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On the dimension of max–min convex sets

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Abstract

We introduce a notion of dimension of max–min convex sets, following the approach of tropical convexity. We introduce a max–min analogue of the tropical rank of a matrix and show that it is equal to the dimension of the associated polytope. We describe the relation between this rank and the notion of strong regularity in max–min algebra, which is traditionally defined in terms of unique solvability of linear systems and the trapezoidal property.
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1. Introduction

The max–min semiring is defined as the unit interval $\mathcal{B} = [0, 1]$ with the operations $a \oplus b := \max(a, b)$, as addition, and $a \otimes b := \min(a, b)$, as multiplication. The operations are idempotent, $\max(a, a) = a = \min(a, a)$, and related to the order:

$$\max(a, b) = b \quad \Leftrightarrow \quad a \leq b \quad \Leftrightarrow \quad \min(a, b) = a. \tag{1}$$

One can naturally extend them to matrices and vectors leading to the max–min (fuzzy) linear algebra of [2,4,12,13,15]. Note that in [15] the authors developed a more general version of max–min algebra over arbitrary linearly ordered set, but we will not follow this generalization here.

We denote by $\mathcal{B}(d, m)$ the set of $d \times m$ matrices with entries in $\mathcal{B}$ and by $\mathcal{B}^d$ the set of d-dimensional vectors with entries in $\mathcal{B}$. Both $\mathcal{B}(d, m)$ and $\mathcal{B}^d$ have a natural structure of semimodule over the semiring $\mathcal{B}$.

A subset $V \subseteq \mathcal{B}^d$ is a *subsemimodule* if $u, v \in V$ imply $u \oplus v \in V$ and $\lambda \otimes v \in V$ for all $\lambda \in \mathcal{B}$. Subsemimodules can be thought of as a max–min analogue of subspaces or convex cones (especially in the context of the present

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paper). In the max–min literature, subsemimodules arise as images of max–min matrices [14] or as eigenspaces. A subsemimodule $V \subseteq \mathcal{B}^d$ is said to be generated by a subset $X \subseteq \mathcal{B}^d$ and it is denoted by $V = \mathrm{span}_{\oplus}(X)$, if it can be represented as a set of all *max–min linear combinations*

$$\bigoplus_{i=1}^{m} \lambda_i \otimes x^i : m \geq 1, \lambda_1, \ldots, \lambda_m \in \mathcal{B}, \tag{2}$$

of all m-tuples of elements $x^1, \ldots, x^m \in X$.

The *max–min segment* between $x, y \in \mathcal{B}^d$ is defined as

$$[x, y]_{\oplus} = \{\alpha \otimes x \oplus \beta \otimes y \mid \alpha, \beta \in \mathcal{B}, \alpha \oplus \beta = 1\}. \tag{3}$$

A set $C \subseteq \mathcal{B}^d$ is called *max–min convex*, if it contains, with any two points x, y, the segment $[x, y]_{\oplus}$ between them. For a general subset $X \subseteq \mathcal{B}^d$, define its *convex hull* $\mathrm{conv}_{\oplus}(X)$ as the smallest max–min convex set containing X, i.e., the smallest set containing X and closed under taking segments (3). As in the ordinary convexity, $\mathrm{conv}_{\oplus}(X)$ is the set of all *max–min convex combinations*

$$\bigoplus_{i=1}^{m} \lambda_i \otimes x^i : m \geq 1,\ \lambda_1, \ldots, \lambda_m \in \mathcal{B}, \bigoplus_{i=1}^{m} \lambda_i = 1, \tag{4}$$

of all m-tuples of elements $x^1, \ldots, x^m \in X$. The max–min convex hull of a finite set of points is also called a *max–min convex polytope*.

The development of max–min convexity has been mostly inspired by new geometric techniques in max–plus (tropical) linear algebra, like those developed in [1,6,7,16]. The development of tropical (max–plus) convexity was started by K. Zimmermann [28], and it gained new impetus after the works of Cohen, Gaubert, Quadrat and Singer [5], and Develin and Sturmfels [6]. This development has led to many theoretical and algorithmic results, and in particular, to new methods describing the solution set of max–plus linear systems of equations [1,16].

K. Zimmermann [29] also suggested to develop the convex geometry over wider classes of semirings with idempotent addition, including the max–min semiring. To the authors' knowledge, the case of max–min semiring did not receive much interest in the past. Some recent developments in max–min convexity include the description of max–min segments [22,26], max–min semispaces [23] and hyperplanes [17], separation and non-separation results [18,19]. See [20] for a survey of max–min convexity that also includes some new results, in particular, colorful extensions of the max–min Carathéodory theorem, as well as some applications of the topological Radon theorem.

The present paper aims to develop a new geometric approach to the well-known notions of strong regularity and matrix rank in max–min algebra. To this end, it seems to be the first paper that connects max–min linear algebra with max–min convexity. Our main result is Theorem 4.5 stating that the "geometric" dimension of a max–min polytope is equal to a max–min analogue of the tropical rank of the matrix whose columns are the "vertices" of that polytope.

Let us make some preliminary observations. Note first that any subsemimodule is a max–min convex set. Moreover, since any max–min convex combination is just a max–min linear combination with one coefficient equal to 1, we obtain that the max–min subsemimodules are precisely the max–min convex sets containing 0. Thus for any $X \subseteq \mathcal{B}^d$, we have

$$\mathrm{span}_{\oplus}(X) = \mathrm{conv}_{\oplus}\big(X, \{0\}\big).$$

We conclude that a finitely generated max–min semimodule can also be described as a max–min polytope with one "vertex" in the origin.

Conversely, if $C \subseteq \mathcal{B}^d$ is a max–min convex set, then

$$V_C := \big\{(\lambda \otimes x, \lambda) \,\big|\, x \in C, \lambda \in \mathcal{B}\big\}$$

is a subsemimodule of $\mathcal{B}^{d+1}$. This construction is called *homogenization*.

Although our interest here is mostly theoretical, it is also motivated by the theory of fuzzy sets [27], which has numerous applications in computer science and decision theory. For example, in [8] Dubois and Prade developed an axiomatic approach to quantitative utility theory. The utility function introduced there relies on the notion of possibilistic mixture, where the possibilistic mixture (which under some natural conditions is also a possibilistic measure [10])

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of the possibilistic measures π_1, π_2 with possibilities α, β, $\max(\alpha, \beta) = 1$, is defined as $\max(\min(\alpha, \pi_1), \min(\beta, \pi_2))$, that is, as a point on the max–min segment $[\pi_1, \pi_2]$. This is a particular case of extended mixtures of decomposable measures (which are a family of set functions encompassing probability measures and necessity and possibility measures as particular cases), as studied in [9] where the application to utility theory is pointed out.

The paper is organized as follows. The structure of max–min segments is revisited in Section 2. A notion of dimension in max–min convexity is introduced and studied in Section 3. Our approach is inspired by a geometric idea behind the notion the tropical rank [7], that is, a tropically convex polytope can be represented as a union of conventionally convex sets, and its dimension can be defined as the greatest dimension of these convex sets. In Section 4 we introduce a notion of strong regularity and a notion of rank for a matrix A over the max–min semiring. We show that the rank of A, as we introduce it, is equal to the dimension of the max–min convex hull of the columns of A. In Section 5 we show that our notion of strong regularity is equivalent to the one traditionally studied in max–min algebra. Thus it is closely related to the unique solvability of max–min linear systems of the type $A \otimes x = b$ and, further, to the trapezoidal property of a matrix as studied, for example, in [2,4,12,14].

