

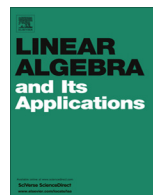


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On common eigenvectors for semigroups of matrices in tropical and traditional linear algebra[☆]



Grigory B. Shpiz^a, Grigory L. Litvinov^b, Sergeĭ N. Sergeev^{c,d,*}

^a Scientific Institute for Nuclear Physics in the Moscow State University, Moscow, Russia

^b The A. A. Kharkevich Institute for Information Transmission Problems RAS and National Research University Higher School of Economics, Moscow, Russia

^c University of Birmingham, School of Mathematics, Edgbaston, Birmingham, UK

^d Moscow Centre for Continuous Mathematical Education, Russia

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ABSTRACT

We prove the existence of a common eigenvector for commutative, nilpotent and quasinilpotent semigroups of matrices with complex or real nonnegative entries both in the conventional and tropical linear algebra.

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1. Preliminaries

We consider semigroups of matrices with entries in (1) the complex field \mathbf{C} , (2) the semifield of nonnegative real numbers with usual arithmetics, (3) the semifield of nonnegative numbers \mathbf{R}_+ (max) with idempotent addition $a \oplus b := \max(a, b)$ and the usual multiplication $ab := a \times b$.

The latter example is known as the max-times semifield; this semifield is isomorphic to the so-called tropical algebra or max-plus algebra. This case is one of our main motivations. As in the case of

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* Corresponding author at: University of Birmingham, School of Mathematics, Edgbaston, Birmingham, UK. Tel.: +44 1214142916. E-mail addresses: shpiz@theory.sinp.msu.ru (G.B. Shpiz), glitvinov@gmail.com (G.L. Litvinov), sergiej@gmail.com (S.N. Sergeev).

usual arithmetics, these semifields naturally extend to matrices and vectors, giving rise to max-linear algebra [1], and to the tropical convex geometry [4] of max-algebraic subspaces of the nonnegative orthant \mathbf{R}_+^n . These subspaces are subsets of \mathbf{R}_+^n closed under taking componentwise maximum of vectors, and multiplication by a nonnegative scalar. They are also known as idempotent linear spaces over $\mathbf{R}_+(\max)$ [7,8], or max cones [2].

We give a general proof that in all these cases, any semigroup of pairwise commuting matrices has a common eigenvector. This can be seen as an extension of a recent result of Katz, Schneider, Sergeev [6]. We also extend this result on existence of a common eigenvector to the case of nilpotent and quasiniptotent semigroup, in full analogy with nilpotent groups. The notion of quasiniptotent semigroup was suggested by G.B. Shpiz. See, e.g., [5,8,9] for other recent results on the existence of a common eigenvector of tropical matrix groups and semigroups.

In the sequel, the cases described above are considered simultaneously. More precisely, we will consider linear operators in spaces (semimodules) $\mathcal{K}^n = \underbrace{\mathcal{K} \times \cdots \times \mathcal{K}}_n$, where \mathcal{K} is one of the following semifields: (1) the complex field \mathbf{C} , or (2) the semifield of nonnegative numbers \mathbf{R}_+ with the usual arithmetics, (3) the max-times semifield $\mathbf{R}_+(\max)$.

We will also consider subspaces of \mathcal{K}^n . In the first case, subspaces of \mathcal{K}^n are just linear subspaces of \mathbf{C}^n in the usual meaning of the word. In the second case, subspaces of \mathcal{K}^n are convex cones in \mathbf{R}_+^n ; in the third case, they are the max cones [2] mentioned above. In any of these cases, a subspace is called nontrivial if it contains a nonzero vector. It will be called closed if it is closed in the sense of the usual Euclidean topology. A nontrivial subspace $W \subseteq \mathcal{K}^n$ is called invariant under the action of a linear operator A , if $AW \subseteq W$, that is, if $Ax \in W$ for all $x \in W$.

