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Matrices that commute with their derivative. On a letter from Schur to Wielandt[☆]

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ABSTRACT

We examine when a matrix whose elements are differentiable functions in one variable commutes with its derivative. This problem was discussed in a letter from Schur to Wielandt written in 1934, which we found in Wielandt's Nachlass. We present this letter and its translation into English. The topic was rediscovered later and partial results were proved. However, there are many subtle observations in Schur's letter which were not obtained in later years. Using an algebraic setting, we put these into perspective and extend them in several directions. We present in detail the relationship between several conditions mentioned in Schur's letter and we focus in particular on the characterization of matrices called Type 1 by Schur. We also present several examples that demonstrate Schur's observations.

1. Introduction

What are the conditions that force a matrix of differentiable functions to commute with its elementwise derivative? This problem, discussed in a letter from Schur to Wielandt [32], has been discussed in a large number of papers [2–4,7,9,11,12,17,19,21–24,27,29,31,33,34]. However, these

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authors were unaware of Schur's letter and did not find some of its principal results. A summary and a historical discussion of the problem and several extensions thereof are presented by Evard in [14,15], where the study of the topic is dated back to the 1940s and 1950s, but in 1946 [12] (translated in Schur's letter shows that it already appeared in Schur's lectures in the 1930s, if not earlier.

The content of the paper is as follows. In Section 2 we present a facsimile of Schur's letter to Wielandt and its English translation. In Section 3 we discuss Schur's letter and we motivate our use of differential fields. In Section 4 we introduce our notation and reprove Frobenius result on Wronskians. In Section 5 we discuss the results that characterize the matrices of Type 1 in Schur's letter and in our main Section 6 we discuss the role played by diagonalizability and triangularizability of the matrix in the commutativity of the matrix and its derivative. We also present several illustrative examples in Section 7 and we state an open problem in Section 8.

2. A letter from Schur to Wielandt

Our paper deals with the following letter from Schur to his Ph.D. student Wielandt. See the facsimile below.

Berlin 21. 7. 34.

Lieber Herr Doktor!

Sie haben ganz recht. Schon für $3 \leq n < 6$ hat nicht jede Lösung der Gleichung $MM' = M'M$ die Form

(1) $M_1 = \sum\limits_{\lambda} f_\lambda C_\lambda,$

wobei die C_λ untereinander vertauschbare konstante Matrizen sind. Man hat noch den Typus

(2) $M_2 = (f_\alpha g_\beta)$ $(\alpha, \beta = 1, 2, \cdots n)$

hervorzuheben, wo $f_1, \dots f_n, g_1, \dots g_n$ beliebige Funktionen sind, die den Bedingungen

$$\sum_\alpha f_\alpha g_\alpha = \sum_\alpha f'_\alpha g_\alpha = 0, \text{ also auch } \sum_\alpha f_\alpha g'_\alpha = 0$$

genügen. In diesem Falle wird $M^2 = MM' = M'M = 0$.

Hierzu kommt dann noch der Typus

(3) $M_3 = \varphi E + M_2$ (M_2 vom Typus (2))

Aus meinen alten Notizen, die ich in der Vorlesung nicht richtig wiedergegeben habe, geht hervor, daß für $n < 6$ jede Lösung von $MM' = M'M$ durch eine konstante Ähnlichkeitstransformation in Matrizen vom Typus (1) oder (3) vollständig zerfällt werden kann. Erst für $n = 6$ gibt es noch andere Fälle.

Dies scheint richtig zu sein. Ich habe aber meine recht mühsamen Rechnungen (für $n = 4$ und $n = 5$) nicht nachgeprüft.

Daß man sich auf den Fall beschränken kann, in dem M nur die eine char. Wurzel 0 besitzt, schloß ich damals einfach so. Besitzt M zwei verschiedene char. Wurzeln, so kann man eine g. rat. Funktion N von M angeben, für die

~~N~~ $N^2 = N$ wird, ohne daß $N = q E$ wird. Auch N ist mit
N' vertauschbar. Aus $N^2 = N$ folgt aber $2NN' = N'$, also
$2N^2 N' = 2NN' = NN'$. Das gibt $2NN = N' = 0$, d. h.
N ist konstant. Man kann nun auf M eine konstante
Ähnlichkeitstransformation anwenden, sodaß anstelle
von N eine Matrix der Form $\begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix}$ tritt. Dies zeigt, daß
M mit Hilfe einer konst. Ähnlichkeitstransf. ~~zu~~ voll-
ständig zerfällt werden kann.

Man wird auf den Typus (2) geführt, indem man
den Fall $M^2 = 0$, $\mathrm{Rg}\, M = 1$ studiert. Schon für $n = 4$
kommen noch die Fälle $M^2 = 0$, $\mathrm{Rg}\, M = 2$, $M^3 = 0$ in Betracht.
Der Typus (1) ist vollständig dadurch charakterisiert,
daß $M, M', M'', \ldots$ untereinander vertauschbar sind.
Dies ist nicht nur notwendig sondern auch hinreichend.
Denn sind unter den n^2 Koeffizienten $f_{\alpha\lambda}$ von M genau
r im Gebiete der Konstanten linear unabhängig, so
kann man

$$M = f_1 C_1 + \cdots + f_r C_r \qquad (C_\varrho \text{ konstante Matrix})$$

schreiben, wobei $f_1, \ldots f_r$ keiner Gleichung $\sum_1 \text{const.}\, f_\alpha = 0$ genügen. Dann wird

$$M^{(\nu)} = f_1^{(\nu)} C_1 + \cdots + f_r^{(\nu)} C_r \qquad (\nu = 0, 1, \ldots r-1)$$

Da die Wronskische Det. $\begin{vmatrix} f_1 & \cdots & f_r \\ f_1' & \cdots & f_r' \\ \cdots & & \cdots \end{vmatrix}$ nicht identisch verschwinden darf, erhält man Gleichungen der Form

$$C_\varrho = \sum_{\sigma=0}^{r-1} \varphi_{\varrho\sigma} M^{(\sigma)}.$$

Sind $M, M', \ldots M^{(r-1)}$ untereinander vertauschbar, so gilt dasselbe auch für $C_1, \ldots C_r$, M ist also vom Typus (1).