2. Max–min segments

In this section we describe general segments in $\mathcal{B}^d$, following [22,26], where complete proofs can be found. Note that the description of the segments in [22,26] is done for the equivalent case where $\mathcal{B} = [-\infty, +\infty]$.

Let $x = (x_1, ..., x_d)$, $y = (y_1, ..., y_d) \in \mathcal{B}^d$, and assume that we are in the *case of comparable endpoints*, say $x \le y$ in the natural order of $\mathcal{B}^d$. Sorting the set of all coordinates $\{x_i, y_i, i = 1, ..., d\}$ we obtain a non-decreasing sequence, denoted by $t_1, t_2, \dots, t_{2d}$. This sequence divides the set $\mathcal{B}$ into $2d + 1$ subintervals $\sigma_0 = [0, t_1], \sigma_1 = [t_1, t_2], ..., \sigma_{2d} = [t_{2d}, 1]$, with consecutive subintervals having one common endpoint.

Every point $z \in [x, y]_\oplus$ is represented as $z = \alpha \otimes x \oplus \beta \otimes y$, where $\alpha = 1$ or $\beta = 1$. However, case $\beta = 1$ yields only $z = y$, so we can assume $\alpha = 1$. Thus z can be regarded as a function of one parameter β, that is, $z(\beta) = (z_1(\beta), ..., z_d(\beta))$ with $\beta \in \mathcal{B}$. Observe that for $\beta \in \sigma_0$ we have $z(\beta) = x$ and for $\beta \in \sigma_{2d}$ we have $z(\beta) = y$. Vectors $z(\beta)$ with β in any other subinterval form a conventional *elementary segment*. Let us proceed with a formal account of all this.

Theorem 2.1. *Let $x, y \in \mathcal{B}^d$ and $x \le y$.*

(i) *We have*

$$[x, y]_\oplus = \bigcup_{\ell=1}^{2d-1} \{z(\beta) \mid \beta \in \sigma_\ell\}, \tag{5}$$

where $z(\beta) = x \oplus (\beta \otimes y)$ and $\sigma_\ell = [t_\ell, t_{\ell+1}]$ for $\ell = 1, \dots, 2d-1$, and $t_1, \dots, t_{2d}$ is the nondecreasing sequence whose elements are the coordinates x_i, y_i for $i = 1, \dots, d$.

(ii) *For each $\beta \in \mathcal{B}$ and i, let $M(\beta) = \{i : x_i \le \beta \le y_i\}$, $H(\beta) = \{i \mid \beta \ge y_i\}$ and $L(\beta) = \{i : \beta \le x_i\}$. Then*

$$z_i(\beta) = \begin{cases} \beta, & \text{if } i \in M(\beta), \\ x_i, & \text{if } i \in L(\beta), \\ y_i, & \text{if } i \in H(\beta), \end{cases} \tag{6}$$

and $M(\beta), L(\beta), H(\beta)$ do not change in the interior of each interval σ_ℓ.

(iii) *The sets $\{z(\beta) \mid \beta \in \sigma_\ell\}$ in (5) are conventional closed segments in $\mathcal{B}^d$ (possibly reduced to a point), described by (6) where $\beta \in \sigma_\ell$.*

For *incomparable endpoints* $x \not\le y$, $y \not\le x$, the description can be reduced to that of segments with comparable endpoints, by means of the following observation.

Theorem 2.2. *Let $x, y \in \mathcal{B}^d$. Then $[x, y]_\oplus$ is the concatenation of two segments with comparable endpoints, namely $[x, y]_\oplus = [x, x \oplus y]_\oplus \cup [y, x \oplus y]_\oplus$.*

All types of segments for $d = 2$ are shown in the right side of Fig. 1.

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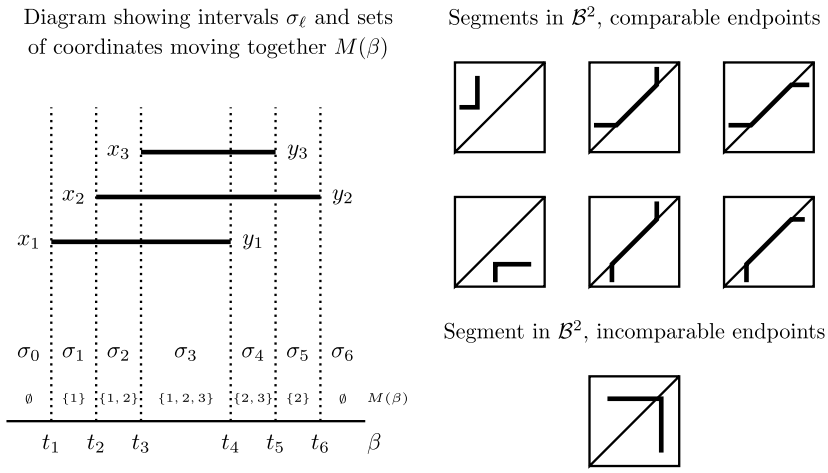

Fig. 1. Max–min segments.

The left side of Fig. 1 shows, for the corresponding segments with comparable endpoints, a diagram, where for $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$, the intervals $[x_1, y_1], [x_2, y_2], [x_3, y_3]$, are placed over one another, and their arrangement induces a tiling of the horizontal axis, which shows the possible values of the parameter β. The partitions of the intervals $[x_i, y_i]$, $1 \le i \le 3$, induced by this tiling are associated with the intervals σ_l, and show the sets of *active indices* i with $z_i(\beta) = \beta$.

Remark 2.3. It follows from the description above that each elementary segment is determined by a partition of the set of coordinates in two subsets. For points in the elementary segment, the coordinates in the first subset are constant and the coordinates in the second subset are all equal to a parameter running over a 1-dimensional interval. Therefore, similarly to the max–plus case (see [21], Remark 4.3) in $\mathcal{B}^d$ there are elementary segments in only $2^d - 1$ directions. Elementary segments are the "building blocks" for the max–min segments in $\mathcal{B}^d$, in the sense that every segment $[x, y] \subseteq \mathcal{B}^d$ is the concatenation of a finite number of elementary subsegments (at most) $2d - 1$, respectively $2d - 2$, in the case of comparable, respectively incomparable, endpoints. In the case of incomparable endpoints, the set of coordinates is partitioned in two subsets of comparable coordinates, say of cardinality d_1, d_2, with $d_1 + d_2 = d$. The first subset determines at most $2d_1 - 1$ elementary segments, and the second set determines at most $2d_2 - 1$ elementary segments, for a total of at most $2d - 2$ elementary segments.

We close this section with an observation which we will need further. In this observation, as in the subsequent parts of the paper, we will use the conventional arithmetic operations $(+, \cdot)$. For a real vector $y = (y_1, \dots, y_d) \in \mathbb{R}^d$, we define the support of y (with respect to the standard basis), as $\mathrm{supp}(y) := \{i \mid y_i \neq 0\}$.

Lemma 2.4. *Let $y \in \mathcal{B}^d$ and let $u \in \mathbb{R}^d$ be a nonnegative real vector with support* $\mathrm{supp}(u) = M$ *such that $y + u \in \mathcal{B}^d$. Then the following are equivalent:*

(i) $[y, y + u]_\oplus$ *contains only vectors $y + u'$ with u' proportional to u;*
(ii) *for all $i, j \in M$ we have $y_i = y_j$ and $u_i = u_j$.*

Proof. (i) $\Rightarrow$ (ii): By contradiction, let the condition of (ii) be violated. Suppose first that $y_i \neq y_j$ for some $i, j \in M$, and let $M' \subseteq M$ be the proper subset of indices attaining $\min_{i \in M} y_i$. By Theorem 2.1 (see also the left part of Fig. 1) it follows that there is a nonnegative vector u' such that $\mathrm{supp}(u') = M'$ and $y + u'$ belongs to the first subsegment of $[y, y + u]_\oplus$. As M' is a proper subset of M, it follows that u' is non-proportional to u.

