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Weakly and strongly stable max-plus matrices

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***Abstract*— A max-plus matrix A is called weakly stable if the orbit of A does not reach an eigenvector of A for any starting vector x unless x is an eigenvector itself. This is in contrast to strongly stable (robust) matrices for which the orbit reaches an eigenvector with any nontrivial starting vector. Max-plus matrices are used to describe multiprocessor interactive systems for which reachability of a steady regime is equivalent to reachability of an eigenvector by a matrix orbit. We prove that an irreducible matrix is weakly stable if and only if its critical graph is a Hamiltonian cycle in the associated graph. We extend this condition to reducible matrices. These criteria can be checked in polynomial time. They complement the known criteria for strong stability which will also be presented.**

I. INTRODUCTION

Consider the system in which processors $P_1,...,P_n$ work interactively and in stages. In each stage all processors simultaneously produce components necessary for the work of some or all other processors in the next stage. Let $x_i(k)$ denote the starting time of the k^{th}stage on P_i $(i=1,...,n)$ and let a_{ij} denote the duration of the operation at which processor P_j prepares the component necessary for processor P_i in the $(k+1)^{st}$ stage $(i,j=1,...,n)$. Then, avoiding any delay, we have for every $i=1,...,n$ and $k=0,1,...$:

$$x_i(k+1)=\max(x_1(k)+a_{i1},...,x_n(k)+a_{in}).$$

We refer to such a system as a *multiprocessor interactive system (MPIS)*. The matrix $A=(a_{ij})$ will be called the *production matrix*. In order to analyze such systems transparently and efficiently, we denote $a\oplus b=\max(a,b)$ and $a\otimes b=a+b$. Then we have for every $i=1,...,n$ and $k=0,1,...$:

$$x_i(k+1)=a_{i1}\otimes x_1(k)\oplus...\oplus a_{in}\otimes x_n(k)$$

If, moreover, the pair of operations $(\oplus,\otimes)$ is extended to matrices and vectors in the same way as in linear algebra then we obtain a compact expression $x(k+1)=A\otimes x(k)$ $(k=0,1,...)$.

We say that an MPIS reaches a *steady regime* [8] if it eventually moves forward in regular steps, that is for some λ and k_0 we have $x(k+1)=\lambda\otimes x(k)$ for all $k\geq k_0$. Equivalently, the time between the starts of consecutive stages will eventually be the same constant for every processor. If this happens then we have $A\otimes x(k)=\lambda\otimes x(k)$ for all $k\geq k_0$ and so $x(k)$ is an eigenvector of A with respect to the pair of operations $(\oplus,\otimes)$ ("eigenvector"), with associated value λ ("eigenvalue"). Reaching stability is a desirable goal and the task of achieving this is of fundamental importance for any MPIS. Since $x(k)=A\otimes x(k-1)=A^2\otimes x(k-2)=...=A^k\otimes x(0)$ for every natural k, the questions that may be of operational interest are:

Q1: *Given $A\in\overline{\mathbb{R}}^{n\times n}$ and $x\in\overline{\mathbb{R}}^{n}$ is there a natural number k such that $A^k\otimes x$ is a max-eigenvector of A?*

Q2: *Characterize matrices A (strongly stable) for which $A^k\otimes x$ is an eigenvector for every x and sufficiently large k.*

Q3: *Characterize matrices A (weakly stable) for which $A^k\otimes x$ is not an eigenvector for any x and any k unless x is an eigenvector itself.*

Question Q1 for irreducible matrices can be answered using an $O\left(n^3\log n\right)$ algorithm [12], see also [4]. It is known that $A^k\otimes x$ always reaches an eigenvector of some power of A and the algorithm in [12] in fact finds the smallest natural number s such that $A^k\otimes x$ is a max-eigenvector of A^s. An efficient characterization of strongly stable matrices is known [6] (note that in that paper strongly stable matrices are called robust), see also [4]. The aim of this presentation is to provide a full and efficient characterization of weakly stable matrices.

