



# Generalizations of bounds on the index of convergence to weighted digraphs



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## ABSTRACT

We study sequences of optimal walks of a growing length in weighted digraphs, or equivalently, sequences of entries of max-algebraic matrix powers with growing exponents. It is known that these sequences are eventually periodic when the digraphs are strongly connected. The transient of such periodicity depends, in general, both on the size of digraph and on the magnitude of the weights. In this paper, we show that some bounds on the indices of periodicity of (unweighted) digraphs, such as the bounds of Wielandt, Dulmage–Mendelsohn, Schwarz, Kim and Gregory–Kirkland–Pullman, apply to the weights of optimal walks when one of their ends is a critical node.

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## 1. Introduction

We show that six known bounds for the index of convergence (transient of periodicity) of an unweighted digraph also apply to weighted digraphs, namely, to transients of rows and columns with critical indices.

The origin of the first of these known bounds lies in Wielandt's well-known paper [25] where an upper bound for the transient of a primitive nonnegative matrix was asserted without proof.<sup>1</sup> Dulmage and Mendelsohn [10] provided a proof of this result by interpreting it in terms of digraphs and they sharpened the result by using as additional information in the hypotheses the length of the smallest cycle of the digraph.<sup>2</sup> Schwarz [18] generalized Wielandt's result to apply to all strongly connected digraphs by using Wielandt's bound for the cyclicity classes of the digraph, see also Shao and Li [23]. Kim's [14] bound encompasses the first three and can be proved using Dulmage and Mendelsohn's bound in the cyclicity classes.

We also generalize another bound by Kim [14], and a bound by Gregory–Kirkland–Pullman [12] which depend on Boolean rank.

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<sup>1</sup> Wielandt's proof was published later in [17].

<sup>2</sup> Denardo [9] later rediscovered their result.

The six bounds mentioned above are stated in [Theorems 2.11](#) and [2.14](#) after the requisite definitions. Our generalizations to weighted digraphs are stated in [Main Theorems 1](#) and [2](#), and subsequently proved in [Sections 3–7](#).

We exploit the natural connection between weighted digraphs and nonnegative matrices in the max (-times) algebra just as the bounds that we take for our starting points connect unweighted digraphs and Boolean matrices.

## 2. Preliminaries and statement of results

### 2.1. Digraphs, walks, and transients

Let us start with some definitions.

**Definition 2.1** (*Digraphs*). A digraph is a pair  $\mathcal{G} = (N, E)$  where  $N$  is a set of nodes and  $E \subseteq N \times N$  is a set of edges.

**Definition 2.2** (*Walks and Cycles*). A walk in  $\mathcal{G}$  is a finite sequence  $W = (i_0, i_1, \dots, i_t)$  of nodes such that each pair  $(i_0, i_1)$ ,  $(i_1, i_2)$ ,  $\dots$ ,  $(i_{t-1}, i_t)$  is an edge of  $\mathcal{G}$  (that is, belongs to  $E$ ). Here, the nodes  $i_0$ , resp.  $i_t$  are the *start* resp. the *end* nodes of the walk.

The number  $t$  is the *length* of the walk, and we denote it by  $\ell(W)$ .

When  $i_0 = i_t$ , the walk is *closed*. If, in a closed walk, none of the nodes except for the start and the end appear more than once, the walk is called a *cycle*. If no node appears more than once, then the walk is called a *path*. A walk is *empty* if its length is 0.

To a digraph  $\mathcal{G} = (N, E)$  with  $N = \{1, \dots, n\}$ , we can associate a Boolean matrix  $A = (a_{i,j}) \in \mathbb{B}^{n \times n}$  defined by

$$a_{i,j} = \begin{cases} 0 & \text{if } (i, j) \notin E \\ 1 & \text{if } (i, j) \in E. \end{cases} \quad (2.1)$$

Conversely, one can associate a digraph to every square Boolean matrix. The connectivity in  $\mathcal{G}$  is closely related to the Boolean matrix powers of  $A$ . By the *Boolean algebra* we mean the set  $\mathbb{B} = \{0, 1\}$  equipped with the logical operations of conjunction  $a \cdot b$  and disjunction  $a \oplus b = \max(a, b)$ , for  $a, b \in \mathbb{B}$ . The Boolean multiplication of two matrices  $A \in \mathbb{B}^{m \times n}$  and  $B \in \mathbb{B}^{n \times q}$  is defined by  $(A \otimes B)_{i,j} := \bigoplus_{k=1}^n (a_{i,k} \cdot b_{k,j})$ , and then we also have Boolean matrix powers  $A^{\otimes t} := \underbrace{A \otimes \dots \otimes A}_{t \text{ times}}$ . The  $(i, j)$ th

entry of  $A^{\otimes t}$  is denoted by  $a_{i,j}^{(t)}$ .

The relation between Boolean powers of  $A$  and connectivity in  $\mathcal{G}$  is based on the following fact:  $a_{i,j}^{(t)} = 1$  if and only if  $\mathcal{G}$  contains a walk of length  $t$  from  $i$  to  $j$ .

Let  $\mathcal{G}$  be a digraph with associated matrix  $A \in \mathbb{B}^{n \times n}$ . The sequence of Boolean matrix powers  $A^{\otimes t}$  is eventually periodic, that is, there exists a positive  $p$  such that

$$A^{\otimes(t+p)} = A^{\otimes t} \quad (2.2)$$

for all  $t$  large enough.

**Definition 2.3** (*Eventual Period*). Each  $p$  satisfying (2.2) is called an *eventual period* of  $A$ .

The set of nonnegative  $t$  satisfying (2.2) is the same for all eventual periods  $p$ .

**Definition 2.4** (*Transient of Digraphs*). The least nonnegative number  $t$  satisfying (2.2) for some (and hence for all)  $p$  is called the *transient (of periodicity) of  $\mathcal{G}$* ; we denote it by  $T(\mathcal{G})$ .

See [4] for general introduction to the theory of digraphs and [15] for a survey on their transients. In the literature,  $T(\mathcal{G})$  is often called the *index of convergence*, or the *exponent* of  $\mathcal{G}$ .

**Definition 2.5** (*Powers of Digraphs*). The digraph associated with  $A^{\otimes t}$  is denoted by  $\mathcal{G}^t$ . Such graphs will be further referred to as the *powers* of  $\mathcal{G}$ .

**Definition 2.6** (*Cyclicity and Primitivity*). For a strongly connected digraph  $\mathcal{G}$ , its *cyclicity* is the greatest common divisor of the lengths of all cycles of  $\mathcal{G}$ . If  $d = 1$ , then  $\mathcal{G}$  is called *primitive*, otherwise it is called *imprimitive*.

The cyclicity  $d$  of  $\mathcal{G}$  can be equivalently defined as the least eventual period  $p$  in (2.2). If  $\mathcal{G}$  is strongly connected, then its cyclicity is the smallest eventual period of its sequence of powers. Let us recall the following basic observation from [4].

**Theorem 2.7** ([4, Theorem 3.4.5]). Let  $\mathcal{G}$  be a strongly connected graph with cyclicity  $d$ . For each  $k \geq 1$ , graph  $\mathcal{G}^k$  consists of  $\gcd(k, d)$  isolated strongly connected components, and every component has cyclicity  $d / \gcd(k, d)$ .

We have an important special case when  $k = d$ .

**Definition 2.8** (Cyclicity Classes). According to Theorem 2.7,  $\mathcal{G}^d$  has exactly  $d$  strongly connected components, each of cyclicity 1. The node sets of these components are called the *cyclicity classes* of  $\mathcal{G}$ .

In terms of walks, nodes  $i$  and  $j$  belong to the same cyclicity class if and only if there is a walk from  $i$  to  $j$  whose length is a multiple of  $d$ . More generally, for each  $i$  and  $j$  there is an integer  $s$  with  $0 \leq s \leq d - 1$  such that the length of every walk connecting  $i$  to  $j$  is congruent to  $s$  modulo  $d$ . This observation defines the *circuit of cyclicity classes*, being crucial for the description of  $\mathcal{G}^t$  in the periodic regime.

Let us introduce the following two definitions. The first of them is given in honor of the first paper on the subject by Wielandt [25].

**Definition 2.9** (Wielandt Number). By *Wielandt number* we mean the following function:

$$Wi(k) := \begin{cases} 0 & \text{if } k = 1, \\ (k - 1)^2 + 1 & \text{if } k > 1. \end{cases} \tag{2.3}$$

**Definition 2.10** (Girth). The *girth* of  $\mathcal{G}$ , denoted by  $g(\mathcal{G})$ , is the smallest length of a nonempty cycle in  $\mathcal{G}$ .

