second and third are proved in the same way. To see that the third statement holds assume, for a contradiction, that $a_{ij} = a_{ip} - \delta_i$. If $fr(a_{ip}) = 0$ then $fr(a_{ij}) = 1 - \delta_i > 0$, a contradiction with A4. If instead $fr(a_{ip}) > 0$ then by A4, $fr(a_{ip}) > \delta_i$. Further (using A3 and Lemma 2.1)

$$\begin{aligned} &a_{ip} > a_{ij} = a_{ip} - \delta_i > \lfloor a_{ip} \rfloor \Rightarrow fr(a_{ip}) > fr(a_{ij}) \text{ and} \\ &fr(a_{ij}) = fr(a_{ip} - \delta_i) = fr(a_{ip}) - \delta_i. \end{aligned}$$

But then

$$\delta_i = fr(a_{ip}) - fr(a_{ij}) = fr(a_{ip} - a_{ij}),$$

a contradiction with A3ii. $\square$

We set

$$r = \sum_{j \in N} |I^{ct}_j|.$$

First assume that $z \in IIm(A) \neq \emptyset$. So there exists $x \in X(A)$ such that $Ax = z$. Observe that the vector

$$x' = (x_1, ..., x_1, x_2, ..., x_2, ..., x_n, ..., x_n)^T \in \overline{\mathbb{R}}^r$$

(where each $x_j, j \in N$ is repeated $|I_j^{ct}|$ times) satisfies $A^{ct}x' = z$.

For the other direction assume that $z \in IIm(A^{ct}), x \in X(A^{ct})$ and let

$$c(1), ..., c(m) \in \{1, ..., r\}$$

be indices of m active columns of A^{ct} such that

$$\bigoplus_{t=1}^{m} A_{c(t)}^{ct} x_{c(t)} = z. \tag{6.1}$$

So $c(1), ..., c(m)$ represent a list of m active columns $A_{c(1)}^{ct}, ..., A_{c(m)}^{ct}$ in A^{ct} with respect to z. Note that there is exactly one active entry in each of the columns in the list as they are column typical.

We prove that there exists some $x' \in \overline{\mathbb{R}}^n$ such that $Ax' = z$ by considering two cases: when the list of these m active columns in A^{ct} contains at most one counterpart of each column $A_j, j \in N$ and when it contains more than one counterpart to some column A_j. To do this we first need the following claim on the active entries of columns in the list.

Claim 6.2. *Let $c(1), ..., c(m)$ represent a list of m active columns of A^{ct} satisfying (6.1). For each $t \in \{1, ..., m\}$ there exists an index $p(t)$ such that the new list of columns $A_{p(1)}^{ct}, ..., A_{p(m)}^{ct}$ satisfies*

$$\bigoplus_{t=1}^{m} A_{p(t)}^{ct} y_{p(t)} = z$$

for some $y \in X(A^{ct})$ where the active entry of each $A_{p(t)}^{ct}$ has not been altered when moving from A to A^{ct}.

Proof.

Fix $t \in \{1, ..., m\}$ and suppose $A_{c(t)}^{ct}$ is a counterpart to A_j for some $j \in N$. Further suppose that the active entry in $A_{c(t)}^{ct}$ is in row i. If $a_{ic(t)}^{ct} = a_{ij}$ then let $p(t) = c(t)$ and $y_{p(t)} = x_{c(t)}$.

If instead $a_{ic(t)}^{ct} = a_{ij} - \delta_i$ then we know $|I_j^{ct}| \geq 2$ and

$$A_{c(t)}^{ct} x_{c(t)} \leq z$$

with equality only in row i. Defining $\mu = x_{c(t)} - \delta_i$, we obtain

$$\begin{aligned}
&(a_{ij} - \delta_i) + (\mu + \delta_i) = z_i, \\
&a_{sj} + (\mu + \delta_i) < z_s \ \forall s \notin I_j^{ct} \text{ and} \\
&(a_{sj} - \delta_s) + (\mu + \delta_i) < z_s \ \forall s \in I_j^{ct} - \{i\}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&a_{ij} + \mu = z_i, \\
&a_{sj} + \mu < z_s \ \forall s \notin I_j^{ct} \text{ and} \\
&(a_{sj} - \delta_s) + \mu < z_s \ \forall s \in I_j^{ct} - \{i\}.
\end{aligned}$$

But this means that there exists a counterpart of A_j in A^{ct}, say A_p^{ct}, such that

$$A_p^{ct} \mu \leq z$$

with equality only for z_i and active entry $a^{ct}_{ip} = a_{ij}$. So set $p(t) = p$ in our choice of columns, and $y_{c(t)} = x_{c(t)} - \delta_i$.

