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On sets of eigenvalues of matrices with prescribed row sums and prescribed graph

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ABSTRACT

Motivated by a work of Boros, Brualdi, Crama and Hoffman, we consider the sets of (i) possible Perron roots of nonnegative matrices with prescribed row sums and associated graph, and (ii) possible eigenvalues of complex matrices with prescribed associated graph and row sums of the moduli of their entries. To characterize the set of Perron roots or possible eigenvalues of matrices in these classes we introduce, following an idea of Al'pin, Elsner and van den Driessche, the concept of row uniform matrix, which is a nonnegative matrix where all nonzero entries in every row are equal. Furthermore, we completely characterize the sets of possible Perron roots of the class of nonnegative matrices and the set of possible eigenvalues of the class of complex matrices under study. Extending known results to the reducible case, we derive new sharp bounds on the set of eigenvalues or Perron roots of matrices when the only information available is the graph of the matrix and the row sums of the moduli of its entries. In the last section of the paper a new constructive proof of the Camion–Hoffman theorem is given.

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1. Introduction

1.1. Background and motivation

The use of the row sums of a matrix to determine nonsingularity or to bound its spectrum has its origins in the 19th century [19, Section 2] and has led to a vast literature associated with the name of Geršgorin and his circles [22]. One of the first observations, due to Frobenius, was that the Perron root $\rho(A)$ (i.e., the biggest nonnegative eigenvalue, or the spectral radius) of a nonnegative matrix $A \in \mathbb{R}_+^{n\times n}$ is bounded by

$$\min_{i=1}^{n} r_i(A) \leq \rho(A) \leq \max_{i=1}^{n} r_i(A) \tag{1}$$

where r_i denotes the ith row sum of the elements of A. If A is irreducible then the inequalities in (1) are strict except when $\min_{i=1}^n r_i(A) = \max_{i=1}^n r_i(A)$.

In a recent development, Al'pin [2], Elsner and van den Driessche [12] sharpened the classical bounds of Frobenius by considering a matrix B which has the same zero-nonzero pattern as A, and whose entries are equal to the row sums of A in the corresponding rows. We formalize this idea in the following definition.

Definition 1.1. For $A \in \mathbb{R}_+^{n\times n}$ we define the *auxiliary matrix* $B = \operatorname{Aux}(A)$ defined by

$$\begin{cases} b_{ij} = \sum_k a_{ik}, & \text{if } a_{ij} \neq 0, \\ b_{ij} = 0, & \text{if } a_{ij} = 0. \end{cases} \tag{2}$$

For a general complex matrix $A \in \mathbb{C}^{n\times n}$, its auxiliary matrix is defined as $\operatorname{Aux}(|A|)$.

Next, recall the concepts of minimal and maximal cycle (geometric) means. For an arbitrary matrix $A \in \mathbb{R}_+^{n\times n}$ these quantities are defined as follows

$$\nu(A) = \min_{(i_1,\ldots,i_\ell)\in C(A)} (a_{i_1 i_2} \cdot a_{i_2 i_3} \cdot \ldots \cdot a_{i_\ell i_1})^{1/\ell},$$

$$\mu(A) = \max_{(i_1,\ldots,i_\ell)\in C(A)} (a_{i_1 i_2} \cdot a_{i_2 i_3} \cdot \ldots \cdot a_{i_\ell i_1})^{1/\ell}, \tag{3}$$

where $C(A)$ denotes the set of cycles of the associated graph. Recall that the directed weighted graph, associated with an arbitrary complex matrix $A \in \mathbb{C}^{nn}$, is defined by the set of nodes $N = \{1, \ldots, n\}$ and set of edges E such that $(i, j) \in E$ if and only if $a_{ij} \neq 0$, in which case edge (i, j) is assigned the weight a_{ij}.

According to Al'pin [2], Elsner and van den Driessche [12], we have

$$\nu(B) \leq \rho(A) \leq \mu(B), \quad B = \mathrm{Aux}(A), \tag{4}$$

for any nonnegative matrix A. If A and hence B are irreducible then either $\nu(B) = \rho(A) = \mu(B)$ or (if $\nu(B) < \mu(B)$) the inequalities in (4) are strict.

Exploiting similar ideas, Boros, Brualdi, Crama and Hoffman [4] investigated a class of complex matrices $A \in \mathbb{C}^{n\times n}$ with prescribed off-diagonal row sums of the moduli of their entries, prescribed associated graph, and prescribed moduli of all diagonal entries. In the case when $\mathcal{G}(A)$ is **s**trongly **c**onnected **w**ith **a**t **l**east **t**wo **cy**cles (scwaltcy), they investigated the existence of a positive vector x satisfying

$$|a_{ii}|x_i \geq \sum_{j\neq i} |a_{ij}|x_j, \quad i = 1, \dots, n \tag{5}$$

for all matrices from the class simultaneously, and described the cases when all inequalities in (5) are strict [4, Theorem 1.1], at least one of the inequalities is strict [4, Theorem 1.2], or all inequalities hold with an equality [4, Theorem 1.3]. These results imply generalizations of Geršgorin's theorem due to Brualdi [5]. Following the statement of [4, Theorem 1.4] the authors provide a detailed outline for the proof that Brualdi's conditions are sharp.

In this paper we mainly deal with the two classes of matrices described in the abstract. These classes are similar to those in [4], but we drop the requirement that $\mathcal{G}(B)$ is scwaltcy. In particular we also handle the reducible (not strongly connected) case. However we do not prescribe the moduli of diagonal entries, and include these moduli in the row sums instead. This allows us, in particular, to combine the problem statement of Boros, Brualdi, Crama and Hoffman [4] with that of Al'pin [2], Elsner and van den Driessche [12] and to generalize all above mentioned results removing the restriction that B is irreducible. The main results of this paper characterize the Perron roots or the sets of eigenvalues of the classes of matrices under consideration.

At the end of the paper we present a new constructive proof of the Camion–Hoffman theorem [9] (see also [11]). This theorem characterizes regularity of a class of complex matrices with prescribed moduli of their entries. These ideas are also presented in a very accessible form in [6, Chapter 7]. The scaling result of Section 2.3 is crucial for our new proof (which also makes use of one of the previously mentioned characterization results). Since we are dealing with complex rather than with nonnegative matrices here, the triangle inequality (implicit in Lemma 4.10) also plays a role.

Other proofs of the Camion–Hoffman theorem have been given by Levinger and Varga [17], and Engel [13].

1.2. Contents of the paper

The rest of this paper is organized as follows. Section 1.3 is a reminder of the Frobenius normal form of nonnegative matrices.

Section 2 is devoted to a form of diagonal similarity scaling called visualization scaling [20] or Fiedler–Pták scaling [15] (see also [1]). Interest in this scaling has been motivated by its use in max algebra, see for example [7] and [8]. Lemmas 2.4 and 2.5 can be used to generalize the simultaneous scaling results of Boros, Brualdi, Crama and Hoffman [4, Theorems 1.1–1.3] to include the reducible case. This also yields a derivation of the bounds of Al'pin, Elsner and van den Driessche (Theorem 2.6). Theorem 2.8 establishes the existence of an advanced visualization scaling, which is applied in the proof of the Camion–Hoffman theorem.

In Section 3 we consider the class of nonnegative matrices with prescribed graph and prescribed row sums. Theorem 3.7 characterizes the set of possible Perron roots of such matrices also when B is reducible. This is one of the main results of this paper. The proof is based on analyzing the sunflower subgraphs of $\mathcal{G}(B)$, a technique well-known in max algebra [16]. As an immediate corollary it follows from Theorem 3.7 that for irreducible B with $\nu(B) < \mu(B)$ and any r, $\nu(B) < r < \mu(B)$ there exists A with $\mathrm{Aux}(A) = B$ such that $\rho(A) = r$.

In Section 4.1 we consider the class of complex matrices with prescribed graph and prescribed row sums of the moduli of their entries. We seek a characterization of the set of nonzero eigenvalues of such matrices, starting with the irreducible case in Theorem 4.4. In this case we show in particular that when B has more than one cycle, the set of possible nonzero eigenvalues of A satisfying $\mathrm{Aux}(A) = B$ consists either of all s satisfying $0 < |s| < \mu(B)$ when $\nu(B) < \mu(B)$, or $0 < |s| \leq \mu(B)$ if $\nu(B) = \mu(B)$. Then, based on the irreducible case the full characterization in the reducible case is given in Theorem 4.9. In addition to this, the occurrence of a 0 eigenvalue is treated in Theorem 4.2.