Hieraus folgt zugleich, daß M dem Typus (1) angehört, wenn M^n die höchste Potenz von M ist, die gleich Null wird. Im Falle $n = 3$ hat man daher nur noch den Typus (2) zu berücksichtigen.

Mit vielen Grüßen

Ihr Schur

Translated into English, the letter reads as follows:

Lieber Herr Doktor! *Berlin, 21.7.34*

You are perfectly right. Already for $3 \leqslant n < 6$ *not every solution of the equation* $MM' = M'M$ *has the form*

$$M_1 = \sum_\lambda f_\lambda C_\lambda, \tag{1}$$

where the C_λ *are pairwise commuting constant matrices. One must also consider the type*

$$M_2 = (f_\alpha g_\beta), \quad (\alpha, \beta = 1, \dots n), \tag{2}$$

where $f_1, \dots f_n, g_1, \dots, g_n$ *are arbitrary functions that satisfy the conditions*

$$\sum_\alpha f_\alpha g_\alpha = \sum_\alpha f'_\alpha g_\alpha = 0$$

and therefore also

$$\sum_\alpha f_\alpha g'_\alpha = 0.$$

In this case we obtain

$$M^2 = MM' = M'M = 0.$$

In addition we have the type

$$M_3 = \phi E + M_2 \tag{3}$$

with M_2 *of type (2).* [1] *From my old notes, which I did not present correctly in my lectures, it can be deduced that for* $n < 6$ *every solution of* $MM' = M'M$ *can be completely decomposed by means of constant similarity transformations into matrices of type (1) and (3). Only from* $n = 6$ *on there are also other cases. This seems to be correct. But I have not checked my rather laborious computations (for* $n = 4$ *and* $n = 5$*).*

I concluded in the following simple manner that one can restrict oneself to the case where M has only one characteristic root (namely 0*): If M has two different characteristic roots, then one can determine a rational function N of M for which* $N^2 = N$ *but not* $N = \phi E$*. Also N commutes with* N'*. It follows from* $N^2 = N$ *that* $2NN' = N'$*, thus* $2N^2N' = 2NN' = NN'$*. This yields* $2NN' = N' = 0$*, i.e., N is constant.*

Now one can apply a constant similarity transformation to M so that instead of N one achieves a matrix of the form

$$\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}.$$

This shows that M can be decomposed completely by means of a constant similarity transformation.

One is led to type (2) by studying the case $M^2 = 0, \operatorname{rank} M = 1$*. Already for* $n = 4$ *also the cases* $M^2 = 0, \operatorname{rank} M = 2, M^3 = 0$ *need to be considered.*

Type (1) is completely characterized by the property that $M, M', M'', \dots$ *are pairwise commuting. This is not only necessary but also sufficient. For, if among the* n^2 *coefficients* $f_{\alpha\beta}$ *of M exactly r are linearly independent over the domain of constants, then one can write*

$$M = f_1 C_1 + \cdots + f_r C_r,$$

(C_s *a constant matrix), where* $f_1, \dots, f_r$ *satisfy no equation* $\sum_\alpha \mathrm{const}\, f_\alpha = 0$*. Then*

$$M^{(\nu)} = f_1^{(\nu)} C_1 + \cdots + f_r^{(\nu)} C_r, \quad (\nu = 1, \dots, r-1).$$

[1] Note that E here denotes the identity matrix.

Since the Wronskian determinant

$$\left|\begin{bmatrix} f_1 & \dots & f_r \\ f_1' & \dots & f_r' \\ & \vdots & \end{bmatrix}\right|$$

cannot vanish identically, one obtains equations of the form

$$C_s = \sum_{\sigma=0}^{r-1} \phi_{s\sigma} M^{(\sigma)}.$$

If $M, M', M'', \dots, M^{(r-1)}$ are pairwise commuting, then the same is true also for $C_1, \dots C_r$ and thus M is of type (1). This implies furthermore that M belongs to type (1) if M^n is the highest[2] *power of M that equals 0. In the case $n = 3$ one therefore only needs to consider type (2).*

With best regards
Yours, Schur

3. Discussion of Schur's letter

This letter was found in Helmut Wielandt's mathematical Nachlass when it was collected by Heinrich Wefelscheid and Hans Schneider not long after Wielandt's death in 2001. We may therefore safely assume that Schur's recent student Wielandt is the "Herr Doktor" to whom the letter is addressed. Schur's letter begins with a reference to a previous remark of Wielandt's which corrected an incorrect assertion by Schur. We can only guess at this sequence of events, but perhaps a clue is provided by Schur's reference to his notes which he did not present correctly in his lectures. Could Wielandt have been in the audience and did he subsequently point out the error? And what was this error? Very probably it was that every matrix of functions that commutes with its derivative is given by (1) (matrices called Type 1), for Schur now denies this and displays another type of matrix commuting with its derivative (called Type 2). He recalls that in his notes he claimed that for matrices of size 5 or less every such matrix is of Type 1, 2 or 3, where Type 3 is obtained from Type 2 by adding a scalar function times the identity. This is not correct because there is also the direct sum of a size 2 matrix of Type 1 and a size 3 matrix of Type 2, we prove this below.