Repeat this for each column in the list. For any unassigned entry of y set $y_l = x_l$. This results in a new list of m distinct columns $A^{ct}_{p(t)}, t \in \{1, ..., m\}$ such that

$$\bigoplus_{t=1}^{m} A^{ct}_{p(t)} y_{p(t)} = z$$

and having active entries which are unaltered from A. It immediately follows that $A^{ct} y = z$ and hence $y \in X(A^{ct})$. $\square$

Hence we can assume that $Ax = z$, and further that there is a list of m active columns $A^{ct}_{c(t)}, t \in \{1, ..., m\}$ satisfying (6.1) with active entries unaltered by some δ_i, i.e. entries such that $a^{ct}_{ij} = a_{ij}$. We use this to describe $x' \in \overline{\mathbb{R}}^n$ such that $Ax' = z$.

Case 1: $(\forall j, l \in \{1, ..., m\}, j \neq l)$ $A^{ct}_{c(j)}$ and $A^{ct}_{c(l)}$ are counterparts to different columns in A.

By rearranging columns in A if necessary, we can assume without loss of generality that $A^{ct}_{c(t)}, t \in \{1, ..., m\}$ is a counterpart to A_t.

Define $x' \in \overline{\mathbb{R}}^n$ by $x'_t = x_{c(t)}, t \in \{1, ..., m\}$ and ε otherwise. Observe that $Ax' \geq z$ since if $a^{ct}_{ic(j)}$ is active in A^{ct} with respect to x then, using Claim 1, $a^{ct}_{ic(j)} + x'_j = a_{ij} + x_j = z_i$. It remains to show $Ax' \leq z$.

Assume there exists $i \in M$, $t \in \{1, ..., m\}$ such that $a_{it} + x'_t > z_i$. Then, by definition of A^{ct}, $a_{it} - \delta_i = a^{ct}_{ic(t)}$ and $a^{ct}_{ic(t)}$ is inactive in $A^{ct} x = z$. Note that $A^{ct}_{c(t)}$ is active in (6.1) so there exists $i' \in M, i' \neq i$ such that $a^{ct}_{i'c(t)} + x_{c(t)} = z_{i'}$ and additionally $a^{ct}_{i'c(t)} = a_{i't}$ by our assumptions on the active entries.

Case 1a: $fr(a_{it}) \neq fr(a_{i't})$

We have

$$a_{it} + x'_t > z_i > a^{ct}_{ic(t)} + x_{c(t)} = a_{it} - \delta_i + x'_t. \tag{6.2}$$

Since $z_i \in \mathbb{Z}$ we deduce

$$\lfloor a_{it} + x'_t \rfloor \geq \lceil a_{it} - \delta_i + x'_t \rceil.$$

This implies that $\delta_i \geq fr(a_{it} + x'_t)$. Further $a_{i't} + x'_t = z_{i'}$ implies $fr(x'_t) = 1 - fr(a_{i't})$. Therefore, using Lemma 2.1,

$$\delta_i \geq fr(a_{it} + x'_t) = fr(fr(a_{it}) + 1 - fr(a_{i't})) = fr(a_{it} - a_{i't})$$

but this is a contradiction with assumption A3i on δ_i.

Case 1b: $fr(a_{it}) = fr(a_{i't})$

Using $a_{it} + x'_t > z_i$ and $a_{i't} + x'_t = z_{i'}$ we get that $a_{it} + x'_t \in \mathbb{Z}$ and therefore $a_{it} + x'_t \geq z_i + 1$. But then $a^{ct}_{ic(t)} + \delta_i + x_{c(t)} \geq z_i + 1$ which is a contradiction with $a^{ct}_{ic(t)} + x_{c(t)} \leq z_i$ as it suggests $\delta_i \geq 1$.

In both subcases we reach a contradiction and therefore $A_t x'_t \leq z$. Since this argument holds for all i we conclude $Ax' \leq z$ and then that $Ax' = z$ as required.

Case 2: $(\exists j \in N)(\exists s, t \in \{1, ..., m\})$ $A^{ct}_{c(s)}$ and $A^{ct}_{c(t)}$ are counterparts of A_j.

We would like to argue that the same idea as in Case 1 holds here, however to do this we must show that when we go from $A^{ct}_{c(s)}$ and $A^{ct}_{c(t)}$ back to A_j the components $x_{c(s)}$ and $x_{c(t)}$ do not cause a problem.