We will be interested in the following bounds on  $T(\mathcal{G})$ .

**Theorem 2.11.** Let  $\mathcal{G}$  be a strongly connected digraph with  $n$  nodes, cyclicity  $d$ , and girth  $g$ . The following upper bounds on the transient of  $\mathcal{G}$  hold:

- (i) (Wielandt [25,17]) If  $d = 1$ , then  $T(\mathcal{G}) \leq Wi(n)$ ;
- (ii) (Dulmage–Mendelsohn [10]) If  $d = 1$ , then  $T(\mathcal{G}) \leq (n - 2) \cdot g + n$ ;
- (iii) (Schwarz [18,23])  $T(\mathcal{G}) \leq d \cdot Wi(\lfloor \frac{n}{d} \rfloor) + (n \bmod d)$ ;
- (iv) (Kim [14])  $T(\mathcal{G}) \leq (\lfloor \frac{n}{d} \rfloor - 2) \cdot g + n$ .

**Remark 2.12.** Clearly, the bound of Schwarz is tighter than the bound of Wielandt in the imprimitive case ( $d > 1$ ), while in the primitive case ( $d = 1$ ) they are equal to each other. The bound of Kim is in the same relation with the bound of Dulmage and Mendelsohn. Further, the bound of Dulmage and Mendelsohn is tighter than that of Wielandt when  $g < n - 1$  and both bounds are equal when  $g = n - 1$ . In the remaining case  $g = n$ , the graph consists of a single Hamiltonian cycle and periodicity starts from the very beginning, i.e.,  $T(\mathcal{G}) = 1$ .

Let us show that the bound of Schwarz can be deduced from the bound of Kim. First, it can be seen that by substituting  $g = d(\lfloor \frac{n}{d} \rfloor - 1)$  in Kim’s bound and using the identity  $n = d \lfloor \frac{n}{d} \rfloor + (n \bmod d)$  we obtain the bound of Schwarz. Hence the bound of Kim is tighter when  $\frac{g}{d} < \lfloor \frac{n}{d} \rfloor - 1$ , and the bounds are equal when  $\frac{g}{d} = \lfloor \frac{n}{d} \rfloor - 1$ .

Consider the remaining case  $\frac{g}{d} = \lfloor \frac{n}{d} \rfloor$ . By definitions of  $g$  and  $d$ , the length of any cycle on  $\mathcal{G}$  equals to  $g + td$  for some  $t \geq 0$ . Since  $\frac{g}{d} = \lfloor \frac{n}{d} \rfloor$ , all cycles are of length  $g$ , thus  $g = d$  and  $\lfloor \frac{n}{d} \rfloor = 1$ . Therefore, both bounds equal to  $(n \bmod d) = n - g$ .

We are also interested in the improvements of Theorem 2.11 in terms of the *factor rank* of a matrix  $A \in \mathbb{B}^{n \times n}$  (also known as the Boolean rank or Schein rank).

**Definition 2.13** (Factor Rank in Boolean Algebra). The *Factor rank* of  $A$  is the least number  $r$  such that

$$A = \bigoplus_{\alpha=1}^r v_\alpha \otimes w_\alpha^T \tag{2.4}$$

with Boolean vectors  $v_1, w_1, \dots, v_r, w_r \in \mathbb{B}^n$ .

The factor rank of  $A$  is at most  $n$  since (2.4) holds when choosing  $r = n$  and the  $w_\alpha$  to be the unit vectors and the  $v_\alpha$  to be the columns of  $A$ .

The following bounds involving the factor rank were established:

**Theorem 2.14.** Let  $\mathcal{G}$  be a strongly connected primitive digraph with girth  $g$ , and let the associated matrix of  $\mathcal{G}$  have factor rank  $r$ . The following upper bounds on the transient of  $\mathcal{G}$  hold:

- (i) (Gregory–Kirkland–Pullman [12])  $T(\mathcal{G}) \leq Wi(r) + 1$ ;
- (ii) (Kim [14])  $T(\mathcal{G}) \leq (r - 2) \cdot g + r + 1$ .

In fact, we will show that the bounds in Theorem 2.14 also hold for non-primitive matrices and that the analogous stronger bounds of Schwarz and Kim with the factor rank instead of  $n$  are true. See Main Theorem 2.

2.2. Weighted digraphs and max algebra

In a *weighted digraph*  $\mathcal{G}$ , every edge  $(i, j) \in E$  is weighted by some *weight*  $a_{i,j}$ . We consider the case of nonnegative weights  $a_{i,j} \in \mathbb{R}_+$  and

**Definition 2.15** (*Weight of a Walk*). The *weight* of a walk  $W = (i_0, i_1, \dots, i_t)$  is the product

$$p(W) := a_{i_0,i_1} \cdot a_{i_1,i_2} \cdots a_{i_{t-1},i_t}. \tag{2.5}$$

Recall that the length of a walk  $W = (i_0, i_1, \dots, i_t)$  is  $\ell(W) = t$ .

Another common definition is letting edge weights be arbitrary reals and the weight of walks be the sum of the weights of its edges. One can navigate between these two definitions by taking the logarithm and the exponential.

By *max algebra* we understand the set of nonnegative real numbers  $\mathbb{R}_+$  equipped with the usual multiplication  $ab := a \cdot b$  and tropical addition  $a \oplus b := \max(a, b)$ . This arithmetic is extended to matrices and vectors in the usual way, which leads to max-linear algebra, i.e. the theory of max-linear systems [1,5]. The product of two matrices  $A \in \mathbb{R}_+^{m \times n}$  and  $B \in \mathbb{R}_+^{n \times q}$  is defined by  $(A \otimes B)_{i,j} := \max_{1 \leq k \leq n} a_{i,k} \cdot b_{k,j}$ , which defines the max-algebraic matrix powers  $A^{\otimes t} := \underbrace{A \otimes \cdots \otimes A}_{t \text{ times}}$ . The  $(i, j)$ th

entry of  $A^{\otimes t}$  will be denoted by  $a_{i,j}^{(t)}$ . Boolean matrices are a special case of max-algebraic matrices.

The walks of maximum weight in  $\mathcal{G}$  are closely related with the entries of max-algebraic powers of the associated nonnegative matrix of weights  $A = (a_{i,j})$ . Conversely, one can associate a weighted digraph  $\mathcal{G}(A)$  to every square max-algebraic matrix  $A$ . The connection between max-algebraic powers and weights of walks is based on the following fact called the *optimal walk interpretation* of max-algebraic matrix powers:  $a_{i,j}^{(t)}$  is the maximum weight of all walks of length  $t$  from  $i$  to  $j$ , or 0 if no such walk exists.

**Definition 2.16** (*Maximum Cycle Geometric Mean*). If  $\mathcal{G}(A)$  has at least one nonempty cycle, then the maximum geometric cycle mean of  $A \in \mathbb{R}_+^{n \times n}$  is equal to

$$\lambda(A) := \max \{ p(C)^{1/\ell(C)} \mid C \text{ is a nonempty cycle in } \mathcal{G}(A) \}. \tag{2.6}$$

Set  $\lambda(A) = 0$  if no nonempty cycle in  $\mathcal{G}(A)$  exists.

**Definition 2.17** (*Critical Graph*). Let  $A \in \mathbb{R}_+^{n \times n}$ . The cycles of  $\mathcal{G}(A)$  at which the maximum geometric cycle mean is attained are called *critical*, and so are all nodes and edges that belong to them. The set of all critical nodes is denoted by  $N_c(A)$  and the set of all critical edges is denoted by  $E_c(A)$ . The *critical graph*, denoted by  $\mathcal{G}^c(A)$ , consists of all critical nodes and all critical edges.

As we have  $\lambda(\alpha \cdot A) = \alpha \cdot \lambda(A)$  for all  $\alpha \in \mathbb{R}_+$  and  $A \in \mathbb{R}_+^{n \times n}$ , we also have  $\lambda(A/\lambda(A)) = 1$  whenever  $\lambda(A) \neq 0$ . It is  $\lambda(A) \neq 0$  if and only if  $\mathcal{G}(A)$  contains a nonempty cycle. In this case, the equality  $A^{\otimes t} = \lambda(A)^t \cdot (A/\lambda(A))^{\otimes t}$  implies that we can indeed assume  $\lambda(A) = 1$  without loss of generality when studying the sequence of max-algebraic matrix powers. We will indeed assume  $\lambda(A) = 1$  in the rest of the paper. It means that we avoid the case when  $\lambda(A) = 0$ . This case is trivial because there are no critical nodes. Moreover, since there are no closed walks on  $\mathcal{G}(A)$ , there are no walks with length more than  $n - 1$ , so  $A^{\otimes n} = 0$ .