We do not know why Schur was interested in the topic of matrices of functions that commute with their derivative, but it is probably a safe guess that this question came up in the context of solving differential equations, at least this is the motivation in many of the subsequent papers on this topic.

As one of the main results of his letter, Schur shows that an idempotent that commutes with its derivative is a constant matrix and, without further explanation, concludes that one can restrict oneself to matrices with a single eigenvalue. The latter observation raises several questions. First, Schur does not say which functions he has in mind. Second, his argument follows from a standard decomposition of a matrix by a similarity into a direct sum of matrices *provided that* the eigenvalues of the matrix are functions of the type considered. But this is not true in general, for example the eigenvalues of a matrix of rational functions are algebraic functions. We wonder whether Schur was aware of this difficulty and we shall return to it at the end of this section.

Then Schur shows that a matrix of size n is of Type 1 if and only if it and its first $n-1$ derivatives are pairwise commutative. His proof is based on a result of Frobenius [16] that a set of functions is linear independent over the constants if and only if their Wronskian determinant is nonzero. Frobenius, like Schur, does not explain what functions he has in mind. In fact, Peano [28] shows that there exist real differentiable functions that are linearly independent over the reals whose Wronskian is 0. This is followed by Bocher [5] who shows that Frobenius' result holds for analytic functions and investigates necessary and sufficient conditions in [6]. A good discussion of this topic can be found in [8].

[2] We think that Schur means *lowest* here.

We conclude this section by explaining how our exposition has been influenced by some of the observations above. As we do not know what functions Schur and Frobenius had in mind, we follow [1] and some unpublished notes of Guralnick [18] and set Schur's results and ours in terms of differential fields (which include the field of rational functions and the quotient field of analytic functions over the real or complex numbers). Since we do not know how Schur concludes that it is enough to consider matrices with a single eigenvalue, we derive our results from standard matrix decomposition (our Lemma 9 below) which does not assume that all eigenvalues lie in the differential field under consideration.

4. Notation and preliminaries

A *differential field* $\mathbb{F}$ is an (algebraic) field together with an additional operation (the derivative), denoted by $'$ that satisfies $(a+b)' = a' + b'$ and $(ab)' = ab' + a'b$ for $a, b \in \mathbb{F}$. An element $a \in \mathbb{F}$ is called a *constant* if $a' = 0$. It is easily shown that the set of constants forms a subfield $\mathbb{K}$ of $\mathbb{F}$ with $1 \in \mathbb{K}$. Examples are provided by the rational functions over the real or complex numbers and the meromorphic functions over the complex numbers.

In what follows we consider a (differential) field $\mathbb{F}$ and matrices $M = [m_{i,j}] \in \mathbb{F}^{n,n}$. The main condition that we want to analyze is when $M \in \mathbb{F}^{n,n}$ commutes with its derivative,

$$MM' = M'M. \tag{4}$$

As $M \in \mathbb{F}^{n,n}$, it has a minimal and a characteristic polynomial, and M is called *nonderogatory* if the characteristic polynomial is equal to the minimal polynomial, otherwise it is called *derogatory*. See [20].

In Schur's letter the following three types of matrices are considered.

Definition 1. Let $M \in \mathbb{F}^{n,n}$. Then M is said to be of

- *Type 1* if
$$M = \sum_{j=1}^{k} f_j C_j,$$
where $f_j \in \mathbb{F}$, and $C_j \in \mathbb{K}^{n,n}$, for $j = 1, \ldots, k$, and the C_j are pairwise commuting;
- *Type 2* if
$$M = fg^T,$$
with $f, g \in \mathbb{F}^n$, satisfying $f^T g = f^T g' = 0$.
- *Type 3* if
$$M = hI + \widetilde{M},$$
with $h \in \mathbb{F}$ and $\widetilde{M}$ is of Type 2.

Schur's letter also mentions the condition that all derivatives of M commute, i.e.,

$$M^{(i)} M^{(j)} = M^{(j)} M^{(i)} \text{ for all nonnegative integers } i, j. \tag{5}$$

To characterize the relationship between all these properties, we first recall several results from Schur's letter and from classical algebra.

Lemma 2. *Let $\mathbb{F}$ be a differential field with field of constants $\mathbb{K}$. Let N be an idempotent matrix in $\mathbb{F}^{n,n}$ that commutes with N'. Then $N \in \mathbb{K}^{n,n}$.*

Proof. (*See Schur's letter.*) It follows from $N^2 = N$ that $2NN' = N'$. Thus $2NN' = 2N^2N' = NN'$ and this implies that $0 = 2NN' = N'$. $\square$

Another important tool in our analysis will be the following result which in its original form is due to Frobenius [16], see Section 3. We phrase and prove the result in the context of differential fields.

Theorem 3. *Consider a differential field $\mathbb{F}$ with field of constants $\mathbb{K}$. Then $y_1, \ldots, y_r \in \mathbb{F}$ are linearly dependent over $\mathbb{K}$ if and only if the columns of the Wronski matrix*

$$Y = \begin{bmatrix} y_1 & y_2 & \ldots & y_r \\ y_1' & y_2' & \ldots & y_r' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(r-1)} & y_2^{(r-1)} & \ldots & y_r^{(r-1)} \end{bmatrix}$$

are linearly dependent over $\mathbb{F}$.