Suppose now that $y_i = y_j$ for all $i, j \in M$ but $u_i \neq u_j$ for some $i, j \in M$. Let $M'' \subseteq M$ be the proper subset of indices attaining $\max_{i \in M} u_i$. By Theorem 2.1 (see also the left part of Fig. 1) it follows that there is a nonnegative vector u'' such that $\mathrm{supp}(u'') = M''$ and $y + u - u''$ belongs to the last subsegment of $[y, y + u]_\oplus$. As M'' is a proper subset of M, it follows that $u - u''$ is non-proportional to u.

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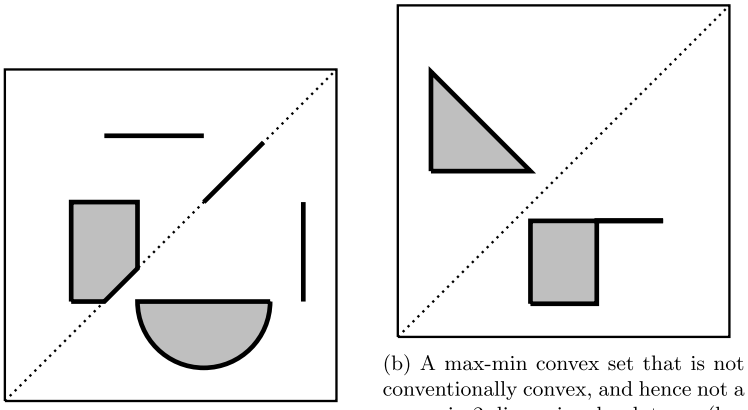

(a) Closed max-min polytropes in $\mathcal{B}^2$.

(b) A max-min convex set that is not conventionally convex, and hence not a max-min 2-dimensional polytrope (below the diagonal), and a conventionally convex set that is not max-min convex in $\mathcal{B}^2$ (above the diagonal).

Fig. 2. Max–min convex sets in $\mathcal{B}^2$.

(ii) $\Rightarrow$ (i): By Theorem 2.1, in this case $[y, y+u]_\oplus$ is just the ordinary segment $\{y + u' \mid u' = \lambda u, 0 \le \lambda \le 1\}$. $\square$

3. Dimension and max–min polytropes

The dimension of a max–min convex set can be introduced in the spirit of the tropical rank, see for instance Develin, Santos and Sturmfels [7, Section 4]. In this set-up we expect polytopes to be representable as complexes of cells that are convex both in the usual and in the new sense. We are interested in the interplay between these convexities, similar to the case of tropical (max–plus) mathematics.

In what follows $\mathcal{B}^d$ has the usual Euclidean topology. If $C \subseteq \mathcal{B}^d$, we denote by $\overline{C}$ the closure of C and by $\mathrm{int}(C)$ the interior of C.

Definition 3.1. A max–min convex set $C \subseteq \mathcal{B}^d$, for $0 \le k \le d$, is called a k-dimensional open (resp. closed) max–min polytrope if it is also a k-dimensional relatively open (resp. closed) conventionally convex set.

This concept is a max–min analogue of the so-called polytropes, i.e., the sets which are (traditionally) convex and tropically convex at the same time, see Joswig and Kulas [11]. Various types of convex sets are shown in Fig. 2.

Definition 3.2. The dimension of a max–min convex set $C \subseteq \mathcal{B}^d$, denoted by $\dim(C)$, is the greatest k such that C contains a k-dimensional open polytrope.

Remark 3.3. Occasionally the notation dim will be also used for the usual dimension of (conventionally) linear spaces and convex sets — making sure that this will not lead to any confusion.

Note that if the max–min convex set $C \subseteq \mathcal{B}^d$ has dimension d, then C has nonempty interior.

In what follows we will make use of the usual linear algebra and the usual convexity. For a convex set $C \subseteq \mathbb{R}^d$, let $C - y := \{z - y : z \in C\}$, and let $\mathrm{Lin}(C - y)$ be the least conventionally linear space containing $C - y$. From the convex analysis, recall that C is relatively open if $C - y$ is open in $\mathrm{Lin}(C - y)$ for some, and hence for all $y \in C$. In this case, for any $u \in \mathrm{Lin}(C - y)$ there is $\epsilon > 0$ such that $y + \epsilon u \in C$ and, conversely, if $y + u \in C$ then $u \in \mathrm{Lin}(C - y)$.

Observe that if C is closed under componentwise maxima $\oplus$, as in the case when it is a polytrope, then for each pair $u, v \in \mathrm{Lin}(C - y)$ we have $y + \epsilon u, y + \epsilon v \in C$ for some $\epsilon > 0$ and $(y + \epsilon u) \oplus (y + \epsilon v) = y + \epsilon(u \oplus v) \in C$, hence $u \oplus v \in \mathrm{Lin}(C - y)$. So $\mathrm{Lin}(C - y)$ is also closed under taking componentwise maxima. In particular, it follows that $\mathrm{Lin}(C - y)$ has a vector whose support contains the support of any other vector in $\mathrm{Lin}(C - y)$, that is, a vector whose support is the *largest* (by inclusion).

The following auxiliary lemma, about the conventional linear algebra, will be needed in the proof of Theorem 3.5.

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Lemma 3.4. *Let $L \subseteq \mathbb{R}^n$ be a linear subspace. Assume that there exists a nonnegative vector $e \in L$ whose support is the largest in L (by inclusion). Then:*

$$\mathrm{Lin}\big(L \cap \mathbb{R}^n_+\big) = L \tag{7}$$

and, in particular, $\dim(\mathrm{Lin}(L \cap \mathbb{R}^n_+)) = \dim(L)$.

Proof. As $L \cap \mathbb{R}^n_+ \subseteq L$, we always have $\mathrm{Lin}(L \cap \mathbb{R}^n_+) = L$, so it suffices to prove that L can be generated by some vectors in $L \cap \mathbb{R}^n_+$, under the given condition.

Let $\{f_1, \dots, f_k\}$ be a basis for L. For every $i = 1, \dots, k$, there exists $m_i > 0$ such that $F := \{f_1 + m_1 e, \dots, f_k + m_k e\}$ is a family of nonnegative vectors in $L \cap \mathbb{R}^n_+$. The family $\tilde{F} := F \cup \{e\}$ is a family of nonnegative vectors in $L \cap \mathbb{R}^n_+$ that generates L (since it generates all the base vectors), so (7) holds. □

The following result investigates some of the interplay between the max–min and conventional convexities. For a monograph in conventional convexity see, e.g., Rockafellar [25].