The following definition is standard.

**Definition 2.18** (*Irreducibility*). A matrix  $A \in \mathbb{R}_+^{n \times n}$  is called *irreducible* if  $\mathcal{G}(A)$  is strongly connected, i.e., if for each  $i, j \in N$  there exists a walk of nonzero weight whose starting node is  $i$  and end node is  $j$ .

$A \in \mathbb{R}_+^{n \times n}$  is called *reducible* if it is not irreducible.

Cohen et al. [8] first proved that the sequence of max-algebraic matrix powers of an irreducible matrix  $A$  with  $\lambda(A) = 1$  is eventually periodic.

**Definition 2.19** (*Transient of Matrices*). The least nonnegative  $t$  satisfying  $A^{\otimes(p+t)} = \lambda(A)^p \cdot A^{\otimes t}$  for some  $p > 0$  is called the *transient* of  $A$  and denoted by  $T(A)$ .

The transient of  $A$  depends not only on the nodes and edges in  $\mathcal{G}(A)$ , but also on the specific weights in  $A$ . It was studied by several authors, including Hartmann and Arguelles [13], Bouillard and Gaujal [3], Soto y Koelemeijer [24], Akian et al. [2, Section 7], and Charron-Bost et al. [7]. However, none of the upper bounds on the transient were generalizations of any of the Boolean bounds of Theorem 2.11 (in the sense that the bounds of that theorem would be immediately recovered when specializing these results to Boolean matrices). To see that  $T(A)$  depends also on the weights of  $A$ , consider

$$A = \begin{pmatrix} 1 & \epsilon \\ \epsilon & 1/e \end{pmatrix},$$

and observe that the transient of  $(a_{2,2}^{(t)})_{t \geq 1}$  is equal to  $\lceil -2 \log \epsilon \rceil$  if  $\epsilon < 1$ .

In the present paper, we generalize all the bounds in [Theorem 2.11](#) to the weighted case. We do this not by giving bounds on the transient of  $A$ , but by giving bounds on the transients of the critical rows and columns of  $A$ . Let us give the corresponding definition.

**Definition 2.20** (Row and Column Transients). Let  $A \in \mathbb{R}_+^{n \times n}$ . The least nonnegative  $t$  satisfying  $a_{k,i}^{(p+t)} = \lambda(A)^p \cdot a_{k,i}^{(t)}$  for a fixed  $k \in N$ , all  $i \in N$  and some  $p > 0$  is called the transient of the  $k$ th row of  $A$  and denoted by  $T_k(A)$ , or just  $T_k$  if the matrix is clear from the context.

Similarly, the least nonnegative  $t$  satisfying  $a_{i,k}^{(p+t)} = \lambda(A)^p \cdot a_{i,k}^{(t)}$  for a fixed  $k \in N$ , all  $i \in N$  and some  $p > 0$  is called the transient of the  $k$ th column of  $A$ .

**Remark 2.21.** This definition makes sense also when  $A$  is reducible. However, in that case the transient is defined only for some rows and columns. Below we are interested only in the case of critical rows and columns, for which the transients always exist regardless of the irreducibility of  $A$ .

**Remark 2.22.** In the Boolean case, all rows and columns are critical, hence we are really generalizing the Boolean bounds. We are motivated by a result of Nachtigall [[16](#)] who showed that the transient of critical rows and columns does not exceed  $n^2$ . Later, Sergeev and Schneider [[21](#)] conjectured that, in fact, this transient should not exceed  $Wi(n)$ . In particular, we prove this conjecture.

The following is the first main result of the paper.

**Main Theorem 1.** Let  $A \in \mathbb{R}_+^{n \times n}$  be irreducible and let  $k$  be a critical node. Denote by  $d$  the cyclicity of  $\mathcal{G}(A)$ , by  $H$  the component of the critical graph  $\mathcal{G}^c(A)$  containing  $k$ , and by  $|H|$  the number of nodes in  $H$ . The following quantities are upper bounds on the transient of the  $k$ th row and the  $k$ th column:

- (i) (Wielandt bound)  $Wi(n)$
- (ii) (Dulmage–Mendelsohn bound)  $(n - 2) \cdot g(H) + |H|$
- (iii) (Schwarz bound)  $d \cdot Wi\left(\lfloor \frac{n}{d} \rfloor\right) + (n \bmod d)$
- (iv) (Kim bound)  $\left(\lfloor \frac{n}{d} \rfloor - 2\right) \cdot g(H) + \min(n, |H| + (n \bmod d))$

The first two bounds also hold in the case when  $A$  is reducible.

**Remark 2.23.** Denote by  $\gamma(\mathcal{G}^c(A))$  the least common multiple of the cyclicities of all strongly connected components of  $\mathcal{G}^c(A)$ . This number is well-known to be the least eventual period of the sequence  $(A^{\otimes t})_{t \geq 1}$  when  $A$  is irreducible (see [[8,5](#)]). It is also the least eventual period of the sequence of submatrices of  $A^{\otimes t}$  extracted from the critical rows or the critical columns (also in the reducible case). For an individual critical row or column, the least eventual period can be shown to be equal to the cyclicity of the component of  $\mathcal{G}^c(A)$  where the index of that row or column lies (see [Remark 3.4](#)).

**Remark 2.24.** As in the Boolean case, the bound of Schwarz (resp. Kim) is tighter than the bound of Wielandt (resp. Dulmage and Mendelsohn) when the corresponding component of  $\mathcal{G}$  is imprimitive. Moreover, the bound of Wielandt is never tighter than that of Dulmage and Mendelsohn when  $g(H) \leq n - 1$ . Unlike for the unweighted graphs, the case  $g(H) = n$  is non-trivial and will be treated below. Likewise, the bound of Schwarz is never tighter than the bound of Kim when  $\frac{g(H)}{d} \leq \lfloor \frac{n}{d} \rfloor - 1$ , but the case  $\frac{g(H)}{d} = \lfloor \frac{n}{d} \rfloor$  has to be treated separately. Here we prefer to deduce the bound of Kim from the bound of Dulmage and Mendelsohn in the same way as the bound of Schwarz is derived from the bound of Wielandt (similar to the approach of Shao and Li [[23](#)]).

**Definition 2.25** (Factor Rank in Max Algebra). In max algebra, factor rank of  $A \in \mathbb{R}_+^{n \times n}$  is the least number  $r$  such that (2.4) holds for some vectors  $v_1, w_1, \dots, v_r, w_r \in \mathbb{R}_+^n$ .

In our next main result, we show that the results of [Main Theorem 1](#) can be improved by means of factor rank, thus obtaining a max-algebraic extension of [Theorem 2.14](#).

**Main Theorem 2.** Let  $A \in \mathbb{R}_+^{n \times n}$  be irreducible. Denote by  $d$  the cyclicity of  $\mathcal{G}(A)$  and by  $r$  the factor rank of  $A$ . Let  $k$  be critical. Denote by  $H$  the component of the critical graph  $\mathcal{G}^c(A)$  containing  $k$ , and let  $h \leq \min(|H|, r)$  be the parameter defined below in (7.4). The following upper bounds on the transient of the  $k$ th row and  $k$ th column hold:

- (i)  $Wi(r) + 1$ ;
- (ii)  $(r - 2) \cdot g(H) + h + 1$ .
- (iii)  $d \cdot Wi\left(\lfloor \frac{r}{d} \rfloor\right) + (r \bmod d) + 1$ ;
- (iv)  $\left(\lfloor \frac{r}{d} \rfloor - 2\right) \cdot g(H) + \min(r, h + (r \bmod d)) + 1$ .

The first two bounds apply to reducible matrices as well.

**Remark 2.26.** While all parameters appearing in the bounds of [Main Theorem 1](#) only depend on the unweighted digraphs underlying  $\mathcal{G}(A)$  and  $\mathcal{G}^c(A)$ , the factor rank  $r$  of [Main Theorem 2](#) depends on the values of  $A$ , i.e. on the weights on  $\mathcal{G}(A)$ .