In Section 4.2 a new proof of the Camion–Hoffman theorem [9] is given, based on the advanced visualization scaling of Section 2.3 and the characterization result of Theorem 4.9.

1.3. Frobenius normal form

Let A be a square nonnegative matrix. If A is irreducible (i.e., the associated digraph is strongly connected) then according to the Perron–Frobenius theorem A has a unique (up to a multiple) positive eigenvector corresponding to the Perron root $\rho(A)$ (which is also the greatest modulus of all eigenvalues of A). If A is reducible then by means of simultaneous permutations of rows and columns or, equivalently, an application of $P^{-1}AP$ similarity where P is a permutation matrix, A can be brought to the following form:

$$\begin{pmatrix} A_1 & 0 & 0 & 0 \\ * & A_2 & 0 & 0 \\ * & * & \ddots & 0 \\ * & * & * & A_m \end{pmatrix},$$

where the square blocks $A_1, \ldots, A_m$ correspond to the maximal strongly connected components of the associated graph. These diagonal blocks $A_1, \ldots, A_m$ will be further referred to as *classes* of A. Note that each class A_i is either a nonzero irreducible matrix, in which case it is called *nontrivial*, or a zero diagonal entry (and then it is called *trivial*). If some component $\mathcal{G}(A_i)$ of the associated graph $\mathcal{G}(A)$ does not have access to any other component, which means that there is no edge connecting one of its nodes to a node in another component, then this component or the corresponding class A_i are called *final*. Otherwise, this component or the corresponding class are called *transient*.

The entries denoted by 0 are actually off-diagonal blocks of zeros of appropriate dimension, and $*$ denote submatrices of appropriate dimensions whose zero-nonzero pattern is unimportant.

2. Visualization scaling

2.1. Visualization of auxiliary matrices

In this section we assume that A is a nonnegative matrix such that $\mathcal{G}(A)$ contains at least one cycle. Let us introduce some terminology related to max algebra and visualization.

Definition 2.1. For a nonnegative matrix A, the critical graph $\mathcal{C}(A) = (N_c(A), E_c(A))$ is defined as the subgraph of $\mathcal{G}(A)$ consisting of all nodes $N_c(A)$ and edges $E_c(A)$ on the cycles whose geometric mean equals $\mu(A)$. These nodes and edges are also called critical. A node is called strictly critical if all edges emanating from it are critical.

Similarly, by anticritical graph we mean the subgraph of $\mathcal{G}(A)$ consisting of all nodes and edges on the cycles whose geometric mean equals $\nu(A)$ (also speaking of anticritical nodes and edges). A node is called strictly anticritical if all edges emanating from it are anticritical.

Definition 2.2. A positive vector x is called a visualizing, resp. strictly visualizing, vector of A if $a_{ij}x_j \leq \mu(A)x_i$ for all $(i,j) \in E(A)$, resp. if also $a_{ij}x_j = \mu(A)$ if and only if (i,j) is critical.

Existence of such vector was proved by Engel and Schneider [14, Theorem 7.2] in the irreducible case, and was extended to reducible matrices in [20].

Definition 2.3. A positive vector x is called an antivisualizing, resp. a strictly antivisualizing, vector of A if $a_{ij}x_j \geq \nu(A)x_i$ for all $(i,j) \in E(A)$, resp. if also $a_{ij}x_j = \nu(A)$ if and only if (i,j) is anticritical.

An existence of such scaling follows from the existence of visualization scaling, applied to a matrix resulting from A after elementwise inversion of the entries.

The following lemmas are based on the results on simultaneous scaling found in [4, Theorems 1.1–1.3]. We make arguments of [4] more precise by basing them on the existence of strictly visualizing vectors [20].

Lemma 2.4. *(Cf. [4].) Let A be a nonnegative matrix and let $B = \mathrm{Aux}(A)$ with $\mu(B) \neq 0$. Let x be a strictly visualizing vector of B. Then we have $Ax \leq \mu(B)x$ and, more precisely, $(Ax)_i = \mu(B)x_i$ if i is a strictly critical node of B and $(Ax)_i < \mu(B)x_i$ otherwise.*

Proof. Assume that $\mu(B) = 1$. Then

$$\begin{aligned} \max_j \frac{b_{ij}x_j}{x_i} &\leq 1 \quad \text{for all } i, \\ \max_j \frac{b_{ij}x_j}{x_i} &= 1 \quad \text{for all critical } i. \end{aligned} \tag{6}$$

If i is strictly critical, we have

$$\forall j : (i,j) \in E(A)\ \frac{b_{ij}x_j}{x_i} = 1, \tag{7}$$

which implies that $x_j = x_k$ for all j and k such that both $(i,j) \in E(A)$ and $(i,k) \in E(A)$. Hence we can take any k with $(i,k) \in E(A)$, and obtain

$$\frac{\sum_j a_{ij}x_j}{x_i} = \frac{(\sum_j a_{ij})x_k}{x_i} = \frac{b_{ik}x_k}{x_i} = 1 \tag{8}$$

If i is not strictly critical then let us denote

$$x_k = \max_j \{x_j \colon (i,j) \in E(A)\}. \tag{9}$$

If i is not critical then

$$\frac{\sum_j a_{ij}x_j}{x_i} \leq \frac{(\sum_j a_{ij})x_k}{x_i} = \frac{b_{ik}x_k}{x_i} < 1 \tag{10}$$

If i is critical (but not strictly) then

$$\exists l, h \quad \frac{b_{il}x_l}{x_i} = 1, \qquad \frac{b_{ih}x_h}{x_i} < 1, \tag{11}$$

which implies $x_l = x_k > x_h$ for these l and h. In particular, note that we have $(i,k) \in E_c(B)$. Hence

$$\frac{\sum_j a_{ij}x_j}{x_i} < \frac{(\sum_j a_{ij})x_k}{x_i} = \frac{b_{ik}x_k}{x_i} = 1. \qquad \square \tag{12}$$

Lemma 2.5. *(Cf. [4].) Let A be a nonnegative matrix and let $B = \mathrm{Aux}(A)$ with $\nu(B) \neq 0$. Let x be a strictly antivisualizing vector of B. Then we have $Ax \geq \nu(B)x$ and, more precisely, $(Ax)_i = \nu(B)x_i$ if i is a strictly anticritical node of B and $(Ax)_i > \nu(B)x_i$ otherwise.*

2.2. Bounds of Al'pin, Elsner, van den Driessche

We call a nonnegative matrix A *truly substochastic*, if $\sum_j a_{ij} \leq 1$ for all i and $\sum_j a_{ij} < 1$ for some i. In a similar way, A is called *truly superstochastic* if $\sum_j a_{ij} \geq 1$ for all i and $\sum_j a_{ij} > 1$ for some i.

The following known result can be now obtained from Lemmas 2.4 and 2.5.

Theorem 2.6. *(See [2], [12, Theorem A].) Let A be an irreducible nonnegative matrix and let $B = \mathrm{Aux}(A)$.*

(i) *If $\mu(B) = \nu(B)$, then A is diagonally similar to a stochastic matrix multiplied by $\mu(B)$. In this case, $\rho(A) = \mu(B) = \nu(B)$.*
(ii) *If $\nu(B) < \mu(B)$, then A is diagonally similar to a truly substochastic matrix multiplied by $\mu(B)$. In this case, $\nu(B) < \rho(A) < \mu(B)$.*

Proof. (i): As B is irreducible and $\mu(B) = \nu(B)$, all nodes of $\mathcal{G}(B)$ are strictly critical. Taking any visualization[2] x of B we have $Ax = \mu(B)x$, which implies $\rho(A) = \mu(B) = \nu(B)$. We also have that $X^{-1}AX$, with $X = \mathrm{diag}(x)$, is a stochastic matrix multiplied by $\rho(A) = \mu(B) = \nu(B)$.

(ii): As $\mu(B) > \nu(B)$, not all nodes of $\mathcal{G}(B)$ are strictly critical. Taking any strictly visualizing vector x of B we have $Ax \leq \mu(B)x$ where $(Ax)_i < \mu(B)x_i$ for some i. We also have that $X^{-1}AX$ with $X = \mathrm{diag}(x)$, is a truly substochastic matrix multiplied by $\mu(B)$, as claimed. As $X^{-1}AX$ is also irreducible, it follows that $\rho(A) = \rho(X^{-1}AX) < \mu(B)$. The inequality $\rho(A) < \mu(B)$ can be also obtained (following an argument found, for instance in [12]) by multiplying the system $Ax \leq \mu(B)x$, where at least one of the inequalities is strict, from the left by a row vector z such that $zA = \rho(A)z$ (which does not have 0 components if A is irreducible).