Theorem 3.5. *Let $d \geq 1, 0 \leq k \leq d$. Let $C \subseteq \mathcal{B}^d$ be a k-dimensional open polytrope. Then for each point $y \in C$ there exist pairwise disjoint index sets $J_1, \dots, J_k \subseteq \{1, \dots, d\}$ and scalars $t_1, \dots, t_k \in \mathcal{B}$ such that*

(i) *$y_\ell = t_i$ for each $\ell \in J_i$ and $i \in \{1, \dots, k\}$;*
(ii) *for some sufficiently small $\epsilon > 0$, the set*

$$B^\epsilon_y(J_1, \dots, J_k) := \bigtimes_{i=1}^{k} \big\{ z^{J_i} \mid z^{J_i}_\ell = s_i,\ \forall \ell \in J_i,\ t_i - \epsilon < s_i < t_i + \epsilon \big\} \bigtimes_{\ell \notin J} \{y_\ell\}, \tag{8}$$

where $J = J_1 \cup \ldots \cup J_k$ and z^{J_i} denotes a (sub)vector with components indexed by J_i, is contained in C.

Proof. Assume C is not a point. Given $y \in C$, consider $\mathrm{Lin}(C - y)$. We will show that it has a nonnegative orthogonal basis. First, observe that the max–min segments connecting y with other points of C give rise to some nontrivial nonnegative vectors in $\mathrm{Lin}(C - y)$. This follows from the description of max–min segments given in Theorem 2.1.

Let us first show that the largest support of nonnegative vectors in $\mathrm{Lin}(C - y)$ is equal to the largest support among all vectors of $\mathrm{Lin}(C - y)$. By contradiction, assume that the largest support of a nonnegative vector is a proper subset $M \subset \{1, \dots, d\}$, achieved by a vector $f \in \mathrm{Lin}(C - y)$, and that there is a vector $g \in \mathrm{Lin}(C - y)$ with some negative coordinates and support $\mathrm{supp}\, g \nsubseteq M$. At least one of the vectors g and $g' := -g$ has some positive coordinates, whose indices do not belong to M. As C is max–min convex, we have $f \oplus g \in \mathrm{Lin}(C - y)$ and $f \oplus g' \in \mathrm{Lin}(C - y)$, and then at least one of the vectors $f \oplus g$ and $f \oplus g'$ is a nonnegative vector whose support strictly includes M, a contradiction.

Thus we can assume that $\mathrm{Lin}(C - y)$ contains nonnegative vectors with the largest support, hence by Lemma 3.4 the linear span of its nonnegative part, the convex cone $K := \mathrm{Lin}(C - y) \cap \mathbb{R}^d_+$, has the same dimension k as $\mathrm{Lin}(C - y)$. As K is closed, by the usual Minkowski theorem [25, Corollary 18.5.1] it can be represented as the set of positive linear combinations of its extremal rays (recall that $w \in K$ is called extremal if $u + v = w$ and $u, v \in K$ imply that u and v are proportional with w), which generate the whole $\mathrm{Lin}(C - y)$. We will prove that the extremal rays of K have pairwise disjoint supports.

By contradiction, let u and v be extremal rays of K, not proportional with each other, with $L := \mathrm{supp}\, u \cap \mathrm{supp}\, v \neq \emptyset$. We can assume that $\mathrm{supp}(u) = \mathrm{supp}(v) = L$ or that $\mathrm{supp}\, u \neq \mathrm{supp}\, v$ and $(\mathrm{supp}\, v) \backslash L \neq \emptyset$. Take $\lambda > 0$ and $\mu > 0$ such that $\lambda v_i > 2u_i$ and $u_i > \mu v_i$ for all $i \in L$. Hence we have $(\lambda v - u)_i > \mu v_i$ for all $i \in L$. The vector $w = \mu v \oplus (\lambda v - u)$ is nonnegative, below λv, and not proportional to v: in the case when $\mathrm{supp}(u) = \mathrm{supp}(v) = L$ it is equal to $\lambda v - u$, and in the other case we have $w_i = \mu v_i$ for $i \in (\mathrm{supp}\, v) \backslash L$ and $w_i > \mu v_i$ for $i \in L$. We see that w and $\lambda v - w$ are in K not being proportional to v, which contradicts that v is extremal.

Thus we have proved that $\mathrm{Lin}(C - y)$ has an orthonormal basis consisting of nonnegative vectors whose supports are pairwise disjoint. The vectors of this basis (no more than d) also generate the cone $K = \mathrm{Lin}(C - y) \cap \mathbb{R}^d_+$ being the extremals of K. Now we use that C is max–min convex and investigate the properties of y and the vectors of that

basis. For a vector u from the basis, there exists $\epsilon > 0$ such that $y + \epsilon u$ belongs to C. From Lemma 2.4 we see that unless all components of u are equal to each other and the corresponding components of y are equal to each other, we can find a vector $u' \leq u$ such that $y + u' \in [y, y + \epsilon u]_{\oplus} \subseteq C$, where u' is non-proportional to u. Then $u = (u - u') + u'$ is not an extremal of K, a contradiction.

So we obtained that for each u in the nonnegative orthogonal basis of $\mathrm{Lin}(C - y)$, all nonzero components of u are equal to each other, and the corresponding coordinates of y are equal to each other. Since the supports of the base vectors are pairwise disjoint, this implies that C contains a set of the form (8). More precisely, if the base vectors are denoted by $g^1, \ldots, g^k$ then we take $J_i = \mathrm{supp}(g^i)$ for $i = 1, \ldots, k$. Since we can find ϵ such that $y + \epsilon g^i \in C$ for all i, we obtain that $B_y^{\epsilon}(J_1, \ldots, J_k) \subseteq C$. □

Definition 3.6. A set of the form (8) will be called a (k-dimensional, open) quasibox.

Lemma 3.7. *A k-dimensional quasibox is a k-dimensional polytrope.*

Proof. A quasibox is obviously conventionally convex, so we only need to show that it is max–min convex. Let $B := B_y^{\epsilon}(J_1, \ldots, J_k)$ be a quasibox defined by (8), $z, \zeta \in B$ and $\tau \in [z, \zeta]_{\oplus}$. Then $\tau_\ell = y_\ell$, $\ell \notin J$ and $t_i - \epsilon < \tau_\ell < t_i + \epsilon$ if $\ell \in J_i$ due to the inequality

$$\min(x, y) \leq \max\big(\min(\alpha, x), \min(\beta, y)\big) \leq \max(x, y),$$

which is true for all $x, y \in \mathcal{B}$ and $\alpha, \beta \in \mathcal{B}$ such that $\max(\alpha, \beta) = 1$, and which can be easily checked by looking at all possible orders on $\{x, y, \alpha, \beta\}$. □

Remark 3.8. As Fig. 2 shows, there are many polytropes that are not quasiboxes.

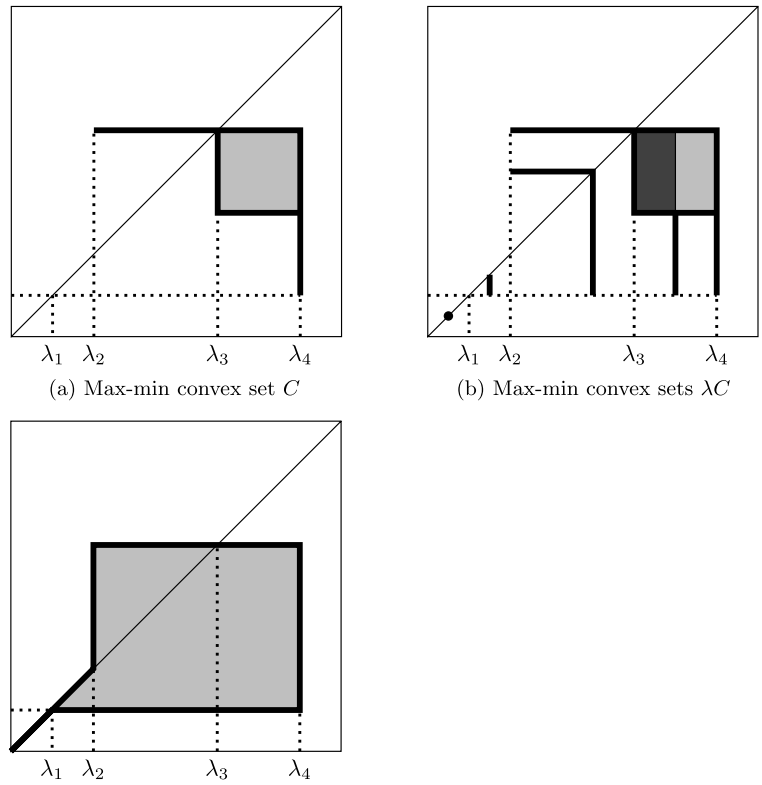
Corollary 3.9. *The dimension* $\dim(C)$ *of a max–min convex set $C \subseteq \mathcal{B}^d$ is equal to the greatest number k such that C contains a k-dimensional open quasibox.*