Proof. We proceed by induction over r. The case $r = 1$ is trivial.

Consider the Wronski matrix Y and the lower triangular matrix

$$Z = \begin{bmatrix} z & 0 & \ldots & 0 \\ c_{2,1}z' & z & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ c_{n,1}z^{(n-1)} & c_{n,2}z^{(n-2)} & \ldots & z \end{bmatrix},$$

with $c_{i,j}$ appropriate binomial coefficients such that

$$ZY = \begin{bmatrix} zy_1 & zy_2 & \ldots & zy_r \\ (zy_1)' & (zy_2)' & \ldots & (zy_r)' \\ \vdots & \vdots & \ddots & \vdots \\ (zy_1)^{(n-1)} & (zy_2)^{(n-1)} & \ldots & (zy_r)^{(n-1)} \end{bmatrix}.$$

Since $\mathbb{F}$ is a differential field, we can choose $z = y_1^{-1}$ and obtain that

$$ZY = \begin{bmatrix} 1 & y_1^{-1}y_2 & \ldots & y_1^{-1}y_r \\ 0 & (y_1^{-1}y_2)' & \ldots & (y_1^{-1}y_r)' \\ \vdots & \vdots & \ddots & \vdots \\ 0 & (y_1^{-1}y_2)^{(n-1)} & \ldots & (y_1^{-1}y_r)^{(n-1)} \end{bmatrix}.$$

It follows that the columns of Y are linearly dependent over $\mathbb{F}$ if and only if the columns of

$$\begin{bmatrix} \left(y_1^{-1}y_2\right)' & \ldots & \left(y_1^{-1}y_r\right)' \\ \vdots & \ddots & \vdots \\ \left(y_1^{-1}y_2\right)^{(n-1)} & \ldots & \left(y_1^{-1}y_r\right)^{(n-1)} \end{bmatrix}$$

are linearly dependent over $\mathbb{F}$, which, by induction, holds if and only if $\left(y_1^{-1}y_2\right)', \ldots, \left(y_1^{-1}y_r\right)'$ are linearly dependent over $\mathbb{K}$, i.e., there exist coefficients $b_2, \ldots, b_r \in \mathbb{K}$, not all 0, such that

$$b_2\left(y_1^{-1}y_2\right)' + \cdots + b_r\left(y_1^{-1}y_r\right)' = 0.$$

Integrating this identity, we obtain

$$b_2\left(y_1^{-1}y_2\right)+\cdots+b_r\left(y_1^{-1}y_r\right)=-b_1$$

for some integration constant $b_1 \in \mathbb{K}$, or equivalently

$$b_1y_1+\cdots+b_ry_r=0. \quad \square$$

Theorem 3 implies in particular that the columns of the Wronski matrix Y are linearly independent over $\mathbb{F}$ if and only if they are linearly independent over $\mathbb{K}$.

Remark 4. Theorem 3 is discussed from a formal algebraic point of view, which however includes the cases of complex analytic functions and rational functions over a field, since these are contained in differential fields. Necessary and sufficient conditions for Theorem 3 to hold for other functions were proved in [6] and discussed in many places, see, e.g. [8,25,26, Ch. XVIII].

5. Characterization of matrices of Type 1

In this section we discuss relationships among the various properties introduced in Schur's letter and in the previous section. This will give, in particular, a characterization of matrices of Type 1.

In his letter, Schur proves the following result.

Theorem 5. *Let $\mathbb{F}$ be a differential field. Then $M \in \mathbb{F}^{n,n}$ is of Type 1 if and only if it satisfies condition (5), i.e., $M^{(i)}M^{(j)} = M^{(j)}M^{(i)}$ for all nonnegative integers i, j.*

Proof. (*See Schur's letter.*) If M is of Type 1, then $M = \sum_{j=1}^{k} f_jC_j$ and the $C_j \in \mathbb{K}^{n,n}$ are pairwise commuting, which immediately implies (5). For the converse, Schur makes use of Theorem 3, since if among the n^2 coefficients $m_{i,j}$ exactly r are linearly independent over $\mathbb{K}$, then

$$M = f_1C_1+\cdots+f_rC_r,$$

with coefficients $C_i \in \mathbb{K}^{n,n}$, where $f_1, \ldots, f_r$ are linearly independent over $\mathbb{K}$. Then

$$M^{(i)} = f_1^{(i)}C_1+\cdots+f_r^{(i)}C_r, \qquad i=1,\ldots,r-1.$$

By Theorem 3, the columns of the associated Wronski matrix are linearly independent, and hence each of the C_i can be expressed as

$$C_i=\sum_{j=0}^{r-1} g_{i,j}M^{(j)}.$$

Thus, if condition (5) holds, then the C_i, $i = 1, \ldots, r$, are pairwise commuting and thus M is of Type 1. $\square$

Using this result we immediately have the following theorem.

Theorem 6. *Let $\mathbb{F}$ be a differential field with field of constants $\mathbb{K}$. If $M \in \mathbb{F}^{n,n}$ is nonderogatory and $MM' = M'M$, then M is of Type 1.*

Proof. If M is nonderogatory then all matrices that commute with M have the form $p(M)$, where p is a polynomial with coefficients in $\mathbb{F}$, see [10,20]. Thus $MM' = M'M$ implies that M' is a polynomial in M. But then every derivative $M^{(j)}$ is a polynomial in M as well and thus (5) holds which by Theorem 5 implies that M is of Type 1. $\square$

The following example from [4,14] of a Type 2 matrix shows that one cannot easily drop the condition that the matrix is nonderogatory.