Both columns have a single active entry in different rows, i_1 and i_2 say.

So, using our assumptions on the active entries,

$$a_{i_1j} + x_{c(s)} = a^{ct}_{i_1c(s)} + x_{c(s)} = z_{i_1} \text{ and } a_{i_2j} + x_{c(t)} = a^{ct}_{i_2c(t)} + x_{c(t)} = z_{i_2},$$
$$\therefore a^{ct}_{i_1c(s)} + x_{c(s)} \geq a^{ct}_{i_1c(t)} + x_{c(t)} \text{ and } a^{ct}_{i_2c(s)} + x_{c(s)} \leq a^{ct}_{i_2c(t)} + x_{c(t)}.$$

Note that if $p, q \in \{1, ..., m\}$ and $A_{c(p)}, A_{c(q)}$ are counterparts to the same column in A then

$$(\forall i \in M)|a^{ct}_{ic(p)} - a^{ct}_{ic(q)}| \in \{0, \delta_i\}.$$

Using this we obtain

$$x_{c(t)} - x_{c(s)} \leq a^{ct}_{i_1c(s)} - a^{ct}_{i_1c(t)} \leq \delta_{i_1} \text{ and } x_{c(s)} - x_{c(t)} \leq a^{ct}_{i_2c(t)} - a^{ct}_{i_2c(s)} \leq \delta_{i_2}$$
$$\therefore -\delta_{i_1} \leq x_{c(s)} - x_{c(t)} \leq \delta_{i_2}.$$

Substituting $x_{c(s)} = z_{i_1} - a_{i_1j}$ and $x_{c(t)} = z_{i_2} - a_{i_2j}$ gives

$$-\delta_{i_1} \leq z_{i_1} - a_{i_1j} - z_{i_2} + a_{i_2j} \leq \delta_{i_2}.$$

Case 2a: $0 = z_{i_1} - a_{i_1j} - z_{i_2} + a_{i_2j}$.

Then $fr(a_{i_1j}) = fr(a_{i_2j})$ and more importantly $0 = z_{i_1} - a_{i_1j} - z_{i_2} + a_{i_2j} = x_{c(s)} - x_{c(t)}$ so $x_{c(s)} = x_{c(t)}$ and there will be no conflict in choosing x'_j. We detail this later.

Case 2b: $0 < z_{i_1} - a_{i_1j} - z_{i_2} + a_{i_2j} \leq \delta_{i_2}$.

Then since $\delta_{i_2} < 1$ and $fr(a_{i_2j}) \neq fr(a_{i_1j})$ we have,

$$\delta_{i_2} \geq z_{i_1} - a_{i_1j} - z_{i_2} + a_{i_2j} = fr(z_{i_1} - a_{i_1j} - z_{i_2} + a_{i_2j}) = fr(a_{i_2j} - a_{i_1j}).$$

But this is a contradiction with assumption A3i on δ. So this case does not occur.

Case 2c: $0 < z_{i_2} - a_{i_2j} - z_{i_1} + a_{i_1j} \leq \delta_{i_1}$.

Similarly as in Case 2b we can reach a contradiction on the size of δ_{i_1}.

Since only Case 2a can occur we conclude that the active entries of $A^{ct}_{c(s)}$ and $A^{ct}_{c(t)}$ correspond to entries of A_j with the same fractional part, and $x_{c(s)} = x_{c(t)}$. This proves that there is no conflict moving from multipliers $x_{c(s)}$ and $x_{c(t)}$ to a single multiplier x_j.

In general, given a list $A^{ct}_{c(1)}, ..., A^{ct}_{c(m)}$ satisfying (6.1) with active entries unaltered by any δ_i, we construct $x' \in \overline{\mathbb{R}}^n$ as follows:

For each $j \in N$

(1) If no column corresponding to A_j in A^{ct} is in the list then let $x'_j = \varepsilon$.

(2) If exactly one column, $A^{ct}_{c(j)}$ say, corresponding to A_j is in the list set $x'_j = x_{c(j)}$.

(3) If more than one column corresponding to A_j in A^{ct} is in the list then choose any of them, $A^{ct}_{c(j')}$ say, and set $x_j = x_{c(j')}$.

Finally $Ax' = z$ can be shown using similar arguments as in Case 1; $Ax' \geq z$ because $Ax' \geq A^{ct}x = z$ and $Ax' \leq z$ because otherwise there would exist i, t such that

$$a_{it} + x'_t > z_i \geq a^{ct}_{ic(t)} + x_{c(t)} \geq a_{it} - \delta_i + x'_t,$$

which is exactly (6.2) and so we can follow the same argument to reach a contradiction with assumption A3i on δ_i.