The eigenvector existence theorems (Lemma 1.2) will be an important ingredient in the proofs below. Note that the spectral theory of max-linear maps is well-known only for the case of nonnegative orthant, while we will need the eigenvector existence in an arbitrary invariant max cone. In the classical nonnegative algebra, such results follow either from the Brouwer fixed point theorem, or by extending Wielandt’s proof of the Perron–Frobenius theorem to cones. In Section 3 we show how these two techniques can be recast in the max-algebraic setting. Let us remark that these techniques have been developed in the framework of more general tropical spaces and over idempotent semirings more general than max-times in [7,8,12,13].

Proof of the following Lemma is elementary, but it is recalled for the reader’s convenience.

Lemma 1.1. *Let $\{W_i\}_{i \geq 1}$ be a sequence of closed nontrivial subspaces of \mathcal{K}^n , where $\mathcal{K} \in \{\mathbf{C}, \mathbf{R}_+, \mathbf{R}_+(\max)\}$, such that $W_1 \supseteq W_2 \supseteq W_3 \supseteq \cdots$. Then the intersection $\bigcap_{i=1}^\infty W_i$ is a closed nontrivial space.*

Proof. For each i , consider intersection of W_i with the sphere $\{y : \|y\| = 1\}$ where $\|\cdot\|$ is, for instance, the usual Euclidean 2-norm. Denote this intersection by S_i . Evidently $0 \notin S_i$ and all S_i are non-empty and compact and satisfy $S_1 \supseteq S_2 \supseteq S_3 \supseteq \cdots$, hence $\bigcap_{i=1}^\infty S_i$ is a compact non-empty set and $\bigcap_{i=1}^\infty W_i$ is a closed nontrivial space. \square

Lemma 1.2. *Let W be a closed subspace of \mathcal{K}^n , invariant under the action of a linear operator A , where $\mathcal{K} \in \{\mathbf{C}, \mathbf{R}_+, \mathbf{R}_+(\max)\}$. Then W contains an eigenvector of A .*

Proof. The case of $\mathbf{R}_+(\max)$ will be treated in Section 3.

In the case when A is a complex-valued matrix and W is a subspace in \mathbf{C}^n , this result is standard. In the case when A is a nonnegative matrix and W is a closed invariant cone of A in \mathbf{R}_+^n , the result follows by a standard application of the Brouwer fixed point theorem. A more precise formulation is known as the finite-dimensional Kreĭn–Rutman theorem or the Perron-Frobenius theorem for cones, see Schneider [10] Theorem 0. \square

We conclude with the following facts, which are evident in all algebraic structures that we consider. Recall that for $\lambda_A \in \mathcal{K}$, the set $\{v \in \mathcal{K}^n : Av = \lambda_A v\}$ is called the eigenspace of A associated with λ_A . Linear operators A and B commute if $AB = BA$, that is, if $A(Bv) = B(Av)$ for all $v \in \mathcal{K}^n$.

Lemma 1.3. Let $W \subseteq \mathcal{K}^n$ be a nontrivial eigenspace of a linear operator A associated with some eigenvalue of A , where $\mathcal{K} \in \{\mathbf{C}, \mathbf{R}_+, \mathbf{R}_+(\max)\}$.

1. W is closed;
2. Let B be a linear operator that commutes with A . Then $BW \subseteq W$.

Proof. The first part follows from the continuity of linear operators in all spaces \mathcal{K}^n that we consider. For the second part, let $v \in \mathcal{K}^n$ satisfy $Av = \lambda_A v$ for some $\lambda_A \in \mathcal{K}$. Then $A(Bv) = B(Av) = B(\lambda_A v) = \lambda_A(Bv)$, thus the eigenspace of A associated with A is invariant under B . \square

2. Main results

We say that a semigroup S of linear operators acting in a subspace $V \subseteq \mathcal{K}^n$ has an eigenvector v , if for each $A \in S$ there is a value $\lambda_A \in \mathcal{K}$ such that $Av = \lambda_A v$, and $v \in V$ is nonzero. We say that S is commutative if $AB = BA$ for all $A, B \in S$.