II. PREREQUISITES

In this section we give the definitions and an overview of those results of max-algebra which will be instrumental for the formulations and proofs of the results of this paper. For the proofs and more information about max-algebra the reader is referred to [1], [2], [11], and [4].

We will use the following notation: As usual $\mathbb{R}$ is the set of real numbers and the symbol $\overline{\mathbb{R}}$ will stand for $\mathbb{R}\cup\{-\infty\}$. If $a,b\in\overline{\mathbb{R}}$ then we set $a\oplus b=\max(a,b)$ and $a\otimes b=a+b$. We denote $-\infty$ by ε. We assume everywhere that $n\geq 1$ is a natural number and denote $N=\{1,...,n\}$. We extend the pair of operations $(\oplus,\otimes)$ to matrices and vectors as in conventional linear algebra. That is if $A=(a_{ij}),\ B=(b_{ij})$ and $C=(c_{ij})$ are matrices of compatible sizes with entries from $\overline{\mathbb{R}}$, we write $C=A\oplus B$ if $c_{ij}=a_{ij}\oplus b_{ij}$ for all $i,j\in N$ and $C=A\otimes B$ if

$$c_{ij}=\bigoplus_{k\in N}a_{ik}\otimes b_{kj}=\max_k(a_{ik}+b_{kj})$$

for all $i,j\in N$. If $\alpha\in\overline{\mathbb{R}}$ then $\alpha\otimes A=(\alpha\otimes a_{ij})$. Although the use of the symbols $\otimes$ and $\oplus$ is common in max-algebra we will use the usual convention of not writing the symbol $\otimes$.

Given $A\in\overline{\mathbb{R}}^{n\times n}$, the symbol $\lambda(A)$ will stand for the *maximum cycle mean* of A, that is: $\lambda(A)=\max_\sigma\mu(\sigma,A)$,where the maximization is taken over all elementary cycles in D_A, and $\mu(\sigma,A)=\frac{w(\sigma,A)}{l(\sigma)}$ denotes the *mean* of a cycle σ.

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Given $A = (a_{ij}) \in \overline{\mathbb{R}}^{n\times n}$ the symbol D_A will denote the associated digraph (N,E,w) where $E = \{(i,j); a_{ij} > \varepsilon\}$ and $w(i,j) = a_{ij}$ for all $(i,j) \in E$. Let $A \in \overline{\mathbb{R}}^{n\times n}$. A cycle σ in D_A is called *critical* if $\mu(\sigma,A) = \lambda(A)$. We denote by $N_c(A)$ the set of *critical nodes,* that is nodes on critical cycles. The *critical digraph* of A is the digraph C_A with the set of nodes N and the set of arcs, notation $E_c(A)$, is the set of arcs of all critical cycles. A strongly connected component of C_A is called *trivial* if it consists of a single node without a loop, *nontrivial* otherwise.

Given $A \in \overline{\mathbb{R}}^{n\times n}$ it is standard [8], [2], [11], [4] in max-algebra to define the infinite series $A^+ = A \oplus A^2 \oplus A^3 \oplus ...$and $A^* = I \oplus A^+ = I \oplus A \oplus A^2 \oplus A^3 \oplus ...$. The max-algebraic *eigenvalue-eigenvector problem* (briefly *eigenproblem*) is the following: *Given $A \in \overline{\mathbb{R}}^{n\times n}$, find all $\lambda \in \overline{\mathbb{R}}$ (eigenvalues) and $x \in \overline{\mathbb{R}}^n, x \neq \varepsilon$ (eigenvectors) such that $Ax = \lambda x$.* This problem has been studied since the work of R. A. Cuninghame-Green. A full solution of the eigenproblem in the case of irreducible matrices has been presented by R. A. Cuninghame-Green [8] and M. Gondran and M. Minoux [10], see also N. N. Vorobyov [14]. The general (reducible) case was first presented by S. Gaubert [9] and R. B. Bapat, D. Stanford and P. van den Driessche [3]. See also [6] and [4]. A key role in the solution of the eigenproblem is played by the maximum cycle mean. It is the biggest eigenvalue for any matrix (therefore called the *principal eigenvalue*) and although for an $n \times n$ matrix there may be up to n other eigenvalues in total, each of them is the maximum cycle mean of some principal submatrix. The maximum cycle mean of a matrix is the only eigenvalue whose associated eigenvectors may be finite. Irreducible matrices have no eigenvalues other than the maximum cycle mean. If the maximum cycle mean is ε then all eigenvectors can be described in a straightforward way; we will therefore usually assume that the maximum cycle mean is finite.