Proof. Let $k = \dim(C)$. Then C contains a k-dimensional (relatively) open polytrope and, by Theorem 3.5, it also contains a k-dimensional open quasibox. A quasibox of greater dimension cannot be contained in C, since any quasibox is a polytrope. □

We now investigate the change of dimension under homogenization. In fact, unlike in the usual convexity or max–plus convexity, the set $\lambda \otimes C := \{\lambda \otimes x \mid x \in C\}$ does not look like a homothety of C, since the multiplication is not invertible. In particular, the dimension can also change. Consider the following example displayed on Fig. 3. Let λ decrease from 1 to 0. Before λ reaches λ_4 we have $\lambda \otimes C = C$. As λ decreases from λ_4 to λ_3, we see that $\lambda \otimes C$ is steadily "swept" towards the origin, but it still has a two-dimensional region so that $\dim(\lambda \otimes C) = 2$. The set $\lambda \otimes C$ becomes one-dimensional at $\lambda = \lambda_3$, consisting of two segments, one horizontal and one vertical. At $\lambda = \lambda_2$ the set $\lambda \otimes C$ becomes a single vertical segment, and at $\lambda = \lambda_1$ it shrinks to a point. The point moves towards the origin along the diagonal as λ gets closer to 0. The last subfigure displays the convex hull $\mathrm{conv}_{\oplus}(0, C)$, which is the least subsemimodule containing C, and also the projection of $V_C \subseteq \mathcal{B}^3$ onto the first $k = 2$ coordinates.

Lemma 3.10. *Let $C \subseteq \mathcal{B}^d$ be a max–min convex set. Then* $\dim(\lambda \otimes C) \leq \dim(C)$ *for all* $0 \leq \lambda \leq 1$.

Proof. Let $k = \dim(\lambda \otimes C)$. Then for some $y \in C$ that satisfies condition (i) of Theorem 3.5, for some numbers $t_1, \ldots, t_k$, and some subsets $J_1, \ldots, J_k$ of $\{1, \ldots, d\}$, the set $\lambda \otimes C$ contains a quasibox $B_y^{\epsilon}(J_1, \ldots, J_k)$ defined by (8). As $B_y^{\epsilon}(J_1, \ldots, J_k) \subseteq \lambda \otimes C$ we obtain that $\lambda \geq t_i + \epsilon$ for all $i = 1, \ldots, k$. Now let $y = \lambda \otimes u$ for some $u \in C$ and consider any point $z \in B_y^{\epsilon}(J_1, \ldots, J_k)$ with $z \geq y$. Since $\lambda \geq t_i + \epsilon$ for all $i = 1, \ldots, k$, we have $u_j = y_j$ for all $j \in J_1 \cup \ldots \cup J_k$, and hence the components $(u \oplus z)_j$ with $j \in J_1 \cup \ldots \cup J_k$ are equal to those of $y \oplus z = z$. The components $(u \oplus z)_j$ with $j \notin J_1 \cup \ldots \cup J_k$ are equal to those of u, due to the fact that in this case the components of z coincide to the components of y and due to the formula $\max(a, \min(a, b)) = a$, which is true for all $a, b \in \mathcal{B}$. So these components of u are independent of z. It follows that the points $u \oplus z$, for $z \geq y$ and $z \in B_y^{\epsilon}(J_1, \ldots, J_k)$, form a set which contains $B_x^{\epsilon/2}(J_1, \ldots, J_k)$, where $x_\ell = t_i + \epsilon/2$ for each $\ell \in J_i$ and $i \in \{1, \ldots, k\}$ and $x_\ell = u_\ell$ for $\ell \notin$



(a) Max-min convex set C (b) Max-min convex sets λC

(c) Max-min convex hull of C and $\mathbf{0}$

Fig. 3. The behavior of $\lambda \otimes C$.

$J_1 \cup \ldots \cup J_k$. However, $z = \lambda \otimes v$ for some $v \in C$ and hence $u \oplus z = u \oplus \lambda v \in C$. It follows that $B_x^{\epsilon/2}(J_1, \ldots, J_k) \subseteq C$ and $\dim C \geq k$. The proof is complete. □

Theorem 3.11. *Let $C \subseteq \mathcal{B}^d$ be a max–min convex set and let $V_C \subseteq \mathcal{B}^{d+1}$ be the homogenization of C. Then* $\dim(V_C) = \dim(C) + 1$.

Proof. We first prove that $\dim(V_C) \leq \dim(C) + 1$. Suppose by contradiction that $\dim(V_C) > \dim(C) + 1$. Then V_C contains a polytrope of dimension at least $\dim(C) + 2$. For some μ, the section of V_C by $\{u \in \mathcal{B}^{d+1} \mid u_{d+1} = \mu\}$ has a nontrivial intersection with that polytrope, and that intersection is a polytrope of dimension at least $\dim(C) + 1$. But the section of V_C by $\{u \in \mathcal{B}^{d+1} \mid u_{d+1} = \mu\}$ is exactly $(\mu \otimes C, \mu)$, and the dimension of $\mu \otimes C$ does not exceed $\dim(C)$ by Lemma 3.10. This contradiction shows that $\dim(V_C) \leq \dim(C) + 1$.

We now prove that $\dim(V_C) \geq \dim(C) + 1$. For this, let $k = \dim(C)$ and let C contain a quasibox $B_y^\epsilon(J_1, \ldots, J_k)$ defined by (8) as in Theorem 3.5. Choosing a small enough ϵ we can assume that $t_i + \epsilon < 1$ for all $i = 1, \ldots, k$. Let J_{k+1} consist of the index $d+1$ and all indices of the components of y that are equal to 1. Choose ϵ such that $1 - 2\epsilon$ is greater than all $t_i + \epsilon$ and any coordinate of y not equal to 1, and set $t_{k+1} := 1 - \epsilon$. Define the components of $\tilde{y} \in \mathcal{B}^{d+1}$ by $\tilde{y}_\ell = t_i$ for each $\ell \in J_i$ and $i \in \{1, \ldots, k+1\}$, and $\tilde{y}_\ell = y_\ell$ otherwise. Then the homogenization of $B_y^\epsilon(J_1, \ldots, J_k)$, which is by definition the set

$$\left\{(\mu \otimes x, \mu) \,\middle|\, x \in B_y^\epsilon(J_1, \ldots, J_k),\ \mu \in \mathcal{B}\right\},$$

contains the quasibox $B_{\tilde{y}}^\epsilon(J_1, \ldots, J_{k+1})$. As this homogenization is contained in V_C, the dimension of V_C is at least $k+1$. □

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4. Dimension equals rank

In the remaining part of the paper, following the parallel with the tropical rank considered by Develin, Santos and Sturmfels [7] in the max–plus algebra, we investigate how our notion of dimension relates with the notion of strong regularity in max–min algebra. For $A \in \mathcal{B}(d, m+1)$, the ith column will be denoted by $A_{\bullet i}$.