The next five sections of the paper contain the proofs of [Main Theorems 1](#) and [2](#). That is, we will prove that  $T_k(A)$  for a critical index  $k$  is less than any of the quantities in [Main Theorems 1](#) and [2](#). Applying the result to the transposed matrix  $A^T$ , we see that the bounds also hold for the transients of the columns.

The proofs do not use the results of [Theorem 2.11](#) or [Theorem 2.14](#) and hence, in particular, we give new proofs for those classical results.

### 2.3. Visualization

In the end of this section, let us recall a result on diagonal matrix scaling, which we will use.

Let  $X$  be an  $n \times n$  nonnegative diagonal matrix, i.e. a matrix whose diagonal entries are positive and whose off-diagonal entries are zero. Consider the operation of *diagonal similarity scaling*  $A \mapsto X^{-1}AX$ , applied to any  $A \in \mathbb{R}_+^{n \times n}$ . It can be checked that the diagonal similarity scaling preserves  $\lambda(A)$  and commutes with max-algebraic matrix powering: for  $B = X^{-1}AX$  we have  $\lambda(A) = \lambda(B)$  and  $B^{\otimes t} = X^{-1}A^{\otimes t}X$ . Hence, to analyze max-algebraic matrix powers we will use a particular “canonical” form that can be always reached by means of a diagonal similarity scaling.

**Definition 2.27** (*Visualization*). A matrix  $A \in \mathbb{R}_+^{n \times n}$  is called *visualized* if it has  $a_{i,j} \leq \lambda(A)$  for all  $i, j \in N$ . For a visualized matrix, it also follows that  $a_{i,j} = \lambda(A)$  for all critical edges  $(i, j) \in E_c$ .

Further if  $a_{i,j} = \lambda(A)$  holds only for all critical edges  $(i, j) \in E_c$ , matrix  $A$  is called *strictly visualized*.

**Remark 2.28.** It is known that every nonnegative matrix with positive maximum geometric cycle mean can be brought to a visualized form by means of a diagonal similarity scaling. Moreover, every nonnegative matrix can be brought to a strictly visualized form [22]. Hence in our analysis of max-algebraic powers, we can assume without loss of generality that  $A$  is visualized (and, moreover, strictly visualized), which we do in the rest of the paper. Since we also assume  $\lambda(A) = 1$ , it means that all entries are between 0 and 1 and critical edges are exactly edges with weight 1.

An early use of visualization scaling (unrelated to max algebra) can be found in Fiedler and Pták [11], and the scaling was studied in more detail in [22]. For a short survey on the use of visualization scaling in max algebra see [20]. Let us conclude with the following observation concerning the visualization of max-algebraic powers. Recall that  $\mathcal{G}^t$  denotes the  $t$ th power of an arbitrary digraph  $\mathcal{G}$ .

**Lemma 2.29** (Cf. [6, Lemma 2.9], [19]). Let  $A \in \mathbb{R}_+^{n \times n}$  and  $t \geq 1$ .

- (i)  $\mathcal{G}^c(A^{\otimes t}) = (\mathcal{G}^c(A))^t$ ,
- (ii) If  $A$  is visualized (or strictly visualized), then so is  $A^{\otimes t}$ .

### 3. Proof of Dulmage–Mendelsohn bound

In this section, we want to prove the second bound of [Main Theorem 1](#). In fact, we first argue that  $T_k(A) \leq n - 1$  when  $a_{k,k} = \lambda(A)$ , from which we get  $T_k(A) \leq g(H)(n - 1)$  for  $H$  a component of the critical graph and  $k$  in a minimal cycle of  $H$ , and finally the general bound.

The proof splits into the following lemmas, all of them apply to any  $A \in \mathbb{R}_+^{n \times n}$  and will be used again later. Without loss of generality, we will assume in the proofs that  $\lambda(A) = 1$  and  $A$  is visualized, that is all edges have weight at most 1 and all critical edges have weight 1. (See [Remark 2.28](#))

The first lemma gives a means to bound the transient of a row of the matrix. More specifically, it states that a row is periodic as soon as its entries are equal to that of a higher matrix power. The proof is a straightforward calculation.

**Lemma 3.1.** Let  $k \in N$ . Further assume that there exist  $r < s$  such that  $a_{k,j}^{(r)} = a_{k,j}^{(s)}$  for all  $j \in N$ . Then  $T_k \leq r$ .

In particular,  $T_k(A) \leq m \cdot T_k(A^{\otimes m})$  for all  $m \geq 1$ .

**Proof.** Set  $p = s - r$  and let  $j \in N$ . Then, for all  $t \geq r$ :

$$a_{k,j}^{(t+p)} = \max_{l \in N} a_{k,l}^{(s)} \cdot a_{l,j}^{(t+p-s)} = \max_{l \in N} a_{k,l}^{(r)} \cdot a_{l,j}^{(t-r)} = a_{k,j}^{(t)}$$

This concludes the proof.  $\square$

The next lemma shows that a stronger form of the Weighted Dulmage–Mendelsohn bound holds if  $k$  lies on a critical cycle of length  $g(H)$ . Its proof uses the fact that  $k$  lies on a critical cycle of length 1 in the digraph of the matrix power  $A^{\otimes g(H)}$ . Denote by  $A_k$  the  $k$ th row of  $A$ .

**Lemma 3.2** (Nachtigall [16]). *Let  $k$  be a critical node on a critical cycle of length  $\ell$ . Then  $T_k \leq (n - 1) \cdot \ell$  and  $\ell$  is an eventual period of  $A_k^{\otimes t}$ .*

**Proof.** Set  $B = A^{\otimes \ell}$ . Then  $b_{k,k} = 1$  and hence  $b_{k,j}^{(t)}$  is non-decreasing with  $t$ . But since we assume  $\lambda(A) = 1, \lambda(B) = 1$  and  $\sup_{t \in \mathbb{N}} B^{\otimes t} = \max_{t=1}^{n-1} B^{\otimes t}$ , so  $b_{k,j}^{(t)}$  is constant for  $t \geq n - 1$ , i.e.  $T_k(A^{\otimes \ell}) \leq n - 1$ .

Lemma 3.1 now concludes the proof.  $\square$

The following result enables us to use the bound of Lemma 3.2 for nodes that do not lie on a critical cycle of minimal length. As usual, we assume that  $\lambda(A) = 1$  and  $A$  is visualized. The proof makes heavy use of this assumption. It is closely related to the circulant symmetries of the critical part of max-algebraic powers in the periodic regime, as described by Butkovič and Sergeev [5, Section 8.3], [19].

**Lemma 3.3.** *Let  $k$  and  $l$  be two indices of  $N_c(A)$ , and suppose that there exists a walk from  $k$  to  $l$ , of length  $r$  and with all edges critical.*

- (i) *If  $t \geq T_l(A)$ , then  $A_k^{\otimes(t+r)} = \lambda(A)^r \cdot A_l^{\otimes t}$ .*
- (ii)  *$T_k(A) \leq T_l(A) + r$ .*

**Remark 3.4.** Since none of the above lemmas assume the irreducibility of  $A$ , they imply the eventual periodicity of all rows and columns with critical indices in the reducible case. Moreover, they show that the cyclicity  $\gamma(H)$  of a strongly connected component  $H$  of  $\mathcal{G}^c(A)$  is an eventual period for the  $A_k^{\otimes t}$  for any  $k \in H$ . It is the least period because when  $A$  is strictly visualized,  $a_{k,k}^{(t)}$  takes the value 1 with least eventual period  $\gamma(H)$ .

**Proof of Lemma 3.3.** Since  $A$  is assumed to be visualized with  $\lambda(A) = 1$ , the existence of the walk with critical edges exactly means  $a_{k,l}^{(r)} = 1$ .

Since each edge of  $\mathcal{G}^c(A)$  belongs to a cycle of  $\mathcal{G}^c(A)$ , there is a walk from  $l$  to  $k$  with critical edges. Let  $s \geq 1$  be its length. We have  $a_{l,k}^{(s)} = 1$ .

Thus, we have

$$a_{l,j}^{(t+r+s)} \geq a_{l,k}^{(s)} \cdot a_{k,j}^{(t+r)} = a_{k,j}^{(t+r)} \geq a_{k,l}^{(r)} \cdot a_{l,j}^{(t)} = a_{l,j}^{(t)}$$

for any  $t$ .