Not all nodes of $\mathcal{G}(B)$ are strictly anticritical, either. Taking any strictly antivisualizing vector y of $B = \mathrm{Aux}(A)$ we have $Ay \geq \nu(B)y$ where $(Ay)_i > \nu(B)y_i$ for some i. We also

[2] Not necessarily strict.

have that $Y^{-1}AY$ with $Y = \operatorname{diag}(y)$, is a truly superstochastic matrix multiplied by $\mu(B)$, as claimed. As $Y^{-1}AY$ is also irreducible, it follows that $\rho(A) = \rho(Y^{-1}AY) > \nu(B)$. The inequality $\rho(A) > \nu(B)$ can be also obtained by multiplying the system $Ay \geq \mu(B)y$, where at least one of the inequalities is strict, from the left by a row vector z such that $zA = \rho(A)z$ (which does not have 0 components if A is irreducible). □

2.3. Sum visualization

Definition 2.7. For $A \in \mathbb{R}_+^{n\times n}$ and $a > 0$, a vector $x \in \mathbb{R}_+^n$ is called an a-sum visualizing vector of A, if the entries of $C = X^{-1}AX$ with $X = \operatorname{diag}(x)$ satisfy $c_{ij} \leq a$ for all i, j and $\sum_j c_{ij} \geq a$ for all i. In this case C is called an a-sum visualization of A.

Recall that we have $\mu(A) \leq \rho(A)$ for any nonnegative matrix. Indeed, since for any positive x and any cycle $(i_1, \dots i_\ell)$ we have that

$$\left(a_{i_1 i_2}\frac{x_{i_2}}{x_{i_1}} \cdot a_{i_2 i_3}\frac{x_{i_3}}{x_{i_2}} \cdot \dots \cdot a_{i_\ell i_1}\frac{x_{i_\ell}}{x_{i_1}}\right)^{1/\ell} \leq \left(\prod_{k\in\{i_1,\dots,i_\ell\}} \sum_j a_{kj}\frac{x_j}{x_k}\right)^{1/\ell},$$

it follows by taking x satisfying $Ax = \rho(A)x$, that $\mu(A) \leq \rho(A)$.

Theorem 2.8. *Let $A \in \mathbb{R}_+^{n\times n}$ be irreducible, and define $\alpha(A)$ as the set of positive numbers a for which an a-sum visualization of A exists. Then $\alpha(A) = [\mu(A), \rho(A)]$.*

Proof. 1. $\alpha(A) \subseteq [\mu(A), \rho(A)]$:
Let $a \in \alpha(A)$ and let $C = X^{-1}AX$ (for some diagonal X) be such that $c_{ij} \leq a$ for all i, j and $\sum_j c_{ij} \geq a$ for all i. Then $\mu(C) \leq a$ and $\rho(C) \geq a$, and as $\mu(A) = \mu(C)$ and $\rho(A) = \rho(C)$ we obtain that $a \in [\mu(A), \rho(A)]$.

2. $[\mu(A), \rho(A)] \subseteq \alpha(A)$:
Let $\mu(A) \leq a \leq \rho(A)$. We can assume without loss of generality (dividing A by a if necessary) that $a = 1$ and $\mu(A) \leq 1 \leq \rho(A)$.

As $\mu(A) \leq 1$, there exists a nonsingular diagonal matrix X such that all entries g_{ij} of $G := X^{-1}AX$ satisfy $0 \leq g_{ij} \leq 1$. Since G is diagonally similar to A, $\rho(A)$ is also the spectral radius of G and hence there exists a vector z whose entries z_i satisfy $1 = \max_i z_i$ and $\sum_j g_{ij}\frac{z_j}{z_i} \geq 1$ for all i.

We will now construct an entrywise nonincreasing sequence of vectors $\{y^{(s)}\}_{s\geq 0}$ bounded from below by z. Such a sequence obviously converges, and as we will argue, the limit denoted by y satisfies $g_{ij}\frac{y_j}{y_i} \leq 1$ for all i, j, and $\sum_j g_{ij}\frac{y_j}{y_i} \geq 1$ for all i (and, obviously, $y \geq z$).

Let us define a continuous mapping $f: (\mathbb{R}_+\backslash\{0\})^n \to (\mathbb{R}_+\backslash\{0\})^n$, by its components

$$f_i(x) = \min\left(x_i, \sum_j g_{ij}x_j\right), \quad i = 1, \dots, n. \tag{13}$$

Now let $y^{(0)} = (1, 1 \dots 1)$ and consider a sequence $\{y^{(s)}\}_{s \geq 0}$ defined by $y^{(s+1)} := f(y^{(s)})$ (that is, the orbit of $y^{(0)}$ under f).

Observe that $y^{(s+1)} \leq y^{(s)}$, as $f(x) \leq x$ for all $x \in (\mathbb{R}_+ \backslash \{0\})^n$.

It follows by induction that $y^{(s)} \geq z$ for all s. The case $s = 0$ is the basis of induction (since $z_i \leq 1$ for all i). We have to show that $y^{(s+1)} \geq z$ knowing that $y^{(s)} \geq z$. It amounts to verifying that $y_k^{(s+1)} \geq z_k$ for the indices k where $y_k^{(s+1)} < y_k^{(s)}$. For such indices we have

$$y_k^{(s+1)} = \sum_j g_{kj} y_j^{(s)} \geq \sum_j g_{kj} z_j^{(s)} \geq z_k.$$

As the sequence $\{y^{(s)}\}_{s \geq 0}$ is nonincreasing and bounded from below, it has a limit which we denote by y. As f is continuous, this limit satisfies $f(y) = y$, which by the definition of f implies that $\sum_j g_{ij} \frac{y_j}{y_i} \geq 1$ for all i.

We now show by induction that $g_{ij} \frac{y_j^{(s)}}{y_i^{(s)}} \leq 1$, for all $i \neq j$ and s. Denote by I_s the set of indices i where $\sum_j g_{ij} y_j^{(s)} < y_i^{(s)}$. Thus $y_i^{(s+1)} = \sum_j g_{ij} y_j^{(s)}$ and $y_i^{(s+1)} < y_i^{(s)}$ for $i \in I_s$, while $y_i^{(s+1)} = y_i^{(s)}$ for $i \notin I_s$.

Observe that $s = 0$ is the basis of induction, so we assume that the claim holds for s and we have to prove it for $s+1$. For $i, j \notin I_s$ the inequality $g_{ij} \frac{y_j^{(s+1)}}{y_i^{(s+1)}} \leq 1$ holds trivially. If $i \in I_s$ then

$$g_{ij} \frac{y_j^{(s+1)}}{y_i^{(s+1)}} \leq g_{ij} \frac{y_j^{(s)}}{y_i^{(s+1)}} = g_{ij} y_j^{(s)} \left(\sum_k g_{ik} y_k^{(s)} \right)^{-1} \leq 1$$

(where the last inequality follows since $g_{ij} y_j^{(s)}$ is just one of the nonnegative terms of the sum in the denominator).

Finally if $i \notin I_s$ and $j \in I_s$: then we have $g_{ij} \frac{y_j^{(s+1)}}{y_i^{(s+1)}} < g_{ij} \frac{y_j^{(s)}}{y_i^{(s+1)}} = g_{ik} \frac{y_j^{(s)}}{y_i^{(s)}} \leq 1$.

Thus the inequalities $g_{ij} \frac{y_j^{(s)}}{y_i^{(s)}} \leq 1$ hold for all $i \neq j$ and s, and this implies that for the limit point y, all the inequalities $g_{ij} \frac{y_j}{y_i} \leq 1$ hold as well. The case $i = j$ is trivial since the inequality $g_{ii} \frac{y_i}{y_i} = g_{ii} \leq 1$ holds for all i.

Let D be the diagonal matrix with $d_{ii} = y_i x_i$ for all i. For the entries c_{ij} of $C = D^{-1} A D$ we have $c_{ij} \leq 1$ for all i, j and $\sum_j c_{ij} \geq 1$ for all i so the theorem is proved. □

Denote by $A^{[-1]} = (a_{ij}^{[-1]})$ the Hadamard inverse of $A \in \mathbb{R}_+^{n \times n}$:

$$a_{ij}^{[-1]} = \begin{cases} \frac{1}{a_{ij}}, & \text{if } a_{ij} > 0, \\ 0, & \text{if } a_{ij} = 0. \end{cases}$$

Observe that $\mu(A^{[-1]}) = (\nu(A))^{-1}$ (however, there is no such inversion for the Perron root), and let us formulate the following corollary of Theorem 2.8.