Definition 4.1. A matrix $A \in \mathcal{B}(k, k+1)$ is called strongly regular if there exists an index $j : 1 \le j \le k+1$, a bijection $\pi: \{1, \dots, k\} \to \{1, \dots, k+1\} \backslash \{j\}$ and coefficients $\lambda_1, \dots, \lambda_{j-1}, \lambda_{j+1}, \dots, \lambda_{k+1} \in \mathcal{B}$ such that in the matrix

$$A[\lambda] := (\lambda_1 \otimes A_{\bullet 1}, \dots, \lambda_{j-1} \otimes A_{\bullet j-1}, A_{\bullet j}, \lambda_{j+1} \otimes A_{\bullet j+1}, \dots, \lambda_{k+1} \otimes A_{\bullet k+1}) \tag{9}$$

the maximum in each row $i \in \{1, \dots, k\}$ equals $\lambda_{\pi(i)}$ and is attained only by the term $\pi(i) \in \{1, \dots, k+1\} \backslash \{j\}$. We will say that the coefficients λ_i and bijection π certify the strong regularity of A.

Remark 4.2. In Definition 4.1, the coefficients λ_i are all nonzero. Furthermore, by slightly decreasing these coefficients we can assume that they are all different and distinct from 1 and the entries of A.

For $A \in \mathcal{B}(d, m+1)$, let $\mathrm{conv}_{\oplus}(A)$ denote the max–min convex hull of the columns of A.

Definition 4.3. Let $A \in \mathcal{B}(d, m+1)$. We call the max–min rank and denote by $\mathrm{rank}(A)$ the largest integer k such that A contains a strongly regular $k \times (k+1)$ submatrix.

Remark 4.4. Note that the definition of strong regularity is introduced here for $k \times (k+1)$ rectangular matrices. A more usual "square" version of this definition will appear in the next section, and we will show that it is equivalent to the one studied in [2,12].

The following theorem can be considered as one of the main result of this paper.

Theorem 4.5. *Let* $A = (a_{ij}) \in \mathcal{B}(d, m+1)$. *Then* $\dim(\mathrm{conv}_{\oplus}(A)) = \mathrm{rank}(A)$.

Proof. We first suppose that A contains a strongly regular $k \times (k+1)$ submatrix, and show that $\dim(\mathrm{conv}_{\oplus}(A))$ is at least k. Without loss of generality we assume that this strongly regular submatrix is extracted from the first k rows and $k+1$ columns of A, and that $j = k+1$ in (9). Let A' be the submatrix of A extracted from the first $k+1$ columns. Since $\mathrm{conv}_{\oplus}(A') \subseteq \mathrm{conv}_{\oplus}(A)$, we have $\dim(\mathrm{conv}_{\oplus}(A')) \le \dim(\mathrm{conv}_{\oplus}(A))$, so it suffices to prove that $\dim(\mathrm{conv}_{\oplus}(A')) \ge k$. For each column $i : 1 \le i \le k$ there is a row where the maximum in $A'[\lambda]$ (9) is attained only by the ith term. We assume that the $\lambda_1, \dots, \lambda_k$ are all different and distinct from the entries of A'. With this, let J_i, for $1 \le i \le k$, be the set of rows of $A'[\lambda]$ where the only maximum is attained by the ith column and equals λ_i. Let $J = J_1 \cup \ldots \cup J_k$ and for $\ell \notin J$, if such indices exist, let α_ℓ be the maximum of the ℓth row of $A'[\lambda]$. Observe that this maximum is equal to an entry of A. For each $i: 1 \le i \le k$, set

$$\begin{aligned} m_i &:= \max\bigl\{\max\{\alpha_\ell : \ell \notin J, \alpha_\ell < \lambda_i\}, \max\{\lambda_s \otimes a_{\ell s} : \ell \in J_i, s \ne i\}\bigr\}, \\ \kappa &:= \min_{1 \le i \le k} (\lambda_i - m_i). \end{aligned} \tag{10}$$

If the set $\{\alpha_\ell : \ell \notin J, \alpha_\ell < \lambda_i\}$ is empty, then we assume that its maximum is zero. Observe that $\max\{\lambda_s \otimes a_{\ell s} : \ell \in J_i, s \ne i\} < \lambda_i$ by the definition of J_i, hence $m_i < \lambda_i$. For any vector $\epsilon = (\epsilon_1, \dots, \epsilon_k)$ such that $0 \le \epsilon_i < \kappa$ for all $i = 1, \dots, k$, define the vector-function $y(\epsilon)$:

$$y_\ell(\epsilon) = \begin{cases} \lambda_i - \epsilon_i, & \text{if } \ell \in J_i \text{ and } 1 \le i \le k, \\ \alpha_\ell, & \text{if } \ell \notin J. \end{cases} \tag{11}$$

Using the definition of κ and the fact that $y(0)$ is the max–min linear combination of the columns of A' with coefficients $\lambda_1, \dots, \lambda_k, 1$, we obtain that $y(\epsilon)$ is the max–min linear combination of the columns of A' with coefficients $\lambda_1 - \epsilon_1, \dots, \lambda_k - \epsilon_k, 1$ for any $\epsilon: 0 \le \epsilon_i < \kappa$ where $i = 1, \dots, k$. Denote $\bar{y} := y(\kappa/2, \dots, \kappa/2)$. Then the quasibox

$$B_{\bar{y}}^{\kappa/2}(J_1,\dots,J_k)=\{y(\epsilon):0<\epsilon_i<\kappa, i=1,\dots,k\},$$

is contained in $\mathrm{conv}_\oplus(A')$ and in $\mathrm{conv}_\oplus(A)$. Since $B_{\bar{y}}^{\kappa/2}(J_1,\dots,J_k)$ is a k-dimensional quasibox, this shows that $\dim(\mathrm{conv}_\oplus(A))\geq k$.

Next, we have to show that given $k=\dim(\mathrm{conv}_\oplus(A))$, there is a strongly regular $k\times(k+1)$ submatrix. By Theorem 3.5, $\mathrm{conv}_\oplus(A)$ contains a k-dimensional quasibox (8). Then taking an element from each J_i, consider the submatrix of A extracted from the corresponding k rows. Denote it by $A''\in\mathcal{B}(k,m+1)$. Assume that the rows are $\{1,\dots,k\}$. We will show that this submatrix contains a strongly regular $k\times(k+1)$ submatrix. Indeed, being equal to the projection of $\mathrm{conv}_\oplus(A)$ onto the first k coordinates $((z_1,\dots,z_k,\dots,z_d)\mapsto(z_1,\dots,z_k))$, $\mathrm{conv}_\oplus(A'')$ contains the projection of the k-dimensional quasibox mentioned above, and this is a usual k-dimensional box. This box contains a point $x=(x_1,\dots,x_k)$ whose all coordinates are different, and distinct from the coefficients of A''. Since $x\in\mathrm{conv}_\oplus(A'')$, possibly permuting the columns of A'' we obtain that $(x_1,\dots,x_k)$ are the row maxima of the matrix

$$(\mu_1\otimes A''_{\bullet 1},\dots,\mu_m\otimes A''_{\bullet m},A''_{\bullet m+1}).$$