Example 7. Let

$$f = \begin{bmatrix} 1 \\ t \\ t^2 \end{bmatrix}, \quad g = \begin{bmatrix} t^2 \\ -2t \\ 1 \end{bmatrix},$$

then $f^T g = 0$ and $f^T g' = 0$, hence

$$M_a := gf^T = \begin{bmatrix} t^2 & t^3 & t^4 \\ -2t & -2t^2 & -2t^3 \\ 1 & t & t^2 \end{bmatrix} \tag{6}$$

is of Type 2. Since M_a is nilpotent with $M_a^2 = 0$ but $M_a \neq 0$ and the rank is 1, it is derogatory. One has

$$M_a' = \begin{bmatrix} 2t & 3t^2 & 4t^3 \\ -2 & -4t & -6t^2 \\ 0 & 1 & 2t \end{bmatrix}, \quad M_a'' = \begin{bmatrix} 2 & 6t & 12t^2 \\ 0 & -4 & -12t \\ 0 & 0 & 2 \end{bmatrix},$$

and thus $M_a M_a' = M_a' M_a = 0$. By the product rule it immediately follows that $M_a M_a'' = M_a'' M_a$, but

$$M_a' M_a'' = \begin{bmatrix} 4t & 0 & -4t^3 \\ -4 & 4t & 12t^2 \\ 0 & -4 & -8t \end{bmatrix} \neq M_a' M_a'' = \begin{bmatrix} -8t & -6t^2 & -4t^3 \\ 8 & 4t & 0 \\ 0 & 2 & 4t \end{bmatrix}.$$

Therefore, it follows from Theorem 5 that M_a is not of Type 1.

For any dimension $n \geqslant 3$, one can construct an example of Type 2 by choosing $f \in \mathbb{F}^n$, setting $F = [f, f']$ and then choosing g in the nullspace of F^T. Then fg^T is of Type 2.

Actually every nilpotent matrix function M of rank one satisfying $MM' = M'M$ is of the form $M = fg^T$ and hence of Type 2. This follows immediately because if $M = fg^T$ and $M^2 = 0$ then $g^T f = 0$ and hence $g^T f' + (g^T)' f = 0$. Then it follows from $MM' = M'M$ that $fg^T \left(f(g^T)' + f'g^T\right) = (g^T f')fg^T = (f(g^T)' + f'g^T)fg^T = (g^T)'ffg^T$ which implies that $g^T f' = f^T g'$ and hence $g^T f' = f^T g' = 0$.

6. Triangularizability and diagonalizability

In his letter Schur claims that it is sufficient to consider the case that $M \in \mathbb{F}^{n,n}$ is triangular with only one eigenvalue. This follows from his argument in case the matrix has its eigenvalues in $\mathbb{F}$, which could be guaranteed by assuming that this matrix is $\mathbb{F}$-diagonalizable or even $\mathbb{F}$-triangularizable. Clearly a sufficient condition for this to hold is that $\mathbb{F}$ is algebraically closed, because then for every matrix in $\mathbb{F}^{n,n}$ the characteristic polynomial splits into linear factors.

Definition 8. Let $\mathbb{F}$ be a differential field and let $\mathbb{H}$ be a subfield of $\mathbb{F}$. Then $M \in \mathbb{F}^{n,n}$ is called $\mathbb{H}$-*triangularizable* (*diagonalizable*) if there exists a nonsingular $T \in \mathbb{H}^{n,n}$ such that $T^{-1}MT$ is upper triangular (diagonal).

Using Lemma 2, we can obtain the following result for matrices $M \in \mathbb{F}^{n,n}$ that commute with their derivative M', which is most likely well known but we could not find a reference.

Lemma 9. *Let $\mathbb{F}$ be a differential field with field of constants $\mathbb{K}$, and suppose that $M \in \mathbb{F}^{n,n}$ satisfies $MM' = M'M$. Then there exists an invertible matrix $T \in \mathbb{K}^{n,n}$ such that*

$$T^{-1}MT = \operatorname{diag}(M_1, \ldots, M_k), \tag{7}$$

where the minimal polynomial of each M_i is a power of a polynomial that is irreducible over $\mathbb{F}$.

Proof. Let the minimal polynomial of M be $\mu(\lambda) = \mu_1(\lambda) \cdots \mu_k(\lambda)$, where the $\mu_i(\lambda)$ are powers of pairwise distinct polynomials that are irreducible over $\mathbb{F}$. Set

$$p_i(\lambda) = \mu(\lambda)/\mu_i(\lambda), \qquad i = 1, \ldots, k.$$

Since the polynomials $p_i(\lambda)$ have no common factor, there exist polynomials $q_i(\lambda)$, $i = 1, \ldots, k$, such that the polynomials $\epsilon_i(\lambda) = p_i(\lambda)q_i(\lambda)$, $i = 1, \ldots, k$, satisfy

$$\epsilon_1(\lambda) + \cdots + \epsilon_k(\lambda) = 1. \tag{8}$$

Setting $E_i = \epsilon_i(M)$, $i = 1, \ldots, k$ and using the fact that $\mu(M) = 0$ yields that