Corollary 2.9. *Let $A \in \mathbb{R}_+^{n\times n}$ be irreducible. The following are equivalent:*

(i) $\frac{1}{a} \in [\frac{1}{\nu(A)}, \rho(A^{[-1]})]$;
(ii) $\exists x > 0$ *such that for* $C = X^{-1}AX$ *with* $X = \operatorname{diag}(x)$ *we have that* $c_{ij} \geq a$ *for all* i, j. *and* $\sum_j \frac{a}{c_{ij}} \geq 1$ *for all* i.

Proof. The corollary follows by elementwise inversion of the nonzero entries and applying Theorem 2.8. □

3. Nonnegative reducible matrices

Here we characterize Perron roots of nonnegative matrices with prescribed row sums and prescribed graph. Section 3.1 is devoted to sunflower graphs, which will be used in the proof of the main result. Section 3.2 contains the main result and example.

3.1. Sunflowers

We introduce the following definition, inspired by description of the Howard algorithm in [10] and [16, Chapter 6].

Definition 3.1. Let $\mathcal{G}$ be a weighted graph. A subgraph $\tilde{\mathcal{G}}$ of $\mathcal{G}$ is called a sunflower subgraph of $\mathcal{G}$ if the following conditions hold:

(i) If a node in $\mathcal{G}$ has an outgoing edge then it has a unique outgoing edge in $\tilde{\mathcal{G}}$.
(ii) Every edge in $\tilde{\mathcal{G}}$ has the same weight as the corresponding edge in $\mathcal{G}$.

It is easy to see [16] that such a digraph can be decomposed into several isolated components, each of them either acyclic or consisting of a unique cycle and some walks leading to it. A sunflower subgraph $\tilde{\mathcal{G}}$ of $\mathcal{G}$ is called a simple γ-sunflower subgraph of $\mathcal{G}$, if γ is the unique cycle of $\tilde{\mathcal{G}}$. The set of all sunflower subgraphs of the weighted digraph $\mathcal{G}(B)$, with full node set $1, \ldots, n$, will be denoted by $\mathcal{S}(B)$.

Denoting by $\mu(\mathcal{G})$ the maximal cycle mean of a subgraph $\mathcal{G} \subseteq \mathcal{G}(B)$, we introduce the following parameters:

$$M(B) := \max_{\mathcal{G}\in\mathcal{S}(B)} \mu(\mathcal{G}), \qquad m(B) := \min_{\mathcal{G}\in\mathcal{S}(B)} \mu(\mathcal{G}). \tag{14}$$

Lemma 3.2. *Let $\mathcal{G}$ be a strongly connected graph. Then, for any cycle γ of $\mathcal{G}$ there exists a simple γ-sunflower subgraph of $\mathcal{G}$.*

Proof. Let $\{1, \ldots, n\}$ be the nodes of $\mathcal{G}$. Suppose that $\{1, \ldots, k\}$ are the nodes in γ, and $k+1, \ldots, n$ are the rest of the nodes.

Observe first that we can construct a simple γ-sunflower on nodes $1, \dots, k$: this is just the cycle γ itself.

The proof is by contradiction. Assume that a simple γ-sunflower $\tilde{\mathcal{G}}$ can be constructed for a subgraph induced by the set of nodes M, which contains the nodes $1, \dots, k$ and is a **proper** subset of $\{1, \dots, n\}$, and that M is a maximal such set. However, since $\mathcal{G}$ is connected, there is a walk W from $\{1, \dots, n\} \backslash M$ to M, and we can pick the last edge of that walk and its last node before it enters M. Adding that node and that edge to $\tilde{\mathcal{G}}$ we increase it while it remains a simple γ-sunflower (of a subgraph induced by a larger node set). The contradiction shows that we can construct a simple γ-sunflower of $\mathcal{G}$. □

Let us also recall the following.

Lemma 3.3. *Let A be a nonnegative square matrix such that the digraph associated with A is a sunflower graph. Then $\rho(A) = \mu(A)$.*

Proof. Clearly, the cycles of $\mathcal{G}(A)$ are exactly the nontrivial classes of the Frobenius Normal Form. Hence it suffices to observe that $\rho(A) = \mu(A)$ if $\mathcal{G}(A)$ is a Hamiltonian cycle γ. Indeed, we can set $x_i = 1$ for any $i \in \gamma$ and then calculating all the rest of coordinates from the equalities $a_{ij}x_j = \mu(A)x_i$ for $a_{ij} \neq 0$. This computation does not lead to a contradiction, since $\mu(A)$ is the cycle mean of γ. □

The following proposition expresses $m(B)$ and $M(B)$ in terms associated with the Frobenius normal form.

Proposition 3.4. *Let B be a nonnegative matrix. Then*

$$M(B) = \mu(B), \qquad m(B) = \max_{N_i \text{ is final}} \nu(B_i). \tag{15}$$

Proof. $M(B)$: It is obvious from (14) that $M(B) \leq \mu(B)$. The reverse inequality $M(B) \geq \mu(B)$ follows since we can take a cycle α of $\mathcal{G}(B)$ whose cycle mean equals $\mu(B)$ and construct a sunflower subgraph of $\mathcal{G}(B)$ that contains α as one of its cycles.

$m(B)$: It is obvious from (14) that $m(B) \geq \max_{N_i \text{ is final}} \nu(B_i)$, since any sunflower subgraph of B contains a cycle in every nontrivial final class. So we show that $m(B) \leq \max_{N_i \text{ is final}} \nu(B_i)$. For this, in each submatrix B_i corresponding to a final class we take a cycle α_i whose mean value is $\nu(B_i)$ and using Lemma 3.2 build a simple α_i sunflower of the strongly connected component associated with B_i. Unite all these sunflowers. If B_i is not final then it has access to another class from some node k_i. In this case build a spanning tree on the nodes of B_i, directed to k_i, and for k_i choose an edge going to another class. Finally, for each trivial node of B_i we choose an arbitrary outgoing edge if it exists. Adjoin these spanning trees and outgoing edges to the above union of simple sunflowers. This leads to a sunflower subgraph $\mathcal{G}$ of $\mathcal{G}(B)$, for which we have $\mu(\mathcal{G}) = \max_{N_i \text{ is final}} \nu(B_i)$, hence $m(B) \leq \max_{N_i \text{ is final}} \nu(B_i)$ and the required equality follows. □

Remark 3.5. Observe that $m(B) = 0$ if and only if all final classes of B are trivial.

A sunflower subgraph which has cycles only in the final classes of $\mathcal{G}(B)$ will be called *thin*. In the proof of Proposition 3.4 we actually established the following result.

Lemma 3.6. *Let $\mathcal{G}$ be a graph where each node has an outgoing edge and let $\mathcal{G}_i$ for $i = 1, \dots, q$ be the nontrivial final components of $\mathcal{G}$.*

For each collection of cycles $\alpha_i \in \mathcal{G}_i$ for $i = 1, \dots, q$, there is a (thin) sunflower subgraph of $\mathcal{G}$ whose cycles are $\alpha_1, \dots, \alpha_q$.

If all final components of $\mathcal{G}$ are trivial then there exists an acyclic sunflower subgraph of $\mathcal{G}$ (i.e., a directed forest).

3.2. Range of the Perron root

For a row uniform nonnegative matrix B, denote

$$\eta(B) := \{\rho(A) \colon A \in \mathbb{R}_+^{n\times n},\ \mathrm{Aux}(A) = B\}. \tag{16}$$

We are going to extend Theorem 2.6 to include the reducible case and describe $\eta(B)$ for a general row uniform nonnegative matrix B.

Theorem 3.7. *Let B be a nonnegative row uniform matrix.*

(i) $\eta(B) \subseteq [m(B), M(B)]$.
(ii) *$M(B) \in \eta(B)$ if and only if there is at least one final class B_i with $\mu(B_i) = \nu(B_i) = M(B)$.*
(iii) *If $m(B) > 0$ then $m(B) \in \eta(B)$ if and only if $\mu(B_i) = \nu(B_i) = m(B)$ for all final (nontrivial) classes N_i attaining the maximum in (15). If $m(B) = 0$ then $m(B) \in \eta(B)$ of and only if $\mathcal{G}(B)$ is acyclic, in which case $\eta(B) = \{0\}$.*
(iv) *If $M(B) = m(B)$ then $\eta(B) = \{m(B)\}$.*
(v) *If $M(B) > m(B)$ then $(m(B), M(B)) \subseteq \eta(B)$.*

Proof. Throughout the proof, let A be such that $\mathrm{Aux}(A) = B$. Let A_i and B_i for $i = 1, \dots, m$ be the classes of the Frobenius normal form of A and B respectively, and let N_i be the corresponding node sets (or classes).