Let $A \in \overline{\mathbb{R}}^{n\times n}$. We denote by $V(A,\lambda)$ the set of all eigenvectors of A corresponding to $\lambda \in \overline{\mathbb{R}}$, by $V(A)$ the set of all eigenvectors of A and by $\Lambda(A)$ the set of all eigenvalues of A.

We now briefly discuss how can all eigenvalues be found for reducible matrices. Every matrix $A = (a_{ij}) \in \overline{\mathbb{R}}^{n\times n}$ can be transformed by simultaneous permutations of the rows and columns in linear time to a *Frobenius normal form*

$$\begin{pmatrix} A_{11} & \varepsilon & ... & \varepsilon \\ A_{21} & A_{22} & ... & \varepsilon \\ ... & ... & ... & ... \\ A_{r1} & A_{r2} & ... & A_{rr} \end{pmatrix},$$

where $A_{11},...,A_{rr}$ are irreducible square submatrices of A. If A is in a Frobenius normal form then the corresponding partition subsets of the node set N of D_A will be denoted as $N_1,...,N_r$ and these sets will be called *classes (of A)*. If A is in the Frobenius normal form then the *condensation digraph*, notation $C(A)$, is the digraph

$$(\{N_1,...,N_r\}, \{(N_i,N_j); (\exists k \in N_i)(\exists \ell \in N_j) a_{k\ell} > \varepsilon\}).$$

Observe that $C(A)$ is acyclic and represents a partially ordered set. Any class that has no incoming arcs in $C(A)$ is called *initial*, similarly for diagonal blocks.

The symbol $N_i \longrightarrow N_j$ means that there is a directed path from a node in N_i to a node in N_j in $C(A)$ (and therefore from each node in N_i to each node in N_j in D_A). We may assume without loss of generality that A is in a Frobenius normal form. It is intuitively clear that all eigenvalues of A are among the unique eigenvalues of diagonal blocks. However, not all of these eigenvalues are also eigenvalues of A. The following key result appeared for the first time independently in the thesis [9] and [3], see also [4] and [6].

Theorem 2.1: (Spectral Theorem) Let $A \in \overline{\mathbb{R}}^{n\times n}$ be a Frobenius normal form. Then

$$\Lambda(A) = \{\lambda; (\exists j)\, \lambda = \lambda(A_{jj}) = \max_{N_i \to N_j} \lambda(A_{ii})\}.$$

If

$$\lambda(A_{jj}) = \max_{N_i \to N_j} \lambda(A_{ii})$$

then A_{jj} (and also N_j or just j) will be called *spectral*.

III. WEAKLY STABLE MATRICES

Given $A \in \overline{\mathbb{R}}^{n\times n}$ and $x \in \overline{\mathbb{R}}^n$ the sequence of vectors $Ax, A^2x, A^3x, ...$ is called the *orbit* (*of A with starting vector x*). For $A \in \overline{\mathbb{R}}^{n\times n}$ we denote by $\text{attr}(A)$ the set of all starting vectors from which the orbit reaches an eigenvector, that is

$$\text{attr}(A) = \left\{x \in \overline{\mathbb{R}}^n; (\exists k \geq 0) A^k x \in V(A)\right\}.$$

The set $\text{attr}(A)$ is called the *attraction set of A*. If $\lambda > \varepsilon$ then $x \in V(A,\lambda)$ implies $Ax \in V(A,\lambda)$ since $A(Ax) = A(\lambda x) = \lambda Ax$ and $Ax \neq \varepsilon$ since $Ax = \lambda x$ and $x \neq \varepsilon$. Hence if A has no ε columns we have $V(A) \subseteq \text{attr}(A) \subseteq \overline{\mathbb{R}}^n - \{\varepsilon\}$.