Iterating the inequality, we see that

$$a_{l,j}^{(t+p(r+s))} \geq a_{k,j}^{(t+r)} \geq a_{l,j}^{(t)} \tag{3.1}$$

for all  $t$  and  $p$ .

If  $p$  is an eventual period of  $a_{l,j}^{(t)}$  and  $t \geq T_l$ , the first and the last entry of (3.1) are equal, so all inequalities of (3.1) are equalities.

It means that the sequences  $\left(a_{k,j}^{(t)}\right)_{t \geq T_l+r}$ ,  $\left(a_{l,j}^{(t-r)}\right)_{t \geq T_l+r}$ ,  $\left(a_{l,j}^{(t)}\right)_{t \geq T_l}$  are identical. Since the last sequence is periodic, both parts of the lemma are proved.  $\square$

**Proof of Dulmage–Mendelsohn bound.** Let  $C$  be a cycle in  $H$  of length  $\ell(C) = g(H)$ . By Lemma 3.2,  $T_k \leq (n - 1) \cdot g(H)$  for all nodes  $k$  of  $C$ .

Let now  $k$  be any node in  $H$ . There exist walks in  $H$  from  $k$  to  $C$  of length at most  $|H| - g(H)$ . Application of Lemma 3.3 now concludes the proof.  $\square$

#### 4. Proof of Kim’s bound

Set  $B = A^{\otimes d}$ . The cyclicity classes of  $\mathcal{G}(A)$  are the strongly connected components of  $\mathcal{G}(B)$ , and  $\mathcal{G}(B)$  is completely reducible, i.e. it has no edge between two different strongly connected components.

It means that, up to reordering the indices,  $B$  is block-diagonal. As in Kim [14], we want to apply the previous bound to the smallest blocks. Then, we will get the general bound thanks to Lemmas 3.1–3.3.

Obviously, any cycle in  $\mathcal{G}(A)$  has to go through every cyclicity class. Thus,  $d$  divides  $g(H)$  and if  $k$  belongs to  $H$  then the girth of its strongly connected components in  $\mathcal{G}^c(B)$  is at most  $g(H)/d$ .

Call a cyclicity class of  $\mathcal{G}(A)$  *small* if it contains the minimal number of nodes amongst cyclicity classes. Let  $m$  be the number of nodes in any small class.

We apply the Dulmage–Mendelsohn bound to the nodes of small classes and then use Lemma 3.3 to extend it to all other critical nodes.

We distinguish the cases (A)  $m \leq \lfloor n/d \rfloor - 1$  and (B)  $m = \lfloor n/d \rfloor$ . Note that  $m \geq \lfloor n/d \rfloor + 1$  is not possible.

In case (B), there are at least  $d - (n \bmod d)$  small classes because otherwise the sum of sizes of cyclicity classes  $C$  would satisfy

$$\begin{aligned} n &= \sum_C |C| > (d - (n \bmod d)) \cdot \lfloor n/d \rfloor + (n \bmod d) \cdot (\lfloor n/d \rfloor + 1) \\ &= d \cdot \lfloor n/d \rfloor + (n \bmod d) = n, \end{aligned} \quad (4.1)$$

a contradiction. Hence every critical node is connected to a small class by a path consisting of critical edges of length at most  $(n \bmod d)$ .

Let us first prove that

$$T_k \leq (\lfloor n/d \rfloor - 2) \cdot g(H) + n. \quad (4.2)$$

In both cases (A) and (B), by the max-algebraic extension of Dulmage and Mendelsohn's bound, we have  $T_k(B) \leq (m - 2) \cdot g(H)/d + m$  for each critical node  $k$  of  $H$  in a small class. Lemma 3.1 then implies  $T_k(A) \leq (m - 2) \cdot g(H) + d \cdot m$  for all critical nodes  $k$  of  $H$  in small classes.

In case (A), a crude estimation for all critical  $k$  in small classes is

$$T_k \leq (m - 1) \cdot g(H) + d \cdot m \leq (\lfloor n/d \rfloor - 2) \cdot g(H) + n - d.$$

Because every critical node has paths consisting of critical edges to a small class of length at most  $d - 1$ , (4.2) follows from Lemma 3.3 in case (A).

In case (B), there is a path of length at most  $(n \bmod d)$  consisting of critical edges to a small class. Hence, again by Lemma 3.3,

$$\begin{aligned} T_k &\leq (\lfloor n/d \rfloor - 2) \cdot g(H) + d \cdot \lfloor n/d \rfloor + (n \bmod d) \\ &= (\lfloor n/d \rfloor - 2) \cdot g(H) + n. \end{aligned}$$

This concludes the proof of (4.2).

It remains to prove

$$T_k(A) \leq (\lfloor n/d \rfloor - 2) \cdot g(H) + |H| + (n \bmod d). \quad (4.3)$$

This is based on the following lemma, which we prove using the same case distinction.

**Lemma 4.1.** *If  $k$  is in a critical closed walk  $C$ , then*

$$T_k(A) \leq (\lfloor n/d \rfloor - 1)\ell(C) + (n \bmod d), \quad (4.4)$$

**Proof.** Let us first notice that  $k$  is in a critical closed walk of  $A^{\otimes d}$  with length  $\ell(C)/d$ .

If  $k$  is in a small class, then  $T_k(A^{\otimes d}) \leq (m - 1)\ell(C)/d$  by Lemma 3.2, thus  $T_k(A) \leq (m - 1)\ell(C)$  by Lemma 3.1.

If  $k$  is in  $C$  but not necessarily in a small class, we distinguish between cases (A) and (B) and apply Lemma 3.3. In case (A),  $m \leq \lfloor n/d \rfloor - 1$ . Recall that  $C$  contains representatives of all cyclicity classes (and the small classes, too), and therefore, each node  $k \in C$  can be connected to a node of  $C$  in a small class by a subpath of  $C$ , with length at most  $\ell(C) - 1$ . Applying Lemma 3.3 we get

$$T_k(A) \leq (\lfloor n/d \rfloor - 2)\ell(C) + \ell(C) - 1 = (\lfloor n/d \rfloor - 1)\ell(C) - 1$$

which implies (4.4). In case (B),  $m = \lfloor n/d \rfloor$  and there is a path from  $k$  to a node of  $C$  in a small class with length at most  $(n \bmod d)$ , so we get exactly (4.4).  $\square$

To conclude the proof of (4.3), we apply Lemma 4.1 to a cycle  $C$  with length  $g(H)$  and notice that for any  $k \in H$ , there is a critical path from  $k$  to  $C$  with length at most  $|H| - g(H)$ . By Lemma 3.3, it implies

$$\begin{aligned} T_k(A) &\leq (\lfloor n/d \rfloor - 1)g(H) + (n \bmod d) + |H| - g(H) \\ &= (\lfloor n/d \rfloor - 2)g(H) + |H| + (n \bmod d) \end{aligned}$$

and (4.3) is proved.

## 5. Proof of Wielandt's bound

If  $g(H) \leq n - 1$ , then Wielandt's bound follows from the Dulmage–Mendelsohn bound. It remains to treat the case that  $g(H) = n$ , i.e.  $g^c(A)$  is a Hamiltonian cycle. We therefore prove a result on cycle removal and insertion (Theorem 5.2) which implies the Wielandt bound for matrices with a critical Hamiltonian cycle (Corollary 5.3).



5.1. Cycle replacement with a Hamiltonian cycle

We recall the following elementary application of the pigeonhole principle.

**Lemma 5.1.** *Let  $x_1, \dots, x_n$  be integers. There exists a nonempty subset  $I$  of  $\{1, \dots, n\}$  such that  $\sum_{i \in I} x_i$  is a multiple of  $n$ .*

**Proof.** Either one of the  $n$  sums  $\sum_{i=1}^k x_i$  with  $1 \leq k \leq n$  is a multiple of  $n$ , or two of these sums are in the same congruence class modulo  $n$ .  $\square$

We will use this lemma to prove:

**Theorem 5.2.** *Let  $\mathcal{G}$  be a digraph with  $n$  nodes. For any Hamiltonian cycle  $C_H$  in  $\mathcal{G}$  and any walk  $W$ , there is a walk  $V$  that has the same start and end nodes as  $W$ , is formed by removing cycles from  $W$  and possibly inserting copies of  $C_H$ , and has a length satisfying  $(n - 1)^2 + 1 \leq \ell(V) \leq (n - 1)^2 + n$  and  $\ell(V) \equiv \ell(W) \pmod n$ .*

**Corollary 5.3.** *If  $A \in \mathbb{R}_+^{n \times n}$  has a critical Hamiltonian cycle and  $\lambda(A) = 1$ , then the transient of  $A$  is at most  $Wi(n)$ .*

**Proof of Theorem 5.2.**  $W$  can be decomposed into a path  $P$  and a collection  $\mathcal{C}$  of cycles. Note that  $P$  is empty when the start and end nodes of  $W$  are the same.