(i): We have to show that $\rho(A) \in [m(B), M(B)]$. Note that for any class A_i of A we have $\mathrm{Aux}(A_i) \le B_i$, and Theorem 2.6 implies that $\rho(A_i) \le \mu(B_i)$, but we do not have $\rho(A_i) \ge \nu(B_i)$ in general. However, $\mathrm{Aux}(A_i) = B_i$ holds for a final class, and hence $\nu(B_i) \le \rho(A_i) \le \mu(B_i)$ for any final class.

With above considerations, the inequality $\rho(A) \le M(B)$ follows since $M(B) = \mu(B)$ and $\rho(A_i) \le \mu(B_i)$ for all classes. To show that $\rho(A) \ge m(B)$ we first define matrix $\tilde{A}$

formed from A by zeroing out all the entries except for the entries in final classes, and we similarly define $\tilde{B} = \mathrm{Aux}(\tilde{A})$. Then we have $\rho(A) \geq \rho(\tilde{A})$ by monotonicity of the spectral radius. Since $m(B) = \max_{N_i \text{ is final}} \nu(B_i)$ by (15) and $\rho(A_i) = \rho(\tilde{A}_i) \geq \nu(\tilde{B}_i) = \nu(B_i)$ for each final class we obtain that $\rho(A) \geq \rho(\tilde{A}) \geq m(B)$, hence the claim.

(ii): When $\mathcal{G}(B)$ is acyclic the proof of (ii) is trivial. If $\mathcal{G}(B)$ is not acyclic then $M(B) = \mu(B) > 0$ and if $\rho(A_i) = \mu(B)$ then A_i must be nontrivial. We first argue that $\rho(A_i) = M(B)$ is impossible if A_i has access to other classes. Indeed, if there is such access then we only have $\mathrm{Aux}(A_i) \leq B_i$ with strict inequalities in some rows. This implies that we can find A'_i such that $\mathrm{Aux}(A'_i) = B_i$ and $A'_i \geq A_i$, with strict inequalities in the same rows. But then we have $\rho(A_i) < \rho(A'_i) \leq \mu(B_i)$ so $\rho(A_i) = \mu(B)$ is impossible. Thus $\rho(A_i) = \mu(B)$ can be attained only in a final class, which happens if and only if $\rho(A_i) = \mu(B_i)$, and by Theorem 2.6, if and only if $\mu(B_i) = \nu(B_i) = \mu(B)$ for one such class.

(iii): In the case when $m(B) = 0$ but B has at least one nontrivial class B_i, we have $\rho(A_i) > 0$ and hence $\rho(A) > 0$ for any A such that $\mathrm{Aux}(A) = B$. Therefore in this case $m(B) = 0 \in \eta(B)$ if and only if all classes of B are trivial (i.e., $\mathcal{G}(B)$ is acyclic).

If $m(B) > 0$, we first show that the given condition is necessary: if $\nu(B_i) < \mu(B_i)$ for at least one of these final classes then we have $\nu(B_i) < \rho(A_i) < \mu(B_i)$ by Theorem 2.6, hence $m(B) < \rho(A_i) \leq \rho(A)$, so $m(B) = \rho(A)$ does not hold. Following the proof of Proposition 3.4, we can construct a thin sunflower subgraph of $\mathcal{G}(B)$ with the cycles attaining $\nu(B_i)$ in all final classes. Denote the matrix associated with this subgraph by C and the submatrices extracted from the node sets N_i by C_i. For each submatrix C_i with N_i not final, we have $\rho(C_i) = 0$. By the continuity of Perron root we can find a small enough ϵ such that $\rho((1-\epsilon)C_i + \epsilon A_i)$ is smaller than $m(B)$ for all classes that are not final and for all classes that are final but have $\nu(B_i) < m(B)$. This is while $\mathrm{Aux}((1-\epsilon)C + \epsilon A) = B$ and, by Theorem 2.6, $\rho((1-\epsilon)C_i + \epsilon A_i) = m(B)$ for all classes where the maximum in (15) is attained. This implies that $\rho((1-\epsilon)C + \epsilon A) = m(B)$, hence the claim.

(iv): By part (i), $\rho(A)$ can be only equal to $m(B) = M(B)$. However, the set of A such that $\mathrm{Aux}(A) = B$ is nonempty for any row uniform B, hence the claim.

(v): Let us first observe that by definition of $m(B)$ and $M(B)$ (14), there exist matrices $\underline{A}$ and $\overline{A}$ whose associated graphs are the sunflower subgraphs of $\mathcal{G}(B)$ attaining the maximum and the minimum value of $\mu(\mathcal{G})$ over all possible sunflower subgraphs of $\mathcal{G}(B)$. By Lemma 3.3 we have that $\rho(\underline{A}) = m(B)$ and $\rho(\overline{A}) = M(B)$.

Now we argue that there exists A_0 with $\mathrm{Aux}(A_0) = B$ and $\rho(A_0)$ arbitrarily close to $m(B) = \rho(\underline{A})$. Indeed, let D be any matrix with $\mathrm{Aux}(D) = B$, and consider the family of matrices $C_\epsilon = (1-\epsilon)\underline{A} + \epsilon D$ for $\epsilon > 0$. Then (for any $\epsilon > 0$) we have $\mathrm{Aux}(C_\epsilon) = B$ and since $\rho(C_\epsilon)$ is a continuous function of ϵ it follows that $\lim_{\epsilon \to 0} \rho(C_\epsilon) = \rho(\underline{A})$. Similarly, there exists A_1 with $\mathrm{Aux}(A_1) = B$ and $\rho(A_1)$ arbitrarily close to $M(B) = \rho(\overline{A})$.

Thus for each ϵ we have some A_0 and A_1 with $\mathrm{Aux}(A_0) = \mathrm{Aux}(A_1) = B$ and $\rho(A_0) < m(B) + \epsilon$ and $\rho(A_1) > M(B) - \epsilon$. For λ, where $0 < \lambda < 1$, let $A_\lambda := \lambda A_1 + (1-\lambda)A_0$

interpolate between A_0 and A_1. Since $\mathrm{Aux}(A_\lambda) = B$ for each λ and $\rho(A_\lambda)$ is continuous in λ, the claim follows. $\square$

As an immediate corollary we obtain the following result in the irreducible case.

Corollary 3.8. *Let B be an irreducible nonnegative row uniform matrix. Then*

(i) *If $\nu(B) < \mu(B)$ then $\eta(B) = (\nu(B), \mu(B))$.*
(ii) *If $\nu(B) = \mu(B)$ then $\{\nu(B)\} = \eta(B) = \{\mu(B)\}$.*

Example. Given an irreducible row uniform matrix $B \in \mathbb{R}_+^{n\times n}$ and a constant $\rho \in (\nu(B); \mu(B)) = (m(B); M(B))$, we describe a method for constructing a matrix A such that $\mathrm{Aux}(A) = B$ and $\rho(A) = \rho$. Take two simple γ-sunflowers: one where γ has cycle mean equal to $\mu(B)$, and the other where γ has cycle mean equal to $\nu(B)$. Denote by A_1 the matrix associated with the first sunflower, and by A_2 the matrix associated with the second sunflower. We have $\rho(A_1) = \mu(B)$ and $\rho(A_2) = \nu(B)$. For the convex combinations of these matrices, we have that $\rho(A_\lambda)$, where $A_\lambda := (1-\lambda)A_1 + \lambda A_2$ and $0 \le \lambda \le 1$, will assume all values between $\nu(B)$ and $\mu(B)$. This follows from the continuity of spectral radius as a function of λ (as in the more general construction above). The value of λ for which $\rho(A_\lambda) = \rho$, can be found from the system $A(\lambda)x = \rho x$, which has $n+1$ variables (n components of x and the parameter λ). However, since x can be multiplied by any scalar, one of the coordinates of x can be chosen equal to 1. Then, for at least one of such choices, the existence of solution is guaranteed.