Following the definitions introduced at the beginning we have that A is strongly stable *(robust* in *[6])* if $\text{attr}(A) = \overline{\mathbb{R}}^n - \{\varepsilon\}$ and it is weakly stable if $\text{attr}(A) = V(A)$. Note that in the definition of attraction sets the exponent k starts from 0 and thus the 1×1 matrix $A = (\varepsilon)$ is both strongly and weakly stable since $V(A) = \overline{\mathbb{R}}^n - \{\varepsilon\} = \text{attr}(A)$.

Theorem 3.1: [7] An irreducible matrix $A \in \overline{\mathbb{R}}^{n\times n}, A \neq \varepsilon$ is weakly stable if and only if its critical digraph is an elementary cycle containing all nodes of the associated digraph, that is C_A is a Hamiltonian cycle in D_A.

It should be emphasized that this theorem has no computational complexity implications on the Hamilton cycle problem as the answer to the question of weak stability is given by checking that a certain graph (which can easily be found) is a Hamiltonian cycle (which is straightforward), rather than by verifying that such a cycle exists. Indeed, it is known [2], [4] that C_A can be found in a polynomial number of steps. It is then easy to check whether C_A is a Hamiltonian cycle in D_A.

We can however formulate a problem in terms of weakly stable matrices equivalent to the Hamilton cycle problem, which thus embodies the hardness of the Hamilton cycle problem: *Given a strongly connected digraph D, is it possible* *to assign the weights to the arcs of D so that the obtained weighted digraph is D_A for some weakly stable matrix A?*

To characterise reducible weakly stable matrices suppose without loss of generality that $A \in \overline{\mathbb{R}}^{n\times n}$ is in the Frobenius normal form and denote $R = \{1,...,r\}$. Let $N_1,...,N_r$ be the partition of N determined by the Frobenius normal form. A class $s \in R$ is called *weakly stable* if A_{ss} is weakly stable.

Theorem 3.2: [7] Let $A \in \overline{\mathbb{R}}^{n\times n}$. Then A is weakly stable if and only if every spectral class is initial and weakly stable.

Corollary 3.3: $A \in \overline{\mathbb{R}}^{n\times n}$ is weakly stable if and only if every initial class is weakly stable and the (unique) eigenvalue of every non-initial class is strictly less than the eigenvalue of any initial class from which it is accessible.

IV. Strongly stable matrices

In order to provide a full picture on reachability in this section we present the main results on strongly stable matrices [6], [4], that is matrices for which $\operatorname{attr}(A) = \overline{\mathbb{R}}^{n} - \{\varepsilon\}$. A digraph D is called *primitive* if in every strongly connected component of D the lengths of all directed cycles are coprime. A matrix $A \in \overline{\mathbb{R}}^{n\times n}$ is called *primitive* if its critical digraph C_A is primitive.

Theorem 4.1: [5] Let $A \in \overline{\mathbb{R}}^{n\times n}$ be irreducible. Then A is strongly stable if and only if A is primitive.

A Frobenius normal form class is called *trivial* [*primitive*] if the corresponding diagonal block is the 1×1 matrix (ε) [primitive].

Theorem 4.2: Let $A \in \overline{\mathbb{R}}^{n\times n}$ be a matrix with no ε columns and in the Frobenius normal form with classes $N_1,...,N_r$ and $R = \{1,...,r\}$. Then A is strongly stable if and only if the following hold:

1) All non-trivial classes $N_1,...,N_r$ are spectral and primitive.
2) For any $i, j \in R$, if N_i,N_j are non-trivial mutually inaccessible classes then $\lambda(N_i) = \lambda(N_j)$.

V. Conclusions and further research

This paper fully characterizes weakly stable matrices, that is matrices whose orbit never reaches an eigenvector unless it starts in one. The characterization enables us to check that a matrix is weakly stable in a polynomial number of steps. Description and classification of all possible limit cycles and their attraction sets are still unexplored.

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