Since x_i are not equal to any entries of A'' and are all different, we obtain that there is a k-element set $N\subseteq\{1,\dots,m\}$ and a bijection $\pi:\{1,\dots,k\}\to N$ such that $x_i=\mu_{\pi(i)}$ for all $i=1,\dots,k$, with all terms except for $\pi(i)$ being less than x_i. This implies that the $k\times(k+1)$ submatrix extracted from rows $1,\dots,k$ and columns $\pi(1),\dots,\pi(k),m+1$ is strongly regular. □

Corollary 4.6. *Let $m\geq d$ and $A\in\mathcal{B}(d,m+1)$. Then $\mathrm{conv}_\oplus(A)$ has nonempty interior if and only if A contains a $d\times(d+1)$ strongly regular submatrix.*

Proof. The corollary follows from Theorem 4.5 and Theorem 3.5. □

5. Strong regularity: the link to max–min algebra

In this section we establish a close relation between our notion of strong regularity and the one usually studied in max–min algebra [2–4,12,14]. With this in mind, let us define the notion of strong regularity for square matrices, as a slight variation of Definition 4.1.

Definition 5.1. A matrix $A\in\mathcal{B}(k,k)$ is called strongly regular if there exists a bijection $\pi:\{1,\dots,k\}\to\{1,\dots,k\}$ and coefficients $\lambda_1,\dots,\lambda_k\in\mathcal{B}$ such that in the matrix

$$A[\lambda]:=(\lambda_1\otimes A_{\bullet 1},\dots,\lambda_k\otimes A_{\bullet k}) \tag{12}$$

the maximum in each row $i\in\{1,\dots,k\}$ equals $\lambda_{\pi(i)}$ and is attained only by the term $\pi(i)\in\{1,\dots,k\}$. We will say that the coefficients λ_i and bijection π certify the strong regularity of A.

Remark 5.2. As in Definition 4.1, the coefficients $\lambda_1,\dots,\lambda_k$ can be assumed to be different from each other, distinct from the entries of A, 0 and 1.

We will show later that this notion coincides with the one studied in max–min algebra. The proof of the following statement is omitted.

Lemma 5.3. *$A\in\mathcal{B}(k,k)$ is strongly regular in the sense of Definition 5.1 if and only if $[A\ \mathbf{0}]$, where $\mathbf{0}$ is the k component column of all zeros, is strongly regular in the sense of Definition 4.1.*

For $A\in\mathcal{B}(m,n)$ define $\hat{A}\in\mathcal{B}(m+1,n)$ by

$$\hat{A}:=\begin{pmatrix}A\\ \mathbf{1}\end{pmatrix}, \tag{13}$$

where $\mathbf{1}$ denotes the n-component row of all ones. Note that if $A\in\mathcal{B}(k,k+1)$ then $\hat{A}\in\mathcal{B}(k+1,k+1)$ is square.

The mapping $A \to \hat{A}$ can be seen as a special case of homogenization. Indeed, if we set $C := \mathrm{conv}_{\oplus}(A)$, then we have $V_C = \mathrm{span}_{\oplus}(\hat{A})$. This has the following immediate corollary.

Corollary 5.4. *$A \in \mathcal{B}(k, k+1)$ is strongly regular (in the sense of Definition 4.1) if and only if $\hat{A} \in \mathcal{B}(k+1, k+1)$ is strongly regular (in the sense of Definition 5.1).*

Proof. By Theorem 4.5, $A \in \mathcal{B}(k, k+1)$ is strongly regular if and only if $\dim(\mathrm{conv}_{\oplus}(A)) = k$, and $\hat{A}$ (that is, $[\hat{A}\ \mathbf{0}]$) is strongly regular if and only if $\dim(\mathrm{conv}_{\oplus}([\hat{A}\ \mathbf{0}])) = \dim(\mathrm{span}_{\oplus}(\hat{A})) = k+1$. Theorem 3.11 implies that these statements are equivalent. □

Definition 5.5. A matrix $A \in \mathcal{B}(m, n)$ is called trapezoidal if the following condition holds:

$$a_{ii} > \bigoplus_{\ell=1}^{i} \bigoplus_{t=\ell+1}^{n} a_{\ell t} \quad \forall i = 1, \ldots, m. \tag{14}$$

We now show that for $A \in \mathcal{B}(k, k)$ our notion of strong regularity is equivalent to the trapezoidal property, and hence it coincides with the strong regularity in max–min algebra introduced in [2].

Remark 5.6. In fact, the equivalence between Definition 5.1 and Definition 5.5 (for square matrices) is known in max–min algebra. It follows, for instance, from Butkovič and Szabo [3, Theorem 2]. However, we prefer to write the proofs of Proposition 5.7 and Theorem 5.8 below for the sake of completeness and convenience of the reader.

Proposition 5.7. *$A \in \mathcal{B}(k, k+1)$ (or $A \in \mathcal{B}(k, k)$) is strongly regular if and only if there exist permutation matrices P and Q such that PAQ is trapezoidal.*

Proof. We can assume that $A \in \mathcal{B}(k, k+1)$, since the other case is reduced to that case by adjoining to $A \in \mathcal{B}(k, k)$ a zero column.

For the "if" part, we can assume that A is trapezoidal. For every row index i, we let $\lambda_i := \alpha_i + \epsilon_i$, where α_i equal the right-hand side of (14) and ϵ_i are such that $\epsilon_1 < \ldots < \epsilon_k$ and $\alpha_i + \epsilon_i < a_{ii}$. Observe that $\alpha_1 \le \ldots \le \alpha_k$, and hence $\lambda_1 < \ldots < \lambda_k$. For $t > i$ we obtain $\lambda_i \otimes a_{ii} > \lambda_t \otimes a_{it}$ since $a_{ii} > \lambda_i > a_{it}$ by construction. For $t < i$ we obtain $\lambda_i \otimes a_{ii} > \lambda_t \otimes a_{it}$ since $\lambda_i \otimes a_{ii} = \lambda_i > \lambda_t$. Thus the coefficients $\lambda_1, \ldots, \lambda_k$ and the identity permutation certify that A is strongly regular.

The "only if" part: Let A be strongly regular. Applying row and column permutations if necessary (which corresponds to taking PAQ as in the claim) we can assume that the strong regularity is certified by the identity permutation $\pi : \{1, \ldots, k\} \to \{1, \ldots, k\}$ and $\lambda_1, \ldots, \lambda_k$, which are distinct from the entries of A and satisfy $0 < \lambda_1 < \lambda_2 < \ldots < \lambda_k < 1$. In particular, we have $\lambda_i < a_{ii}$ for all i. For each i, we then have $\lambda_i > a_{it}$ for all $t > i$, since $\lambda_i > \lambda_t \otimes a_{it}$ and $\lambda_i < \lambda_t$. Hence we also have $a_{ii} > \lambda_i > \lambda_\ell > a_{\ell t}$ for all $\ell < i$ and $\ell < t$. Thus the trapezoidal property follows. □

Recall that by Corollary 5.4, $A \in \mathcal{B}(k, k+1)$ is strongly regular if and only if $\hat{A}$ is strongly regular. In fact, this is also easy to see by means of the trapezoidal property. We now conclude with the following observation, which is similar to [2, Theorem 3].