For example, consider

$$B = \begin{pmatrix} 0 & 8 & 8 & 0 & 8 \\ 2 & 0 & 0 & 2 & 2 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 3 & 3 \\ 0 & 3 & 0 & 0 & 0 \end{pmatrix} \tag{17}$$

We see that the cycle $(1, 2)$ is critical, with the cycle mean $\mu(B) = 4$, and the cycle $(2, 5)$ is anticritical with the cycle mean $\nu(B) = \sqrt{6}$. For the matrices A_1 and A_2 associated with the corresponding sunflower graphs, we can take

$$A_1 = \begin{pmatrix} 0 & 8 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \end{pmatrix} \tag{18}$$

Equation $A_\lambda x = \rho x$, where we put $x_1 = 1$, can be written as

$$\begin{pmatrix} 0 & 8 & 0 & 0 & 0 \\ 2-y & 0 & 0 & 0 & y \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \rho \begin{pmatrix} 1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}, \tag{19}$$

where $y \in [0, 2]$ (so that $y = 2\lambda$). Observe that A_λ is irreducible, so the existence of a solution with $x_1 = 1$ (as well as a solution with any other component set to 1) is guaranteed.

System (19) can be solved explicitly. Indeed, from the first equation of this system we have $8x_2 = \rho$ so $x_2 = \rho/8$, from the third equation we have $2 = \rho x_3$ so $x_3 = 2/\rho$, from the fourth and the fifth equation we have $3x_2 = 3\rho/8 = \rho x_4 = \rho x_5$ so $x_4 = 3/8 = x_5$. Using the second equation of the system, we obtain $2 - y + (3/8)y = \rho x_2 = (\rho^2)/8$. Thus $y = \frac{16-\rho^2}{5}$.

4. Complex matrices

In Section 4.1 we characterize the set of eigenvalues of complex matrices with prescribed graph and prescribed row sums of the moduli of their entries.

In Section 4.2, a new proof of the Camion–Hoffman theorem is presented.

4.1. Complex matrices with prescribed row sums of moduli

Definition 4.1. For B a row uniform nonnegative matrix, let $\sigma(B)$ denote the set $\sigma(B) = \{\lambda\colon \exists A \in \mathbb{C}^{n\times n}, \mathrm{Aux}(|A|) = B, \det(A - \lambda I) = 0\}$.

Here $|A|$ denotes the matrix whose entries are the moduli of (complex) entries of A.

We first consider the conditions when $0 \in \sigma(B)$. In what follows, the imaginary number "i" is denoted by $\Im$. By a *generalized diagonal product* of B we mean a product of the form $\prod_{i=1}^{n} b_{i\sigma(i)}$ where σ is an arbitrary permutation of $\{1, \ldots, n\}$.

Theorem 4.2. *Let B be a row uniform nonnegative matrix. Then the following are equivalent:*

(i) $0 \in \sigma(B)$.
(ii) *The number of generalized nonzero diagonal products of B is not* 1.

Proof. Suppose the number of generalized nonzero diagonal products of B is one. Let A be such that $\mathrm{Aux}(A) = B$. The determinant of A equals the signed sum of the nonzero generalized diagonal products of A. Since all but one of the generalized diagonal products of A are zero, we have $\det(A) \neq 0$. Thus $0 \notin \sigma(B)$.

Suppose that B has no generalized nonzero diagonal products and let A be such that $\mathrm{Aux}(A) = B$. Since all the generalized diagonal products of A are zero, we have $\det(A) = 0$ and $0 \in \sigma(B)$.

Suppose that B has two or more nonzero generalized diagonal products. Let us permute the columns of B in order to put one of the generalized diagonal products on the (main) diagonal. In other words, consider BP where P is a permutation matrix and all diagonal entries of BP are nonzero. We have $\mathrm{Aux}(A) = B$ if and only if $\mathrm{Aux}(AP) = BP$, and $\det(A) = \det(AP)$, therefore $0 \in \sigma(B)$ if and only if $0 \in \sigma(BP)$. As BP has at least one nonzero diagonal product different from the main diagonal, the Frobenius normal form of BP has a nontrivial diagonal block of dimension greater than 1.

Denote the index set of that block by M, and let us take any row uniform nonnegative matrix $D = (d_{kl})$ such that $\mathrm{Aux}(D) = BP$. For each $k \in M$, denote by n_k the number of outgoing edges of the kth node in M in the associated digraph of BP that go to the nodes in M. As the block is irreducible and has all diagonal entries nonzero, we have $n_k > 1$. Let t_k be a bijection between the outgoing edges of k and $\{1, 2, \dots, n_k\}$, and define matrix $C = (c_{kl})$ by

$$c_{kl} = \begin{cases} d_{kl} \exp(\Im \frac{t_k(l)2\pi}{n_k}), & \text{if } k, l \in M \text{ and } d_{kl} \neq 0 \\ d_{kl}, & \text{otherwise.} \end{cases} \tag{20}$$

Then $\mathrm{Aux}(|C|) = BP$. In addition $C_{MM} v = 0$ where C_{MM} is the principal submatrix of C extracted from rows and columns with indices in M, and v is the vector with all components equal to 1. This implies that $\det(C) = \det(C_{MM}) = 0$ so $0 \in \sigma(BP)$ and $0 \in \sigma(B)$. □

We now describe $\sigma(B)\backslash\{0\}$ starting from the irreducible case.

Definition 4.3. An irreducible matrix B is called unicyclic if $\mathcal{G}(B)$ consists of a single Hamiltonian cycle, and multicyclic otherwise.

Theorem 4.4. *Let B be a row uniform nonnegative irreducible matrix.*

(i) *If B is unicyclic then $\sigma(B) = \{s: |s| = \mu(B)\}$.*
(ii) *If B is multicyclic and $\nu(B) < \mu(B)$ then $\sigma(B) \setminus \{0\} = \{s: 0 < |s| < \mu(B)\}$.*
(iii) *If B is multicyclic and $\nu(B) = \mu(B)$ then $\sigma(B) \setminus \{0\} = \{s: 0 < |s| \leq \mu(B)\}$.*

Proof. (i): In this case, all complex matrices A satisfying $\mathrm{Aux}(|A|) = B$ are formed by multiplying the entries of B (that is, the entries of its only cycle) by some complex numbers of modulus 1. The claim follows.

(ii), (iii): We first show that $\sigma(B)$ is contained in the above mentioned intervals. For that we first recall a known result of Frobenius (see e.g. [3, p. 31, Theorem 2.14]) that for any square complex matrix A, we have

$$\rho(A) := \max\{|\lambda| > 0 \colon \det(A - \lambda I) = 0\} \le \rho(|A|). \tag{21}$$

As $\mathrm{Aux}(|A|) = B$, Theorem 3.7 implies that $\rho(|A|) \le \mu(B)$ if $\mu(B) \in \eta(B)$ and $\rho(|A|) < \mu(B)$ if $\mu(B) \notin \eta(B)$. Combining these inequalities with (21), we have the desired inclusion.

We are left to show that each number in the intervals can be realized as an eigenvalue of a complex matrix A with $\mathrm{Aux}(|A|) = B$. Select $\lambda \in (0, \mu(B))$ if $\mu(B) \notin \eta(B)$ or $\lambda \in (0, \mu(B)]$ if $\mu(B) \in \eta(B)$.

If $\lambda \in \eta(B)$ where $\eta(B) = \{\mu(B)\}$ if $\nu(B) = \mu(B)$ or $\eta(B)$ an interval whose interior is $(\nu(B), \mu(B))$, then there is an irreducible nonnegative matrix E such that $\mathrm{Aux}(E) = B$ with $\lambda = \rho(A)$.

In the remaining case $\lambda \le \nu(B)$ we will construct a row uniform matrix H so that $\nu(H) \le \lambda \le \mu(H)$. Since B has at least two cycles and it is irreducible, there exists a row with index belonging to one of those cycles and with at least two nonzero elements one of which must be on that cycle. Let t be the index of such row. Consider a cycle α going through that row, with cycle mean c and length ℓ. If we have $c \le \lambda$, it follows that $\nu(B) \le \lambda \le \mu(B)$ and we select $H = B$. If $c > \lambda$ then we multiply all entries of row t by z such that $c \cdot z^{1/\ell} = \lambda$. Let H be the resulting matrix, so we have $0 < \nu(H) \le \lambda \le \mu(H)$.