Let  $\mathcal{B}$  be a result of recursively removing from  $\mathcal{C}$  sets of cycles whose combined length is a multiple of  $n$ . By Lemma 5.1,  $|\mathcal{B}| \leq n - 1$ . Also,  $\ell(C) \leq n - 1$  for all  $C \in \mathcal{B}$ .

Let us build the walk  $V$  as follows. If  $P$  intersects all cycles of  $\mathcal{B}$  (Case (C)), then we successively insert all such cycles in  $P$ . Otherwise (Case (D)), we first insert  $C_H$  into  $P$ , getting  $\tilde{P}$  and then insert all cycles of  $\mathcal{B}$  into  $\tilde{P}$ .

In case (C), we have

$$\begin{aligned} \ell(V) &= \ell(P) + \sum_{\alpha \in \mathcal{B}} \ell(C_\alpha) \\ &\leq (n - 1) + (n - 1) \cdot (n - 1) \\ &< (n - 1)^2 + n. \end{aligned}$$

In case (D), there exists some  $\hat{C} \in \mathcal{B}$  such that  $\ell(P) + \ell(\hat{C}) \leq n - 1$ , so that

$$\begin{aligned} \ell(V) &= \ell(C_H) + \ell(P) + \ell(\hat{C}) + \sum_{\substack{C \in \mathcal{B} \\ C \neq \hat{C}}} \ell(C) \\ &\leq n + (n - 1) + (n - 1) \cdot (n - 2) \\ &= (n - 1)^2 + n. \end{aligned}$$

Moreover,  $\ell(V) \equiv \ell(W) \pmod n$  by construction in both cases.

This concludes the proof, because if  $V$  is too short, we just insert copies of  $C_H$  into it.  $\square$

5.2. Proof of Wielandt’s bound with a critical Hamiltonian cycle

In this section, we prove Corollary 5.3. Because the critical graph contains a Hamiltonian cycle, it is strongly connected and  $n$  is an eventual period of  $A^{\otimes t}$ .

Let  $i, j \in N$  and let  $t \geq Wi(n)$ . We show that  $a_{i,j}^{(t)} = a_{i,j}^{(s(t))}$  where  $s(t) = Wi(n) + ((t - Wi(n)) \bmod n)$ . Because  $s(t) = s(t')$  whenever  $t \equiv t' \pmod n$ , this suffices for the proof.

If  $t = s(t)$  the result is obvious. Otherwise, let  $W$  be a maximum weight walk of length  $t$  from  $i$  to  $j$ , i.e.  $p(W) = a_{i,j}^{(t)}$ . We apply Theorem 5.2 to walk  $W$  and the critical Hamiltonian cycle  $C_H$ . By Theorem 5.2, there is a walk  $V$  from  $i$  to  $j$ , obtained from  $W$  by deleting some cycles and possible inserting copies of  $C$ , with length satisfying  $Wi(n) \leq \ell(V) \leq Wi(n) + n - 1$  and  $\ell(V) \equiv \ell(W) \pmod n$ . In other words,  $\ell(V) = s(t)$ . Because we assume  $\lambda(A) = 1$ , the weight of  $V$  satisfies  $p(V) \geq p(W)$  and hence

$$a_{i,j}^{(s(t))} \geq p(V) \geq p(W) = a_{i,j}^{(t)}. \tag{5.1}$$

Since  $t \geq Wi(n)$  and  $s(t) \equiv t \pmod n$ , there exists some  $r \geq 0$  such that  $t - s(t) = r \cdot n$ . Hence

$$a_{i,j}^{(t)} \geq a_{i,j}^{(s(t))} \cdot a_{j,j}^{(r \cdot n)} = a_{i,j}^{(s(t))} \tag{5.2}$$

because  $a_{j,j}^{(n)} = 1$ . Combination of (5.1) and (5.2) concludes the proof.

**Remark 5.4.** To our knowledge, the results of this section are new. However, let us remark that the method of cycle replacement using Lemma 5.1 was invented by Hartmann and Arguelles [13], who also used it to derive (less precise) transience bounds for sequences of optimal walks.

### 6. Proof of Schwarz’s bound

Schwarz’s bound is derived from Wielandt’s bound as Kim’s bound is derived from Dulmage–Mendelsohn’s bound in Section 4.

Again, call a cyclicity class of  $\mathcal{G}(A)$  *small* if it contains the minimal number of nodes amongst cyclicity classes. Let  $m$  be the number of nodes in any small class and set  $B = A^{\otimes d}$ .

For each critical node  $k$  in a small cyclicity class, we have  $T_k(B) \leq \text{Wi}(m)$  by Wielandt’s bound. Lemma 3.1 hence implies  $T_k(A) \leq d \cdot \text{Wi}(m)$  for all critical nodes  $k$  in small classes.

We distinguish the cases (A)  $m \leq \lfloor n/d \rfloor - 1$  and (B)  $m = \lfloor n/d \rfloor$ . Note that  $m \geq \lfloor n/d \rfloor + 1$  is not possible.

In case (A), a crude estimation for all critical  $k$  in small classes is

$$T_k \leq d \cdot \text{Wi}(m) \leq d \cdot \text{Wi}(\lfloor n/d \rfloor) - d.$$

Observe that each critical node can be connected to a small class, by a path of length at most  $d - 1$  consisting of critical edges only. So in case (A), the theorem just follows from Lemma 3.3.

In case (B), there are at least  $d - (n \bmod d)$  small classes because otherwise (4.1) yields a contradiction. In this case, each critical node can be connected to a node from a small class by a path consisting only of critical edges, of length at most  $(n \bmod d)$ . Hence, by Lemma 3.3,

$$T_k \leq d \cdot \text{Wi}(\lfloor n/d \rfloor) + (n \bmod d).$$

This concludes the proof.

### 7. Proof of the bounds involving the factor rank

In this section, we prove Main Theorem 2. Let  $v_\alpha, w_\alpha \in \mathbb{R}_+^n$ , for  $\alpha = 1, \dots, r$ , be the vectors in factor rank representation (2.4). Further, let  $V$  and  $W$  be the  $n \times r$  matrices whose columns are vectors  $v_\alpha$  and  $w_\alpha$  for  $\alpha = 1, \dots, r$ , and consider the  $(n + r) \times (n + r)$  matrix  $Z$  defined by

$$Z = \begin{pmatrix} 0_{n \times n} & V \\ W^T & 0_{r \times r} \end{pmatrix}, \tag{7.1}$$

Then we have

$$Z^{\otimes 2} = \begin{pmatrix} A & 0_{n \times r} \\ 0_{r \times n} & B \end{pmatrix}, \tag{7.2}$$

where the  $r \times r$  matrix  $B$  is given by

$$b_{\alpha, \beta} = \bigoplus_{i=1}^n w_{\alpha, i} \cdot v_{\beta, i}, \quad \text{for } \alpha, \beta = 1, \dots, r. \tag{7.3}$$

We will apply the bounds of Main Theorem 1 to the critical nodes of  $B$  and transfer the result to the critical nodes of  $A$ , thanks to the following observation.

**Lemma 7.1.** *If  $(k, n + \beta)$  is an edge of  $\mathcal{G}^c(Z)$ , then  $T_k(A) \leq T_\beta(B) + 1$ .*

**Proof.** By Eq. (7.2) and Lemma 3.1,  $T_{n+\beta}(Z) \leq 2T_{n+\beta}(Z^{\otimes 2}) \leq 2T_\beta(B)$ , thus  $T_k(Z) \leq 2T_\beta(B) + 1$  by Lemma 3.3. But Eq. (7.2) now implies  $T_k(A) \leq \lceil T_k(Z)/2 \rceil \leq \lceil (2T_\beta(B) + 1)/2 \rceil = T_\beta(B) + 1$ .  $\square$

To use this lemma, we need to study the links between  $\mathcal{G}^c(Z)$ ,  $\mathcal{G}^c(A)$  and  $\mathcal{G}^c(B)$ .

The next observation is useful for the case of Kim’s and Schwarz’s bounds, where  $A$  is assumed to be irreducible.