Theorem 2.1. Let S be a commutative semigroup of linear operators leaving a closed subspace $V \subseteq \mathcal{K}^n$ invariant, where $\mathcal{K} \in \{\mathbf{C}, \mathbf{R}_+, \mathbf{R}_+(\max)\}$. Then S has an eigenvector $v \in V$.

Proof. Let us consider the collection of all closed nontrivial subspaces of V invariant under all operators of S . Such subspaces will be called invariant by abuse of terminology. This collection contains V , so it is non-empty. For any $\mathcal{K} \in \{\mathbf{C}, \mathbf{R}_+, \mathbf{R}_+(\max)\}$, the intersection of any chain of closed invariant spaces is a closed invariant space, and it is nontrivial by Lemma 1.1. Then we can apply Zorn's Lemma to get a nontrivial minimal (under inclusion) closed invariant space W . Consider a subspace of W consisting of all eigenvectors of a matrix $A \in S$ associated with an eigenvalue λ_A . Denote this subspace by W' . By Lemma 1.2 W' contains a nonzero vector. Observe that W' is the intersection of W with the eigenspace of A associated with λ_A . By Lemma 1.3, W' is a closed invariant subspace of W , and the minimality implies $W = W'$. As we took an arbitrary $A \in S$, it follows that for all $A \in S$ the subspace W contains (besides the zero vector) only eigenvectors of A associated with some eigenvalue λ_A . All these eigenvectors are common eigenvectors of all matrices in S . \square

For a semigroup S and a natural number k , define its subsemigroup $S^{(k)}$ consisting of all $A \in S$ that can be represented as $B_1 \cdots B_k$ with $B_1, \dots, B_k \in S$. Indeed, this is a subsemigroup since $B_1 \cdots B_k C_1 \cdots C_k$ can be represented, for instance, as $(B_1 \cdots B_k C_1) \cdots C_k$. For a similar reason, we have $S^{(k)} \subseteq S^{(k-1)}$ for all $k > 1$. S is called *quasinilpotent* if $S^{(k)}$ is a commutative semigroup for some k . Recall that a semigroup S with a zero element 0 is *nilpotent* if $S^{(k)} = \{0\}$ for some k . Of course, every commutative or nilpotent semigroup is quasinilpotent.

Theorem 2.2. Let S be a quasinilpotent semigroup of linear operators leaving a closed subspace $V \subseteq \mathcal{K}^n$ invariant, where $\mathcal{K} \in \{\mathbf{C}, \mathbf{R}_+, \mathbf{R}_+(\max)\}$. Then S has an eigenvector $v \in V$.

Proof. Let $S^{(t)}$, for some $t \geq 1$, be a commutative semigroup. By Theorem 2.1, $S^{(t)}$ has an eigenvector. It suffices to prove that for $k > 1$, if $S^{(k)}$ has an eigenvector then so does $S^{(k-1)}$, and then a straightforward induction can be applied.

Let u be an eigenvector of $S^{(k)}$ and suppose that there exists $A \in S^{(k)}$, for which this eigenvector is associated with a nonzero eigenvalue λ . For each $B \in S^{(k-1)}$, we have $BA \in S^{(k)}$ (as $A \in S$), and hence $BAu = \mu u$ for some μ . Substituting $Au = \lambda u$ we obtain $Bu = \lambda^{-1} \mu u$, so u is also an eigenvector of $S^{(k-1)}$.

Otherwise, suppose that there is $u \in V, u \neq 0$, such that Au is zero for all $A \in S^{(k)}$. If we also have Bu zero for all $B \in S^{(k-1)}$ then u is an eigenvector of $S^{(k-1)}$ (with a common eigenvalue equal to zero). Otherwise, take $B' \in S^{(k-1)}$ such that $B'u$ is nonzero, but then $BB'u$ is still zero for all $B \in S^{(k-1)}$ since $BB' \in S^{(k)}$. So $B'u$ is an eigenvector of $S^{(k-1)}$ (with a common eigenvalue equal to zero).