If $\nu(H) < \mu(H) \le \nu(B) < \mu(B)$ and $\lambda = \mu(H)$ then $\mu(H)$ is the new mean value of the cycle α, which previously had $c > \lambda$. In this case, the corresponding factor $z < 1$ can be slightly increased so that $\nu(H) < \lambda < \mu(H)$ is satisfied. If $\nu(H) < \mu(H)$ and $\lambda = \nu(H)$, then multiplying the row t by a value $1 - \epsilon$ for small enough ϵ we can also ensure that $\nu(H) < \lambda < \mu(H)$.

Thus we can assume that $\nu(H) = \lambda = \mu(H)$ or $\nu(H) < \lambda < \mu(H)$, where H is obtained from B by multiplying the row t with at least two nonzero entries by a nonnegative scalar $z \le 1$. Then by Theorem 3.7, there is a nonnegative matrix E with an eigenvector v such that $Ev = \lambda v$ and $\mathrm{Aux}(E) = H$, where row t has at least two nonzero entries that we denote by e_{tk} and e_{tl}. Since E is irreducible, all components of v are positive. We now modify row t of E to form a matrix C such that $\mathrm{Aux}(C) = B$ and $Cv = \lambda v$. Let x be such that

$$\sum_{s \ne k,l} e_{ts} + \sqrt{e_{tk}^2 + (x/v_k)^2} + \sqrt{e_{tl}^2 + (x/v_l)^2} = b_{tk}. \tag{22}$$

It can be observed that this equation can be explicitly resolved with respect to x.

$$c_{rs} = \begin{cases} e_{tk} - \Im(x/v_k), & \text{if } r = t,\ s = k; \\ e_{tl} + \Im(x/v_l), & \text{if } r = t,\ s = l; \\ e_{rs}, & \text{otherwise.} \end{cases}$$

Then $\mathrm{Aux}(|C|) = B$ and $Cv = \lambda v$, so λ is an eigenvalue of C. The claim follows. $\square$

We call a class of complex matrices *regular* if all matrices in the class are nonsingular.

Corollary 4.5. *Let B be an irreducible row uniform nonnegative multicyclic matrix with all diagonal elements equal to* 0*. Let $\Gamma(B)$ consist of the set of all complex matrices $I - A$ with* $\operatorname{Aux}(|A|) = B$*.*

(i) *If $\mu(B) < 1$ then $\Gamma(B)$ contains only regular matrices.*
(ii) *If $\mu(B) = 1$ then $\Gamma(B)$ contains only regular matrices if and only if $\nu(B) < 1$.*
(iii) *If $\mu(B) > 1$ then $\Gamma(B)$ contains a singular matrix.*

Proof. $\Gamma(B)$ contains a singular matrix if and only if $1 \in \sigma(B)$. By Theorem 4.4 this happens if and only if either $\mu(B) > 1$ or $\mu(B) = 1 = \nu(B)$. This establishes all the claims. □

Remark 4.6. As noted in the abstract and introduction of [4], the theorems in that paper imply Brualdi's [5] conditions for the non-singularity of matrices and show that they are sharp. There is no essential difference or simplification in assuming that the main diagonal of the matrices considered there is the identity, and in that case the spectral content of [4, Theorems 1.1–1.4] is recaptured by Corollary 4.5 via standard Geršgorin theory, e.g. [21]. More precisely, Corollary 4.5(i) corresponds to Theorem 1.1 of [4], Corollary 4.5(ii) corresponds to Theorems 1.2 and 1.3, and Corollary 4.5(iii) corresponds to Theorem 1.4.

For $A \in \mathbb{R}_+^{n\times n}$, index set K and row uniform matrix B we write $\operatorname{Aux}(A) \lneq^K B$ when the following conditions hold.

(a) For $\operatorname{Aux}(A) = \tilde{B} = (\tilde{b}_{ij})$ we have $\tilde{b}_{ij} = 0 \Leftrightarrow b_{ij} = 0$ for all i, j.
(b) For all $i \in K$ we have $\tilde{b}_{ij} < b_{ij}$ for all j where $b_{ij} > 0$.
(c) For all $i \notin K$ and all j we have $\tilde{b}_{ij} = b_{ij}$.

We will also need the following variation of Definition 4.1.

Definition 4.7. For B a row uniform matrix, let $\tilde{\sigma}_K(B)$ denote the set $\tilde{\sigma}_K(B) = \{\lambda: \exists A \in \mathbb{C}^{n\times n}, \operatorname{Aux}(|A|) \lneq^K B, \det(A - \lambda I) = 0\}$.

The following corollary of Theorem 4.4 is immediate.

Corollary 4.8. *Let B be a row uniform nonnegative irreducible matrix. Then for any non-empty index set K, $\tilde{\sigma}_K(B)\backslash\{0\} = \{s: 0 < |s| < \mu(B)\}$.*

Proof. Let us analyze the following three cases.

Case 1: $\mu(B) > \nu(B)$. There exists an A such that $\mu(\operatorname{Aux}(|A|))$ is arbitrarily close to $\mu(B)$ and $\nu(\operatorname{Aux}(|A|)) < \mu(\operatorname{Aux}(|A|))$, and for each A with $\operatorname{Aux}(|A|) \lneq^K B$ we have $\nu(\operatorname{Aux}(|A|)) < \mu(B)$.

Case 2: $\mu(B) = \nu(B)$ and each cycle contains an index from K. In this case $\mu(\mathrm{Aux}(|A|))$, where $\mathrm{Aux}(|A|) \lneq^K B$, assumes all values in $(0, \mu(B))$.

Case 3: $\mu(B) = \nu(B)$ and there is a cycle avoiding the nodes with indices in K. In this case $\mu(\mathrm{Aux}(|A|)) = \mu(B)$ for all A with $\mathrm{Aux}(A) \lneq^K B$, but $\nu(\mathrm{Aux}(|A|)) < \mu(\mathrm{Aux}(|A|))$ for all such matrices.

In all three cases we obtain the claim by applying Theorem 4.4 to all $\mathrm{Aux}(|A|)$ satisfying $\mathrm{Aux}(|A|) \lneq^K B$. □

We are now ready to deal with the general reducible case.

Theorem 4.9. *Let B be a row uniform nonnegative matrix, and let*

$$\tilde{M}(B) := \max\bigl\{\mu(B_i) \text{ where } B_i \text{ is a transient class or a final multicyclic class of } B\bigr\}. \tag{23}$$

Then

(i) *If $\tilde{M}(B)$ is attained at some final multicyclic class B_s with $\nu(B_s) = \mu(B_s)$ then*

$$\begin{aligned}
&\sigma(B)\backslash\{0\} \\
&\quad = \bigl\{s\colon 0 < |s| \le \tilde{M}(B)\bigr\} \\
&\qquad \cup \bigcup_i \bigl\{s\colon |s| = \mu(B_i), B_i \text{ is a final unicyclic class and } \mu(B_i) > \tilde{M}(B)\bigr\}.
\end{aligned} \tag{24}$$

(ii) *Otherwise,*

$$\begin{aligned}
&\sigma(B)\backslash\{0\} \\
&\quad = \bigl\{s\colon 0 < |s| < \tilde{M}(B)\bigr\} \\
&\qquad \cup \bigcup_i \bigl\{s\colon |s| = \mu(B_i), B_i \text{ is a final unicyclic class and } \mu(B_i) \ge \tilde{M}(B)\bigr\}.
\end{aligned} \tag{25}$$

Proof. It is known that λ is an eigenvalue of a matrix $A \in \mathbb{C}^{n\times n}$ if and only if $\det(A - \lambda I) = 0$, which implies that the spectrum of $A \in \mathbb{C}^{n\times n}$ (i.e., the set of eigenvalues of A) is the union of spectra of its nontrivial classes in the Frobenius normal form. Furthermore, if a principal submatrix A_s corresponds to a transient class then it can be any matrix satisfying $\mathrm{Aux}(|A_s|) \lneq^{K_s} B_s$, where K_s is the (non-empty) set of indices of all nodes in this transient class that have a connection to another class. Observe that the entries in different rows of matrices with the same $\mathrm{Aux}(|A|)$ vary independently and hence the same is true about the sets of rows belonging to different classes. Therefore $\sigma(B)\backslash\{0\}$ can be found as union of $\sigma(B_i)\backslash\{0\}$ over all final classes B_i and $\tilde{\sigma}_{K_s}(B_s)\backslash\{0\}$ over all transient classes B_s, for some non-empty index sets K_s. Using Theorem 4.4 and Corollary 4.8 and taking the above mentioned union, it can be verified that $\sigma(B)\backslash\{0\}$ is as claimed. □

Example. To illustrate the last theorem, let us consider the following row uniform matrices:

$$B = \begin{pmatrix} 5 & 0 & 0 & 0 & 0 \\ 4 & 0 & 4 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 3 & 0 & 0 & 3 & 3 \\ 0 & 3 & 0 & 3 & 3 \end{pmatrix}, \qquad C = \begin{pmatrix} 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 3 & 3 \end{pmatrix}$$

That is, C is formed from B by cutting all connections between the classes.