Theorem 5.8. *Let $A \in \mathcal{B}(d, k+1)$ with $d \ge k$. Then the following are equivalent:*

(i) *there exists a vector $b \in \mathcal{B}^d$ such that the system $A \otimes x = b$ has a positive solution, which is also the unique solution that satisfies $\bigoplus_{i=1}^{k+1} x_i = 1$;*
(ii) *there exists a vector $\hat{b} \in \mathcal{B}^{d+1}$ such that the system $\hat{A} \otimes x = \hat{b}$ has a positive solution, which is also the unique solution to that system;*
(iii) *$\hat{A}$ contains a $(k+1) \times (k+1)$ strongly regular submatrix;*
(iv) *A contains a $k \times (k+1)$ strongly regular submatrix.*

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Proof. (i) $\Rightarrow$ (ii): Take $\hat{b} = (b1)^T$. Observe that $\bigoplus_{i=1}^{k+1} x_i = 1$ is satisfied for any solution of $\hat{A} \otimes x = \hat{b}$, and then (i) shows that it is unique.

(ii) $\Rightarrow$ (iii): For this we can exploit, e.g., [2, Theorem 3].

(iii) $\Leftrightarrow$ (iv): Equivalence between these statements follows from Theorem 4.5 and Theorem 3.11. Alternatively, we can use the existence of permutation matrices P and Q such that A or $\hat{A}$ have a trapezoidal submatrix (Proposition 5.7).

(iv) $\Rightarrow$ (i): Let $\lambda_1, \dots, \lambda_k$ and the identity permutation certify the strong regularity of the $k \times (k+1)$ submatrix extracted from the first k rows of A. Assume that the values of $\lambda_1, \dots, \lambda_k$ are all different and distinct from the entries of A, as well as 0 and 1. Define the components of b to be the maxima in the rows of $A[\lambda]$, then the first k components of b are equal to $\lambda_1, \dots, \lambda_k$.

Let A' be the strongly regular $k \times (k+1)$ submatrix extracted from the first k rows of A. The corresponding subvector of b is $b' = (\lambda_1, \dots, \lambda_k)$, and $x = (\lambda_1, \dots, \lambda_k, 1)$ is a solution to $A' \otimes x = b'$ and $A \otimes x = b$. As the entries of b' are all different from the entries of A', any other solution y with a component equal to 1 contains all these components $\lambda_1, \dots, \lambda_k, 1$, possibly permuted. However, then $x \oplus y$ is also a solution where some of the components λ_i are lost, since we chose them to be all different. This is a contradiction, which shows that $A' \otimes x = b'$ is uniquely solvable with $(\lambda_1, \dots, \lambda_k, 1)$ (requiring one 1 component), which implies the same for $A \otimes x = b$. □

In particular, $A \in \mathcal{B}(d, k+1)$ contains a strongly regular $k \times (k+1)$ submatrix if and only if there exist permutation matrices P and Q such that $P\hat{A}Q$ contains a trapezoidal $(k+1) \times (k+1)$ submatrix. To find such a submatrix, that is, to verify that the equivalent conditions of Theorem 5.8 hold, we can apply the strongly polynomial algorithm of [2].

We conclude with two sufficient conditions for a matrix to have low rank.

Proposition 5.9. *If $A \in \mathcal{B}(d, m+1)$ is such that for any $\lambda_1, \dots, \lambda_{m+1} \in \mathcal{B}$ with $\lambda_j = 1$ for some $j \in \{1, \dots, m+1\}$ there exist k columns such that the maximum in every row of $A[\lambda]$ (9) is attained in one of these k columns, then* $\dim(\mathrm{conv}_\oplus(A)) \le k$.

Proof. Observe that there is no regular $s \times (s+1)$ submatrix with $s > k$, and apply Theorem 4.5. □

Corollary 5.10 *(Sufficient condition for* $\dim(\mathrm{conv}_\oplus(A)) \le 2$*). Let $A = (a_{ij}) \in \mathcal{B}(d, m)$ satisfy*

$$\max_{1 \le k \le d} a_{ki} \le \min_{1 \le k \le n} a_{k,i+1}, \quad \forall i: 1 \le i \le m. \tag{15}$$

Then $\dim(\mathrm{conv}_\oplus(A)) \le 2$.

Proof. Let x be a max–min convex combination of the columns of A, with coefficients $\lambda_1, \dots, \lambda_m$ such that $\lambda_j = 1$ for some $j \in \{1, \dots, m\}$. Consider the matrix $A[\lambda]$ (9). We will show that there are two columns where all row maxima of (9) are attained. For this, let I be the set of column indices i where $\lambda_i > \min_k a_{ki}$, and let J be the complement of this set.

Considering the submatrix of $A[\lambda]$ (9) extracted from the columns in I we see that all row maxima are attained in the column with the biggest index. All coefficients of a column in J are equal to each other. Therefore, in the submatrix of $A[\lambda]$ extracted from the columns in J there is also a column where all row maxima are attained. This column and the column with the biggest index in I are the two columns where all row maxima of $A[\lambda]$ are attained (possibly, there may be other such columns, but they are redundant). By Proposition 5.9 this shows that $\dim(\mathrm{conv}_\oplus(A)) \le 2$. □

Example 5.11. The max–min polytope $\mathrm{conv}_\oplus(A) \subseteq \mathcal{B}^3$ generated by the matrix

$$A = \begin{pmatrix} .01 & .02 & .03 & .04 \\ .05 & .06 & .07 & .08 \\ .09 & .10 & .11 & .12 \end{pmatrix} \tag{16}$$

has non-empty interior, meaning that $\dim(\mathrm{conv}_\oplus(A)) = 3$. To see that A is strongly regular, choose $j = 1$ and $\lambda_2 = .10, \lambda_3 = .07, \lambda_4 = .04$. A trapezoidal form of A can be obtained by reversing the order of columns:

$$A = \begin{pmatrix} .04 & .03 & .02 & .01 \\ .08 & .07 & .06 & .05 \\ .12 & .11 & .10 & .09 \end{pmatrix}. \tag{17}$$

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Example 5.12. The max–min polytope $\mathrm{conv}_{\oplus}(A) \subseteq \mathcal{B}^3$ generated by the matrix

$$A = \begin{pmatrix} .01 & .04 & .07 & .10 \\ .02 & .05 & .08 & .11 \\ .03 & .06 & .09 & .12 \end{pmatrix} \tag{18}$$

has $\dim(\mathrm{conv}_{\oplus}(A)) = 2$. The inequality $\dim(\mathrm{conv}_{\oplus}(A)) \leq 2$ follows from Corollary 5.10, as condition (15) is satisfied for all $i: 1 \leq i \leq 3$. A regular 2×3 submatrix can be extracted from rows 1 and 3 and columns $1, 3, 4$: set $j = 1$, $\lambda_3 = .09$ and $\lambda_4 = .08$.

6. Concluding remarks

In this paper we introduce the notion of dimension of a max–min convex set and show that it is equivalent to a notion of matrix rank based on the strong regularity for the matrices in max–min algebra [2].

Since the max–min convex combinations also appear as mixtures of possibilistic measures in the framework of Dubois and Prade's possibility theory [8], this paper can be seen as a contribution towards the geometry of such mixtures. More generally, it might be interesting to look for more applications of max–min convexity in the possibility theory.

From the theoretical perspective, we developed a max–min analogue of the geometric interpretation of tropical rank. In max–plus (tropical) convexity, such interpretation also involves the notion of tropical cellular decomposition [6], whose max–min analogue is still unknown.

Furthermore, it is plausible that the results of this paper might be generalizable to the setting of [15], and also to the L-convexities and biconvexities of [24].

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