We proved the claim in all possible cases. \square

Remark 2.3. It is well known that a connected nilpotent or solvable complex matrix Lie group has a common eigenvector (the Lie and Engel theorems, see, e.g. [11]). For abstract nilpotent (but not for solvable) groups of linear continuous operators in a tropical linear Archimedean space the corresponding result was proved in [8].

3. Proofs of Lemma 1.2 for max cones

In this section we give two proofs of Lemma 1.2 for max cones in \mathbf{R}_+^n . The first proof is based on the Brouwer fixed point theorem following the idea of the proof of [13] Theorem 1, and the second proof can be seen as a max-algebraic analogue of Wielandt’s proof of the Perron-Frobenius theorem being an adaptation of the argument of Shpiz [12] Theorem 3.

3.1. Application of the Brouwer fixed-point theorem

Let W be a closed max cone in \mathbf{R}_+^n and let

$$S = W \cap \{x \mid \sum_{i=1}^n x_i = 1\}. \tag{1}$$

Here, \sum is the ordinary sum (not to confuse with the componentwise maximum). Denote by $\text{conv}(S)$ the convex hull of S , i.e., the smallest conventionally convex set containing S . As it is known from convex analysis, this is the same as the set of all convex combinations $\sum_{i=1}^m \mu_i x^i$, where $x^1, \dots, x^m \in S$, all μ_i are nonnegative and satisfy $\sum_{i=1}^m \mu_i = 1$. Since W is closed, S is compact, and $\text{conv}(S)$ is a compact convex subset of $\{x \mid \sum_{i=1}^n x_i = 1\}$.

A nonlinear projector $P_W : \mathbf{R}_+^n \rightarrow W$ (see [3,7]) can be defined by

$$P_W(y) := \sup\{x \in W \mid x \leq y\}, \tag{2}$$

where \sup denotes the componentwise supremum. As W is closed, this supremum is reached. In particular, $P_W(x) = x$ for all $x \in W$.

Lemma 3.1. P_W is a continuous operator on \mathbf{R}_+^n and $P_W(z) \neq 0$ for any nonzero $z \in \text{conv}(S)$.

Proof. A proof of continuity of P_W can be found, for instance, in [3] Theorem 3.11. To prove the second part of the claim, it is sufficient to find, for each nonzero $z \in \text{conv}(S)$, a vector $y \in W$ with $0 \neq y \leq z$. For this, recall that z can be represented as $z = \sum_{i=1}^m \mu_i x^i$ for some $x^1, \dots, x^m \in W$, where all μ_i are nonnegative and satisfy $\sum_{i=1}^m \mu_i = 1$. Take for y the componentwise maximum of $\mu_1 x^1, \dots, \mu_m x^m$, then $y \leq z, y \neq 0$ and $y \in W$. \square

Proof of Lemma 1. We can assume that $A \otimes x \neq 0$ for all $x \in W$, otherwise there exists an eigenvector of A associated with the zero eigenvalue.

Consider the set S in (1), its conventional convex hull $\text{conv}(S)$ and the mappings $\pi : \text{conv}(S) \rightarrow S$ and $\gamma : S \rightarrow S$ defined by

$$\pi(x) := (P_W x) / \sum_{i=1}^n (P_W x)_i \quad \gamma(x) := (A \otimes x) / \sum_{i=1}^n (A \otimes x)_i. \tag{3}$$

By the continuity of A and since $A \otimes x \neq 0$ on S , γ is a continuous mapping of S in itself. By the continuity of P_W and since $P_W x \neq 0$ on $\text{conv}(S)$ (see Lemma 3.1), π is a continuous mapping of $\text{conv}(S)$ in itself. Also note that $\pi(x) = x$ for $x \in S$. The composition $\gamma \pi$ is continuous on $\text{conv}(S)$. Applying the Brouwer fixed point theorem to $\gamma \pi$ acting on $\text{conv}(S)$ (which is a compact convex set)

we obtain a point x satisfying $\gamma\pi(x) = x$. But then we have $x \in S$, since $\gamma\pi(x) \in S$. Since $\pi(x) = x$ for $x \in S$, we obtain $\gamma(x) = x$, so $x \in S \subseteq W$ is a nonzero eigenvector of A .