$$E_1 + \cdots + E_k = I, \tag{9}$$

$$E_iE_j = 0, \quad i, j = 1, \ldots, k, \quad i \neq j, \tag{10}$$

$$E_i^2 = E_i, \quad i = 1, \ldots, k. \tag{11}$$

The last identity follows directly from (9) and (10). Since the E_i are polynomials in M and $MM' = M'M$, it follows that the E_i commute with E_i', $i = 1, \ldots k$. Hence, by Lemma 2, $E_i \in \mathbb{K}^{n,n}$, $i = 1, \ldots, k$. By (9), (10), and (11), $\mathbb{K}^n$ is a direct sum of the ranges of the E_i and we obtain that, for some nonsingular $T \in \mathbb{K}^{n,n}$,

$$\widetilde{E}_i := T^{-1}E_iT = \operatorname{diag}(0, I_i, 0), \qquad i = 1, \ldots, k,$$

where the I_i are identity matrices of the size equal to the dimension to the range of E_i. This is a consequence of the fact that E_i is diagonalizable with eigenvalues 0 and 1. Since each E_i commutes with M, we obtain that

$$\begin{aligned}\widetilde{M_i} &:= T^{-1}E_iMT \\ &= T^{-1}E_iME_iT \\ &= \operatorname{diag}(0, I_i, 0)T^{-1}MT \operatorname{diag}(0, I_i, 0) \\ &= \operatorname{diag}(0, M_i, 0), \qquad i = 1, \ldots, k.\end{aligned}$$

Now observe that

$$\widetilde{E}_i\mu_i(\widetilde{M}_i)\widetilde{E}_i = T^{-1}\epsilon_i(M)\mu_i(M)\epsilon_i(M)T = 0,$$

since $\epsilon_i(\lambda)\mu_i(\lambda) = \mu(\lambda)q_i(\lambda)$. Hence $\mu_i(M_i) = 0$ as well. We assert that $\mu_i(\lambda)$ is the minimal polynomial of M_i, for if $r(M_i) = 0$ for a proper factor $r(\lambda)$ of $m_i(\lambda)$ then $r(M)\Pi_{j\neq i}\mu_j(M) = 0$, contrary to the assumption that $\mu(\lambda)$ is the minimal polynomial of M. □

Lemma 9 has the following corollary, which has been proved in a different way in [1,18].

Corollary 10. *Let $\mathbb{F}$ be a differential field with field of constants $\mathbb{K}$. If $M \in \mathbb{F}^{n,n}$ satisfies $MM' = M'M$ and is $\mathbb{F}$-diagonalizable, then M is $\mathbb{K}$-diagonalizable.*

Proof. In this case, the minimal polynomial of M is a product of distinct linear factors and hence, the minimal polynomial of each M_i occurring in the proof of Lemma 9 is linear. Therefore, each M_i is a scalar matrix. □

We also have the following corollary.

Corollary 11. *Let $\mathbb{F}$ be a differential field with field of constants $\mathbb{K}$. If $M \in \mathbb{F}^{n,n}$ satisfies $MM' = M'M$ and is $\mathbb{F}$-diagonalizable, then M is of Type 1.*

Proof. By Corollary 10, $M = T^{-1}\,\mathrm{diag}(m_1, \ldots, m_n)T$ with $m_i \in \mathbb{F}$ and nonsingular $T \in \mathbb{K}^{n,n}$. Hence

$$M = \sum_{i=1}^{n} m_i T^{-1} E_{i,i} T$$

where $E_{i,i}$ is a matrix that has a 1 in position (i, i) and zeros everywhere else. Since all the matrices $E_{i,i}$ commute, M is of Type 1. $\square$

Remark 12. Any $M \in \mathbb{F}^{n,n}$ that is of rank one, satisfies $MM' = M'M$ and is not nilpotent, is of Type 1, since in this case M is $\mathbb{F}$-diagonalizable. This follows by Corollary 11, since the minimal polynomial has the from $(\lambda - c)\lambda$ for some $c \in \mathbb{F}$. This means in particular for a rank one matrix $M \in \mathbb{F}^{n,n}$ to be of Type 2 and not of Type 1 it has to be nilpotent.

For matrices that are just triangularizable the situation is more subtle. We have the following theorems.

Theorem 13. *Let $\mathbb{F}$ be a differential field with an algebraically closed field of constants $\mathbb{K}$. If $M \in \mathbb{F}^{n,n}$ is Type 1, then M is $\mathbb{K}$-triangularizable.*

Proof. Any finite set of pairwise commutative matrices with elements in an algebraically closed field may be simultaneously triangularized, see e.g., [30, Theorem 1.1.5]. Under this assumption on $\mathbb{K}$, if M is Type 1, then it follows that the matrices $C_i \in \mathbb{K}^{n,n}$ in the representation of M are simultaneously triangularizable by a matrix $T \in \mathbb{K}^{n,n}$. Hence T also triangularizes M. $\square$

Theorem 13 implies that Type 1 matrices have n eigenvalues in $\mathbb{F}$ if $\mathbb{K}$ is algebraically closed and it further immediately leads to a Corollary of Theorem 6.

Corollary 14. *Let $\mathbb{F}$ be a differential field with field of constants $\mathbb{K}$. If $M \in \mathbb{F}^{n,n}$ is nonderogatory, satisfies $MM' = M'M$ and if $\mathbb{K}$ is algebraically closed, then M is $\mathbb{K}$-triangularizable.*

Proof. By Theorem 6 it follows that M is Type 1 and thus the assertion follows from Theorem 13. $\square$

7. Matrices of small size and examples

Example 7 again shows that it is difficult to drop some of the assumptions, since this matrix is derogatory, not of Type 1, and not $\mathbb{K}$-triangularizable.