This ends the proof of Theorem 3.7.

6.2. The choice of δ_i. Given $A \in \overline{\mathbb{R}}^{m\times n}$ we show how to choose $0 < \delta_i < 1, i \in M$ satisfying A1-A4 in $\mathcal{O}((mn)^2)$ time.

We achieve this by showing there exists $g \in (0,1)$ such that any choice of $\delta_i, i \in M$ satisfying $(\forall i)\delta_i < g$ will satisfy A3 and A4. It follows from A3 and A4 that satisfying A1 is trivial. First we prove that A2 follows from A1, A3 and A4. It may be useful to recall Lemma 2.1 on the basic properties of $fr(\cdot)$.

Claim 6.3. *If $0 < \delta_i < 1, i \in M$ satisfy A1, A3 and A4 then $\delta_i, i \in M$ satisfy A2.*

Proof. Assume that A^{ct} is not column typical, thus $\exists j \in N - J^{ct}(A)$. Then there exist $i, t \in M, i \neq t$ such that

$$fr(a_{ij} - \delta_i) = fr(a_{tj} - \delta_t). \tag{6.3}$$

Since $\delta_i \neq \delta_t$ we conclude, using A4, that $fr(a_{ij}) \neq fr(a_{tj})$. Assume without loss of generality that $fr(a_{ij}) > fr(a_{tj})$, therefore

$$fr(a_{ij} - \delta_i) = fr(a_{ij}) - \delta_i \text{ and } fr(a_{tj} - \delta_t) = \begin{cases} fr(a_{tj}) - \delta_t; & \text{if } fr(a_{tj}) > 0; \\ 1 - \delta_t; & \text{otherwise.} \end{cases}$$

Case 1: $fr(a_{tj}) > 0$

Substituting into (6.3) we obtain, using $fr(a_{ij}) > fr(a_{tj})$,

$$fr(a_{ij} - a_{tj}) = fr(a_{ij}) - fr(a_{tj}) = \delta_i - \delta_t \Rightarrow \delta_i > fr(a_{ij} - a_{tj})$$

which contradicts A3.

Case 2: $fr(a_{tj}) = 0$

From (6.3) we get $fr(a_{ij}) - \delta_i = 1 - \delta_t$ which implies $1 - \delta_t < fr(a_{ij})$. But then $\delta_t > 1 - fr(a_{ij}) = fr(a_{tj} - a_{ij})$, a contradiction with A3. ∎

We now consider how to choose δ_i such that A3 and A4 hold.

Let

$$\begin{aligned} F &= \{fr(a_{ij}) : i \in M, j \in N\} - \{0\}, \\ F' &= \{1 - fr(a_{ij}) : i \in M, j \in N\} - \{0\} \text{ and} \\ G &= \{fr(f + f') : f \in F, f' \in F'\} - \{0\}. \end{aligned}$$

So $|F|, |F'| \leq mn$ and $|G| \leq (mn)^2$.

Consider satisfying A3:

To satisfy A3i for each column j of A we need to exclude any $fr(a_{ij} - a_{tj}) \neq 0$ from our choice of δ_i. By Lemma 2.1

$$fr(a_{ij} - a_{tj}) = fr(fr(a_{ij}) - fr(a_{tj}))$$

and hence these excluded values are contained in G. The same argument holds for rows so A3ii is also satisfied by excluding values from G.

To satisfy A4 we additionally exclude the values from $F \cup F'$ from our choice of δ_i.

Now let g be the minimum of the at most $(mn)^2 + 2mn$ values from

$$F \cup F' \cup G.$$

Then any choice of distinct δ_i satisfying $0 < \delta_i < g$ will satisfy our assumptions.

7. Conclusion. We have shown that the problem of determining whether a matrix has a non empty set of integer images can be reduced to the problem of determining whether a column typical matrix has a non empty set of integer images. If the matrix is square then the column typical version of the problem can be solved in polynomial time, which, on the one hand, gives hope that maybe the integer image problem is polynomially solvable. On the other hand we show that similar problems are hard. The problem of finding an integer image of a column typical matrix is equivalent to determining whether a column typical matrix has an integer image for which there is exactly one active entry per row. The complexity of this problem for column typical matrices remains unresolved but if we remove the assumption that the matrix is column typical, we find that the problem of determining whether a general matrix has an integer image with exactly one active entry per row is NP-hard.

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