3.2. Algebraic proof

We adapt a proof of the eigenvector existence theorem of Shpiz [12].

The max-algebraic matrix powers of $A \in \mathbf{R}_+^{n \times n}$ will be denoted by $A^{\otimes t} := \underbrace{A \otimes \cdots \otimes A}_t$. We have

$A^{\otimes 0} = I$, the identity matrix. We recall that the max-algebraic matrix multiplication is homogeneous: $A \otimes (rx) = rA \otimes x$ for all $x \in \mathbf{R}_+^n$ and $r \in \mathbf{R}_+$, monotone: $x \leq y \Rightarrow A \otimes x \leq A \otimes y$ for all $x, y \in \mathbf{R}_+^n$, and continuous. By the continuity of max-algebraic matrix multiplication, the extended max-linearity

$$A \otimes \bigoplus_{\mu \in S} v^\mu = \bigoplus_{\mu \in S} A \otimes v^\mu \tag{4}$$

holds for $A \in \mathbf{R}_+^{n \times n}$ and any set $\{v^\mu\}_{\mu \in S} \subseteq \mathbf{R}_+^n$ bounded from above. The \bigoplus sign denotes the componentwise supremum of a bounded (but possibly infinite) subset of \mathbf{R}_+^n .

We will use the notation

$$[v : w] = \max\{\lambda \mid \lambda w \leq v\} = \min_{i: w_i \neq 0} v_i w_i^{-1}. \tag{5}$$

for $v, w \in \mathbf{R}_+^n$, where $w \neq 0$. Note that $[v : w] \cdot w \leq v$.

The proof of Lemma 1.2 is preceded by the following technical result.

Lemma 3.2 (Shpiz [12], Lemma 2). *Let W be a nontrivial minimal (under inclusion) closed max cone in \mathbf{R}_+^n invariant under the max-algebraic multiplication by a matrix $A \in \mathbf{R}_+^n$. Assume that $A^{\otimes t}v \neq 0$ for any nonzero $v \in W$ and any $t \geq 1$. For every integer $t \geq 1$,*

- (i) $[v : A^{\otimes t} \otimes v]$ is the same for all $v \in W, v \neq 0$.
- (ii) $[v : A^{\otimes t} \otimes v] = [v : A \otimes v]^t$ holds for all $v \in W, v \neq 0$.

Proof.

- (i) For each $v \in W, v \neq 0$,

$$W_{v,t} := \{w \mid [v : A^{\otimes t} \otimes v] \cdot A^{\otimes t} \otimes w \leq w\} \tag{6}$$

is a closed max cone invariant under A . It is nontrivial (i.e., not reduced to the zero vector) since $v \in W_{v,t}$. By the minimality of W we obtain $W_{v,t} = W$, and using (5) this implies that $[v : A^{\otimes t} \otimes v] \leq [w : A^{\otimes t} \otimes w]$ for all nonzero $v, w \in W$. Swapping v and w we get the reverse inequality, which proves the claim.

- (ii) Fix a nonzero $v \in W$, and denote $r := [v : A^{\otimes t} \otimes v]^{1/t}$. It will be shown that $[v : A \otimes v] = r$. The inequality $[v : A \otimes v] \leq r$ is relatively easy: iterating $[v : A \otimes v] \cdot A \otimes v \leq v$ one obtains that $[v : A \otimes v]^t \cdot A^{\otimes t} \otimes v \leq v$, and this implies $[v : A \otimes v]^t \leq [v : A^{\otimes t} \otimes v]$.

For the reverse inequality, consider the vector $w := \bigoplus_{\ell \geq 0} r^\ell A^{\otimes \ell} \otimes v$. It will be shown a bit later that the sequence of vectors $\{r^\ell A^{\otimes \ell} \otimes v\}_{\ell \geq 0}$ is bounded, so its supremum is finite. Applying rA to w and using the extended max-linearity of A (4) we get

$$rA \otimes w = \bigoplus_{\ell \geq 1} r^\ell A^{\otimes \ell} \otimes v \leq w.$$

It follows that $r \leq [w : A \otimes w]$, and since $[w : A \otimes w] = [v : A \otimes v]$ by part (i), we obtain $r \leq [v : A \otimes v]$ and thus $r = [v : A \otimes v]$.