**Lemma 7.2.** *If  $A$  is irreducible, then so are  $Z$  and  $B$ . Moreover  $\mathcal{G}(B)$  and  $\mathcal{G}(A)$  have the same cyclicity.*

**Proof.** As  $A$  is irreducible, there exists a walk in  $\mathcal{G}(Z^{\otimes 2})$ , and hence in  $\mathcal{G}(Z)$ , between every pair of nodes in  $\{1, \dots, n\}$ . None of the vectors  $v_\alpha, w_\alpha$  for  $\alpha = 1, \dots, r$  is zero by the minimality of  $r$ , i.e. every node in  $\{n + 1, \dots, n + r\}$  has an incoming and an outgoing neighbor in  $\{1, \dots, n\}$ . Hence there exists a walk between every pair of nodes in  $\mathcal{G}(Z)$ .

By Theorem 2.7,  $\mathcal{G}(Z^{\otimes 2})$  has at most 2 strongly connected components with the same cyclicity. By Eq. (7.2), it has at least 2 components, one of them is  $\mathcal{G}(A)$  and the second one is isomorphic to  $\mathcal{G}(B)$ , hence these are the two components given by Theorem 2.7. In particular,  $B$  is irreducible and the graphs of  $A$  and  $B$  have the same cyclicity.  $\square$

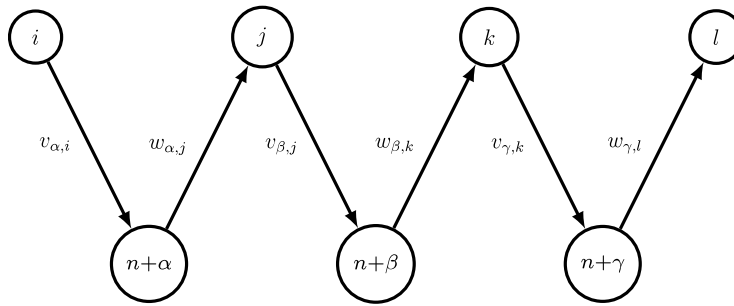


Fig. 1. A walk in  $\mathcal{G}(Z)$ .

By construction,  $\mathcal{G}(Z)$  is a bipartite graph, so every walk in  $\mathcal{G}(Z)$  alternates between nodes in  $\{1, \dots, n\}$  and nodes in  $\{n + 1, \dots, n + r\}$ . Fig. 1 depicts an example of a walk in  $\mathcal{G}(Z)$ .

As all closed walks in  $\mathcal{G}(Z)$  are of even length, the cyclicity of any component of  $\mathcal{G}^c(Z)$  is even, i.e. it is divisible by two. Hence each component  $G$  of  $\mathcal{G}^c(Z)$  splits into two components of  $(\mathcal{G}^c(Z))^2$  such that the (disjoint) union of their node sets is exactly the node set of  $G$  (e.g., apply Theorem 2.7 with  $k = 2$  and even  $\sigma$ ). Following [6] we call these two components *related*. For a component  $H$  of  $(\mathcal{G}^c(Z))^2$ , the related component will be denoted by  $H'$ .

Each closed walk of  $\mathcal{G}(Z)$  and, therefore, each component of  $\mathcal{G}^c(Z)$  contains nodes both from  $\{1, \dots, n\}$  and from  $\{n + 1, \dots, n + r\}$ . Hence, if  $H$  and  $H'$  is a pair of related components of  $(\mathcal{G}^c(Z))^2$  then one of them (say,  $H$ ) contains a node in  $\{1, \dots, n\}$  and the other ( $H'$ ) contains a node in  $\{n + 1, \dots, n + r\}$ . Since there are no edges between the two components of  $\mathcal{G}(Z^{\otimes 2})$ ,  $H$  is a subgraph of  $\mathcal{G}(A)$  and  $H'$  is a subgraph of  $\mathcal{G}(B)$ . Further as  $(\mathcal{G}^c(Z))^2 = \mathcal{G}^c(Z^{\otimes 2})$  (by Lemma 2.29),  $H$  and  $H'$  are components of  $\mathcal{G}^c(Z^{\otimes 2})$ . As  $\mathcal{G}^c(Z^{\otimes 2})$  consists of only such components and the cycles not belonging to such components have a strictly smaller geometric mean, it follows that  $H$  is a component of  $\mathcal{G}^c(A)$ ,  $H'$  is a component of  $\mathcal{G}^c(B)$  and, moreover,  $\mathcal{G}^c(A)$  and  $\mathcal{G}^c(B)$  do not have components that are not formed this way.<sup>3</sup>

**Lemma 7.3.** *Let  $H$  be a component of  $\mathcal{G}^c(A)$ .*

- (i)  $g(H) = g(H')$ .
- (ii) *If  $k$  belongs to a closed walk  $C$  on  $H$ , then there are edges  $(n + \alpha, k)$  and  $(k, n + \beta) \in \mathcal{G}^c(Z)$ , such that  $\alpha$  and  $\beta$  belong to a closed walk  $\tilde{C}$  in  $H'$ , with  $\ell(C) = \ell(C')$ .*

**Proof.** Take a closed walk  $C$  on  $H$ . Each edge of  $C$  results from a path of  $\mathcal{G}^c(Z)$  of length 2, and inserting this path in  $C$  we obtain a closed walk of  $\mathcal{G}^c(Z)$  (see Fig. 2, left). This walk contains nodes from both  $H$  and  $H'$ . In  $Z^{\otimes 2}$  it splits in two closed walks of  $\mathcal{G}^c(Z^{\otimes 2})$  of the same length (see Fig. 2, right). One of these closed walks is  $C$  and the other is a closed walk  $\tilde{C}$  of  $H'$  (since  $H$  and  $H'$  are isolated in  $\mathcal{G}^c(Z^{\otimes 2})$ ).

This implies  $g(H') \leq g(H)$ , and the reverse inequality follows by symmetry, hence part (i).<sup>4</sup> It also follows that each node of the original cycle in  $H$  has neighbors belonging to a closed walk in  $H'$ . Since each node of  $H$  lies on a cycle, we have part (ii). □

Define

$$h = \min(|H|, |H'|). \tag{7.4}$$

We are ready for the proof of Main Theorem 2.

**Proof of Main Theorem 2.** Let  $Z$  and  $B$  be the matrices defined in (7.1) and (7.3). Let  $k$  be an index in  $H$  (belonging to  $\{1, \dots, n\}$ ). By Lemma 7.3 part (ii), there is an edge of  $\mathcal{G}^c(Z)$  connecting it to some node  $n + \beta$ , for  $\beta \in \{1, \dots, r\}$ , which belongs to  $H'$ . For each bound of Main Theorem 1, an application of Lemma 7.1 yields a version of the corresponding bound of Main Theorem 2 on  $T_k$ , where  $d$  is the cyclicity of  $B$ , and we have  $g(H')$  instead of  $g(H)$  and  $|H'|$  instead of  $h$ .

However,  $g(H') = g(H)$  and  $\mathcal{G}(B)$  has the same cyclicity as  $\mathcal{G}(A)$  (when  $A$  and hence also  $B$  are irreducible), so it only remains to explain why we have  $h$  (and not  $|H'|$ ), in the factor rank versions of the bounds of Dulmage and Mendelsohn, and Kim. The following argument accounts for both cases (set  $d = 1$  for Dulmage and Mendelsohn's bound).

First, let  $k$  belong to a cycle  $C$  with length  $g(H)$ . By Lemma 7.3 part (i),  $\beta$  belongs to a closed walk  $\tilde{C}$  with length  $g(H) = g(H')$  in  $H'$ , so we can apply Lemma 4.1 to  $\beta$  and get  $T_\beta(B) \leq g(H)(\lfloor r/d \rfloor - 1) + (r \bmod d)$  (for the bound of Kim), or apply Lemma 3.2 and get  $T_\beta(B) \leq g(H)(r - 1)$  (for the bound of Dulmage and Mendelsohn). By Lemma 7.1, we get

$$T_k(A) \leq g(H)(\lfloor r/d \rfloor - 1) + (r \bmod d) + 1.$$

<sup>3</sup> In fact, we also have  $\lambda(A) = \lambda(B) = \lambda(Z^{\otimes 2}) = (\lambda(Z))^2$ .

<sup>4</sup> A similar argument shows that for any strongly connected graph  $\mathcal{G}$ , all components of  $\mathcal{G}^k$  have the same girth.