One might be tempted to conjecture that any $M \in \mathbb{F}^{n,n}$ that is $\mathbb{K}$-triangularizable and satisfies (4) is of Type 1 but this is so only for small dimensions and is no longer true for large enough n, as we will demonstrate below. Consider small dimensions first.

Proposition 15. *Consider a differential field $\mathbb{F}$ of functions with field of constants $\mathbb{K}$. Let $M = [m_{i,j}] \in \mathbb{F}^{2,2}$ be upper triangular and satisfy $M\,M' = M'\,M$. Then M is of Type 1.*

Proof. Since $M\,M' = M'\,M$, we obtain

$$m_{1,2}(m'_{1,1} - m'_{2,2}) - m'_{1,2}(m_{1,1} - m_{2,2}) = 0,$$

which implies that $m_{1,2} = 0$ or $m_{1,1} - m_{2,2} = 0$ or both are nonzero and $\frac{d}{dt}\left(\frac{m_{1,1}-m_{2,2}}{m_{1,2}}\right) = 0$, i.e., $cm_{1,2} + (m_{1,1} - m_{2,2}) = 0$ for some nonzero constant c.

If $m_{1,1} = m_{2,2}$ or $m_{1,2} = 0$, then M, being triangular, is obviously of Type 1. Otherwise

$$M = m_{1,1}I + m_{1,2}\begin{bmatrix} 0 & 1 \\ 0 & c \end{bmatrix}$$

and hence again of Type 1 as claimed. □

Proposition 15 implies that 2×2 $\mathbb{K}$-triangularizable matrices satisfying (4) are of Type 1.

Proposition 16. *Consider a differential field $\mathbb{F}$ with an algebraically closed field of constants $\mathbb{K}$. Let $M = [m_{i,j}] \in \mathbb{F}^{2,2}$ satisfy $M\,M' = M'\,M$. Then M is of Type 1.*

Proof. If M is $\mathbb{F}$-diagonalizable, then the result follows by Corollary 11. If M is not $\mathbb{F}$-diagonalizable, then it is nonderogatory and the result follows by Corollary 14. □

Example 17. In the 2×2 case, any Type 2 or Type 3 matrix is also of Type 1 but not every Type 1 matrix is Type 3.

Let $M = \phi I_2 + fg^T$ with

$$\phi \in \mathbb{F}, \quad f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \quad g = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \in \mathbb{F}^2$$

be of Type 3, i.e., $f^Tg = {f'}^Tg = f^Tg' = 0$.

If $f_2 = 0$, then M is upper triangular and hence by Proposition 15, M is of Type 1. If $f_2 \neq 0$, then with

$$T = \begin{bmatrix} 1 & -f_1/f_2 \\ 0 & 1 \end{bmatrix}$$

we have

$$TMT^{-1} = \phi I_2 + \begin{bmatrix} 0 & 0 \\ f_2g_1 & 0 \end{bmatrix} = \phi I_2 + f_2g_1\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

since $f_1g_1 + f_2g_2 = 0$, and hence M is of Type 1.

However, if we consider

$$M = \phi I_2 + f\begin{bmatrix} 0 & c \\ 0 & d \end{bmatrix}$$

with ϕ, f nonzero functions and c, d nonzero constants, then M is Type 1 but not Type 3.

Proposition 18. *Consider a differential field $\mathbb{F}$ of functions with field of constants $\mathbb{K}$. Let $M = [m_{i,j}] \in \mathbb{F}^{3,3}$ be $\mathbb{K}$-triangularizable and satisfy $M\,M' = M'\,M$. Then M is of Type 1.*

Proof. Since M is $\mathbb{K}$-triangularizable, we may assume that it is upper triangular already and consider different cases for the diagonal elements. If M has three distinct diagonal elements, then it is $\mathbb{K}$-diagonalizable and the result follows by Corollary 11. If M has exactly two distinct diagonal elements, then it can be transformed to a direct sum of a 2×2 and 1×1 matrix and hence the result follows by Proposition 15. If all diagonal elements are equal, then, letting $E_{i,j}$ be the matrix that is zero except for the position (i, j), where it is 1, we have $M = m_{1,1}I + m_{1,3}E_{1,3} + \widetilde{M}$, where $\widetilde{M} = m_{1,2}E_{1,2} + m_{2,3}E_{2,3}$ also satisfies (4). Then it follows that $m_{1,2}m'_{2,3} = m'_{1,2}m_{2,3}$. If either $m_{1,2} = 0$ or $m_{2,3} = 0$, then we immediately have again Type 1, since $\widetilde{M}$ is a direct sum of a 2×2 and a 1×1 problem. If both are nonzero, then $\widetilde{M}$ is nonderogatory and the result follows by Theorem 6. In fact, in this case $m_{1,2} = cm_{2,3}$ for some $c \in \mathbb{K}$ and therefore

$$M = m_{1,1}I + m_{1,3}E_{1,3} + m_{2,3}\begin{bmatrix} 0 & c & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

which is clearly of Type 1. $\square$

In the 4×4 case, if the matrix is $\mathbb{K}$-triangularizable, then we either have at least two different eigenvalues, in which case we have reduced the problem again to the case of dimensions smaller than 4, or there is only one eigenvalue, and thus without loss of generality M is nilpotent. If M is nonderogatory then we again have Type 1. If M is derogatory then it is the direct sum of blocks of smaller dimension. If these dimensions are smaller than 3, then we are again in the Type 1 case. So it remains to study the case of a block of size 3 and a block of size 1. Since M is nilpotent, the block of size 3 is either Type 1 or Type 2. In both cases the complete matrix is also Type 1 or Type 2, respectively.