To complete the proof, it remains to show that the sequence $\{r^\ell A^{\otimes \ell} \otimes v\}_{\ell \geq 0}$ is bounded. For this, it is observed, by representing $\ell = tp + q$ with $p \geq 0$ and $0 \leq q < t$, by repeatedly applying $[v : A^{\otimes t} \otimes v]A^{\otimes t} \otimes v \leq v$ and using the homogeneity and monotonicity of max-algebraic matrix multiplication, that

$$r^\ell A^{\otimes \ell} \otimes v = [v : A^{\otimes t} \otimes v]^{p+(q/t)} A^{\otimes tp+q} \otimes v \leq [v : A \otimes v]^{q/t} A^{\otimes q} \otimes v.$$

Thus $r^\ell A^{\otimes \ell} \otimes v$ can be bounded by the componentwise maximum of all vectors $[v : A \otimes v]^{q/t} \cdot A^{\otimes q} \otimes v$, over q from $0 \leq q < t$. \square

Proof of Lemma 2 [12, Theorem 3]. If $A^{\otimes t} \otimes v = 0$ for some nonzero $v \in W$ and $t \geq 1$ then A has an eigenvector with zero eigenvalue.

Assume that $A^{\otimes t} v \neq 0$ for any nonzero $v \in W$ and any $t \geq 1$. We can also assume without loss of generality that W is a minimal (under inclusion) closed max cone invariant under the max-algebraic multiplication by A , otherwise we can use Lemma 1.1 and select such a max cone by means of Zorn’s Lemma. Consider the set $C = \{x \in W : \max_{i=1}^n x_i = 1\}$. This set is closed under taking componentwise maxima and compact, hence C contains the greatest point $\max C$ with respect to the usual componentwise (partial) order in \mathbf{R}_+^n . Set $v := \max C$ and let $\alpha_t := \max_{i=1}^n (A^{\otimes t} \otimes v)_i$ for any $t \geq 1$, then $\max_{i=1}^n (\alpha_t^{-1} A^{\otimes t} \otimes v)_i = 1$ and hence $\alpha_t^{-1} A^{\otimes t} \otimes v \leq v$ by the definition of v . If $\kappa > \alpha_t^{-1}$ then $\max_{i=1}^n (\kappa A^{\otimes t} \otimes v)_i = \kappa \alpha_t > 1$ while $\max_{i=1}^n v_i = 1$ so $\kappa A^{\otimes t} \otimes v \not\leq v$. This implies that $\alpha_t^{-1} = [v : A^{\otimes t} \otimes v]$, and then $\alpha_t^{-1} = [v : A \otimes v]^t = \alpha_1^{-t}$ by Lemma 3.2.

Consider the sequence of vectors $\{\alpha_1^{-t} A^{\otimes t} \otimes v\}_{t \geq 0}$, which is the same as $\{\alpha_t^{-1} A^{\otimes t} \otimes v\}_{t \geq 0}$. On one hand, the greatest component of each vector is 1, by the above. On the other hand, iterating $v \geq \alpha_1^{-1} A \otimes v$ we obtain that $v \geq \alpha_1^{-1} A \otimes v \geq \alpha_1^{-2} A^{\otimes 2} \otimes v \geq \dots$, hence $\{\alpha_1^{-t} A^{\otimes t} \otimes v\}_{t \geq 0}$ has the limit, which we denote by u . By the continuity u satisfies $\alpha_1^{-1} A \otimes u = u$ and $\max_{i=1}^n u_i = 1$, in particular $u \neq 0$. Since C is closed, $u \in C \subseteq W$. Hence u is an eigenvector of A belonging to W . The proof is complete.

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