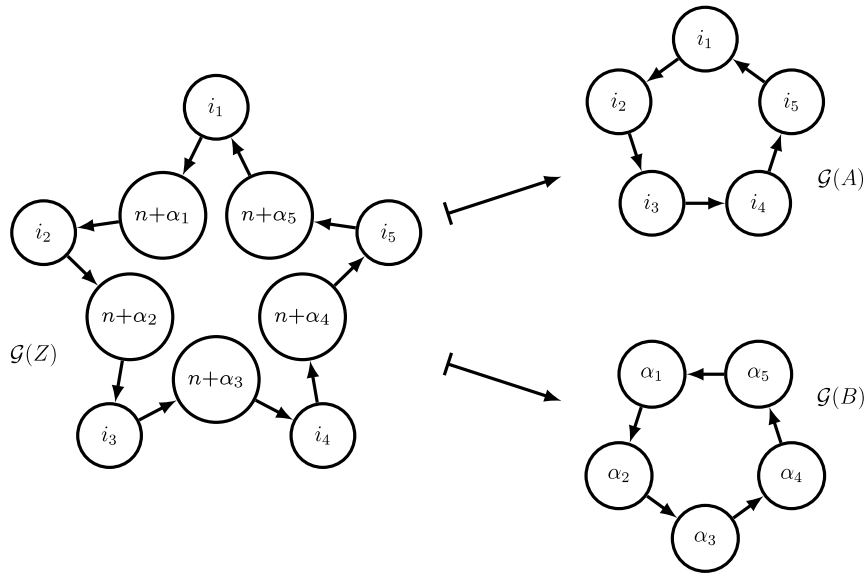


Fig. 2. Correspondence between closed walks of  $\mathcal{G}(Z)$ ,  $\mathcal{G}(A)$  and  $\mathcal{G}(B)$ .

Second, if  $k$  does not belong to such a cycle, we can apply Lemma 3.3, because  $k$  is connected to  $C$  by a path on  $H$  of length at most  $|H| - g(H)$ . Hence we obtain

$$T_k(A) \leq g(H)(\lfloor r/d \rfloor - 2) + (r \bmod d) + |H| + 1$$

and the factor rank versions of Kim’s and Dulmage–Mendelsohn’s bounds. The proof is complete.  $\square$

**8. On the precision of the bounds**

Since the bounds of Main Theorem 1 are extensions of the bounds on Boolean matrices and the latter are known to be exact (see for instance [15] and the references therein), so are the bounds of Main Theorem 1. However, the max-algebraic case is richer, and some natural questions arise. A general question is when these bounds are attained. To begin with, can these bounds be attained by the matrices whose critical graph does not attain the corresponding “Boolean” bound, or can the bounds be attained when not all the nodes are critical.

The easiest way to produce max-plus examples from Boolean ones is to use the semigroup morphism  $\phi_0 : \mathbb{R}_+^{n \times n} \rightarrow \mathbb{B}^{n \times n}$  that maps  $A$  to its pattern  $B = \phi_0(A)$  such that  $b_{ij} = 0$  if and only if  $a_{ij} = 0$  and  $b_{ij} = 1$  otherwise. Since it is a morphism, we have:

**Lemma 8.1.** *Let  $A, B \in \mathbb{R}_+^{n \times n}$  have the same pattern and let  $A$  be Boolean. Then  $T_k(A) \leq T_k(B)$  for all  $k = 1, \dots, n$ .*

To illustrate the use of Lemma 8.1, consider an example from the work of Schwarz [18], attaining the corresponding bound. It is a strongly connected graph consisting of two cycles, of lengths 6 and 4, displayed in the left part of Fig. 3. The greatest transient of a row of the associated Boolean matrix is  $T_4(A) = 11$ , which is equal to Schwarz’s bound of Theorem 2.11 with  $n = 7$  and  $d = 2$ . Now, let  $B$  be a matrix with pattern  $A$  such that node 4 is critical. On one hand, Main Theorem 1 ensures that  $T_4(B) \leq 11$ . On the other hand, by Lemma 8.1, we have  $T_4(B) \geq T_4(A) = 11$ . Thus Schwarz’s bound is attained by  $T_4(B)$ . In particular, consider any nonnegative matrix  $B$  where all entries of the bigger cycle are equal to 1, and the two remaining nonzero entries are less than or equal to 1. The associated digraphs of  $A$  and  $B$  are displayed in Fig. 3. It can be checked by direct computation that  $T_4(B) = 11$ . More examples of this kind can be constructed using the work of Shao and Li [23]. Observe that not all the nodes of the graph on the right-hand side of Fig. 3 are critical, but it does attain the greatest possible transient of critical rows, because node 4 is critical.

Another way to produce examples is to use the map  $\phi_1 : \mathbb{R}_+^{n \times n} \rightarrow \mathbb{B}^{n \times n}$  that maps  $A$  to  $B = \phi_1(A)$  such that  $b_{ij} = 1$  if and only if  $a_{ij} = 1$  and  $b_{ij} = 0$  otherwise. It is not a morphism on  $\mathbb{R}_+^{n \times n}$  but it is not difficult to check (generalizing Lemma 2.29) that it is a morphism on the following semigroups of  $\mathbb{R}_+^{n \times n}$ , defined for each subset  $X$  of  $N$ :

$$\mathcal{S}_X = \{A \in \mathbb{R}_+^{n \times n} \mid \lambda(A) = 1, A \text{ is strictly visualized, } N_c(A) = X\}$$

For any matrix  $A$  in  $\mathcal{S}_X$  the critical edges have weight 1, so that  $\mathcal{G}^c(A) = \mathcal{G}^c(\phi_1(A))$ . Obviously, if  $\lambda(A) = 1$  and  $A$  is strictly visualized,  $A \in \mathcal{S}_{N_c(A)}$ , so that any such matrix satisfies  $T_k(A) \geq T_k(\phi_1(A))$ . It extends to general matrices in the following way.

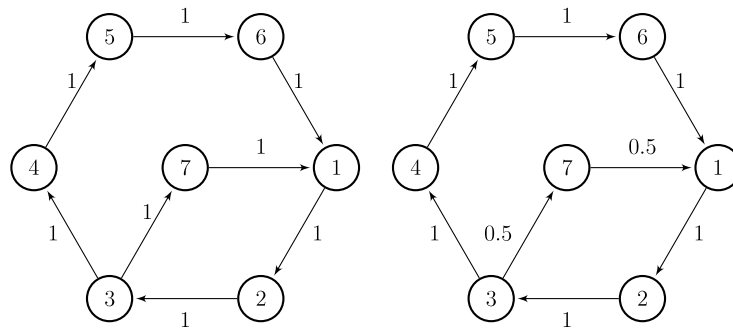


Fig. 3. Schwarz's example (left) and its max-algebraic version (right).

For  $A \in \mathbb{R}_+^{n \times n}$ , let  $A^C$  be the *critical matrix* of  $A$ , with entries

$$a_{ij}^C = \begin{cases} 1, & \text{if } (i, j) \in E_c(A), \\ 0, & \text{otherwise.} \end{cases}$$

**Lemma 8.2** (Cf. [6, Corollary 2.9]).  $T_k(A^C) \leq T_k(A)$  for all  $k \in N_c(A)$ .

Lemma 8.2 shows that if an unweighted digraph  $\mathcal{G}^c$  on  $n$  nodes attains a given bound for some  $T_k$ , then any  $n \times n$  matrix with entries in  $[0, 1]$ , whose critical graph is  $\mathcal{G}^c$ , attains it as well (for the same  $k$ ). In the following example, the first matrix attains Wielandt's bound ( $T_5(A) = 17$ ) second matrix attains Dulmage and Mendelsohn's bound ( $T_4(B) = 14$ ), since their critical graphs attain the corresponding Boolean bounds.

$$A = \begin{pmatrix} 0 & 1 & 0.1 & 0.2 & 0 \\ 0 & 0 & 1 & 0 & 0.3 \\ 0 & 0.4 & 0 & 1 & 0 \\ 1 & 0 & 0.5 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0.9 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0.4 & 0.5 & 0 \\ 0.1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0.2 & 1 & 0.7 \\ 0 & 0.9 & 0 & 0 & 1 \\ 1 & 0.2 & 0.6 & 0.7 & 0 \end{pmatrix}.$$

We conclude that the two lemmas provide us with some classes of matrices attaining the bounds of Main Theorem 1. However, this characterization is far from being complete and leaves vast possibilities of research.

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