The following example shows that $\mathbb{K}$-triangularizability is not enough to imply that the matrix is Type 1.

Example 19. Consider the 9×9 block matrix

$$\hat{M} = \begin{bmatrix} 0 & M_a & 0 \\ 0 & 0 & M_a \\ 0 & 0 & 0 \end{bmatrix},$$

where M_a is the Type 2 matrix from Example 7. Then $\hat{M}$ is nilpotent upper triangular and not of Type 1, 2, or 3, the latter two facts due to its $\mathbb{F}$-rank being 2.

Already in the 5×5 case, we can find examples that are none of the (proper) types.

Example 20. Consider $M = T^{-1}\operatorname{diag}(M_1, M_2)T$ with $T \in \mathbb{K}^{n,n}$, $M_1 \in \mathbb{F}^{3,3}$ of Type 2 (e.g., take $M_1 = M_a$ as in Example 7) and $M_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then clearly M is not of Type 1 and it is not of Type 2, since it has an $\mathbb{F}$-rank larger than 1. By definition it is not of Type 3 either. Clearly examples of any size can be constructed by building direct sums of smaller blocks.

Schur's letter states that for $n \geqslant 6$ there are other types. The following example demonstrates this.

Example 21. Let M_a be the Type 2 matrix in Example 7 and form the block matrix

$$A = \begin{bmatrix} M_a & I \\ 0 & M_a \end{bmatrix}.$$

Direct computation shows $AA' = A'A$ but $A'A'' \neq A''A$. Furthermore $A^3 = 0$ and A has $\mathbb{F}$-rank 3. Thus A is neither Type 1, Type 2 nor Type 3 (the last case need not be considered, since A is nilpotent). We also note that $\operatorname{rank}(A'') = 6$. We now assume that $\mathbb{K}$ is algebraically closed and we show that A is not $\mathbb{K}$-similar to the direct sum of Type 1 or Type 2 matrices.

To obtain a contradiction, we assume that (after a $\mathbb{K}$-similarity) $A = \operatorname{diag}(A_1, A_2)$ where A_1 is the direct sum of Type 1 matrices (and hence Type 1) and A_2 is the direct sum of Type 2 matrices that are not Type 1. Since A is not Type 1, A_2 cannot be the empty matrix. Since the minimum size of a Type 2 matrix that is not Type 1 is 3 and its rank is 1 it follows that A cannot be the sum of Type 2 matrices that are not Type 1. Hence the size of A_1 must be larger or equal to 1 and, since A_1 is nilpotent, it follows that $\operatorname{rank}(A_1) < \operatorname{size}(A_1)$. Since A_1 is $\mathbb{K}$-similar to a strictly triangular matrix, it follows that $\operatorname{rank}(A_1'') < \operatorname{size}(A_1)$. Hence $\operatorname{rank}(A'') = \operatorname{rank}(A_1'') + \operatorname{rank}(A_2'') < 6$, a contradiction.

Example 22. If the matrix $M = \sum_{i=0}^{r} C_i t^i \in \mathbb{F}^{n,n}$ is a polynomial with coefficients $C_i \in \mathbb{K}^{n,n}$, then from (4) we obtain a specific set of conditions on sums of commutators that have to be satisfied. For this we just compare coefficients of powers of t and obtain a set of quadratic equations in the C_i, which has a clear pattern. For example, in the case $r = 2$, we obtain the three conditions $C_0C_1 - C_1C_0 = 0$, $C_0C_2 - C_2C_0 = 0$ and $C_1C_2 - C_2C_1 = 0$, which shows that M is of Type 1. For $r = 3$ we obtain the first nontrivial condition $3(C_0C_3 - C_3C_0) + (C_1C_2 - C_2C_1) = 0$.

We have implemented a Matlab routine for Newton's method to solve the set of quadratic matrix equations in the case $r = 3$ and ran it for many different random starting coefficients C_i of different dimensions n. Whenever Newton's method converged (which it did in most of the cases) it converged to a matrix of Type 1. Even in the neighborhood of a Type 2 matrix it converged to a Type 1 matrix. This suggests that the matrices of Type 1 are generic in the set of matrices satisfying (4). A copy of the Matlab routine is available from the authors upon request.

8. Conclusion

We have presented a letter of Schur's that contains a major contribution to the question when a matrix with elements that are functions in one variable commutes with its derivative. Schur's letter precedes many partial results on this question, which is still partially open. We have put Schur's result in perspective with later results and extended it in an algebraic context to matrices over a differential field. In particular, we have presented several results that characterize Schur's matrices of Type 1. We have given examples of matrices that commute with their derivative which are of none of the Types 1, 2 or 3.We have shown that matrices of Type 1 may be triangularized over the constant field (which implies that their eigenvalues lie in the differential field) but we are left with an open problem already mentioned in Section 3.

Open Problem 23. Let M be a matrix in a differential field $\mathbb{F}$, with an algebraically closed field of constants, that satisfies $MM' = M'M$. Must the eigenvalues of M be elements of the field $\mathbb{F}$?

For example, if M is a polynomial matrix over the complex numbers must the eigenvalues be rational functions? We have found no counterexample.

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