The moduli of the eigenvalues in $\sigma(B)$ assume all the values in $(0, 4) \cup \{5\}$. Note that $\tilde{M}(B) = \max\{3, 4\}$, but $4 \notin \sigma(B)$ because the class extracted from rows and columns 2 and 3 is transient ($b_{12} > 0$). Therefore condition (ii) of Theorem 4.9 is used in computing $\sigma(B)$.

The moduli of eigenvalues in $\sigma(C)$ assume all values in $(0, 3] \cup \{4\} \cup \{5\}$. Here $\tilde{M}(C) = 3$, which is the maximum cycle mean of the only final class which is multicyclic. As the means of all cycles in that class are equal to each other, the value of $\tilde{M}(C)$ belongs to $\sigma(C)$.

4.2. Camion–Hoffman theorem

We now will apply Theorem 2.8 and Theorem 4.9 to provide a new proof for a theorem of Camion and Hoffman [9].

Let us first recall the following known facts and a definition:

Lemma 4.10. *(See [9].) Let $a_1, \ldots, a_n$ be nonnegative numbers such that each number does not exceed the sum of other numbers. Then there exist complex numbers $c_1, \ldots, c_n$ such that $|c_i| = a_i$ for $i = 1, \ldots, n$ and $c_1 + \ldots + c_n = 0$.*

Corollary 4.11. *(See [9].) Let the entries of $A = (a_{ij}) \in \mathbb{R}_+^{n\times n}$ satisfy $a_{ii} = 1$, $\sum_{j\neq i} a_{ij} \geq 1$ for all i and $a_{ij} \leq 1$ for all i, j. Then there exists a complex matrix C with $|C| = A$ and* $\det(C) = 0$.

Proof. Since the conditions of Lemma 4.10 are satisfied for $a_{i1}, \ldots, a_{in}$ for all i, there exists a complex matrix C with $|C| = A$ such that $\sum_j c_{ij} = 0$ for all i. This implies $\det(C) = 0$. □

Definition 4.12. A matrix $A = (a_{ij}) \in \mathbb{R}_+^{n\times n}$ is called strictly diagonally dominant if $a_{ii} > \sum_{j\neq i} a_{ij}$ for all i.

We will investigate the following matrix class:

Definition 4.13. For $A \in \mathbb{R}_+^{n\times n}$ define $\Omega(A) = \{E : |e_{ij}| = a_{ij}\ 1 \leq i, j \leq n\}$.

Theorem 4.14 *(Camion–Hoffman). For $A \in \mathbb{R}_+^{n\times n}$ the following are equivalent:*

(i) *$\Omega(A)$ does not contain a singular matrix;*
(ii) *There exists a permutation matrix P and a diagonal matrix D such that PAD is strictly diagonally dominant;*
(iii) *There exists a permutation matrix P and nonsingular diagonal matrices D_1, D_2 such that all diagonal entries of D_1PAD_2 are equal to 1 and $\mu(\mathrm{Aux}(D_1PAD_2 - I)) < 1$.*

Proof. (i) $\Rightarrow$ (ii): Assume that $\Omega(A)$ is regular. Let P be a permutation matrix such that the diagonal product of PA is greater then or equal to any generalized diagonal product of A. Since A is nonsingular the diagonal elements of $E = PA$ are nonzero. Let D be the diagonal matrix with entries equal to the inverse of the corresponding diagonal elements of PA. Since all diagonal entries of PAD are equal to 1, for any cycle α we can find a generalized diagonal product of PAD equal to the product of the entries of α. Since any generalized diagonal product of PAD is less than or equal to 1, it follows that $\mu(PAD) = 1$.

We will now establish that $\rho(PAD - I) < 1$. The proof is by contradiction. Assume that $\rho(PAD - I) \geq 1$. Then $PAD - I$ has a class B such that $\rho(B) \geq 1$ and for all i we have $b_{ii} = 0$. Since $\mu(PAD - I) \leq 1$ we also have $\mu(B) \leq 1$. Applying Theorem 2.8 to B, we obtain a diagonal nonnegative matrix Y such that matrix $E := Y^{-1}(B + I)Y$ has entries satisfying $0 \leq e_{ij} \leq 1$ and $e_{ii} = 1$ for all i, j, and $\sum_{k\neq i} e_{ik} \geq 1$ for all i. By Corollary 4.11 there is a matrix $H = (h_{ij})$ with complex entries satisfying $|H| = E$ and $\det(H) = 0$. Replacing the class $B + I$ in PAD by YHY^{-1} we obtain a matrix G with $\det(G) = 0$ and $|G| \in \Omega(PAD)$. As P is a permutation matrix and D diagonal, there is a bijective correspondence between $\Omega(PAD)$ and $\Omega(A)$ in which the singularity and nonsingularity are preserved. This contradicts that $\Omega(A)$ does not contain a singular matrix and hence $\rho(PAD - I) < 1$.

Since $\rho(PAD - I) < 1$, there exists a diagonal matrix Z such that $Z^{-1}(PAD - I)Z$ has all row sums strictly less than 1, see [3, Chapter 6] or [18] for a detailed argument. (Such a diagonal matrix Z can be constructed using Perron eigenvectors of nontrivial classes.) As all row sums in the matrix $Z^{-1}(PAD - I)Z = Z^{-1}PADZ - I$ are strictly less than 1, it follows that the matrix $PADZ$ is strictly diagonally dominant, with P a permutation matrix and DZ a diagonal matrix, as required.

(ii) $\Rightarrow$ (iii): If PAD is strictly diagonally dominant then there is a diagonal matrix D_1 such that the diagonal entries of D_1PAD are equal to 1 and the row sums of $D_1PAD - I$ are strictly less than 1. As each entry in $\mathrm{Aux}(D_1PAD - I)$ is strictly less than 1, we also have $\mu(\mathrm{Aux}(D_1PAD - I)) < 1$ as claimed.

(iii) $\Rightarrow$ (i): The proof is by contradiction. Assume that (iii) holds but (i) does not hold. That is, assume that there exists a permutation matrix P and nonsingular diagonal matrices D_1, D_2 such that $\mu(\mathrm{Aux}(D_1PAD_2 - I)) < 1$, and that (in contradiction with (i)) there exists $C \in \Omega(A)$ with $\det(C) = 0$. Then $\mu(\mathrm{Aux}(D_1P|C|D_2 - I)) = \mu(\mathrm{Aux}(D_1PAD_2 - I)) < 1$, and by Theorem 4.9 we have $1 \notin \sigma(\mathrm{Aux}(D_1PAD_2 - I))$.

However, we have $\det(D_1PCD_2) = 0$, and we can multiply the rows of D_1PCD_2 by some complex numbers with moduli 1 to obtain a matrix with zero determinant and with all diagonal entries equal to -1. Adding the identity matrix to this matrix we obtain a matrix in the class $\Omega(D_1PAD_2 - I)$, for which 1 is an eigenvalue. The set of eigenvalues of matrices in $\Omega(D_1PAD_2 - I)$ is a subset of $\sigma(\mathrm{Aux}(D_1PAD_2 - I))$, so $1 \in \sigma(\mathrm{Aux}(D_1PAD_2 - I))$, a contradiction. □

Let us also reformulate the Camion–Hoffman theorem in terms of M-matrices and comparison matrices. Recall that a real matrix B is a nonsingular M-matrix if $B = \rho I - C$ where C is a nonnegative matrix and the Perron root of C is strictly less than ρ (see [3] for many other equivalent definitions). For a nonnegative matrix $A \in \mathbb{R}_+^{n\times n}$, its comparison matrix $E = \mathrm{comp}(A)$ has entries $e_{ii} = a_{ii}$ for $i = 1, \ldots, n$ and $e_{ij} = -a_{ij}$ for $i \neq j$.

Theorem 4.15. *For a nonnegative matrix A, the following are equivalent:*

(i) *$\Omega(A)$ does not contain a singular matrix,*
(ii) *For P a permutation matrix corresponding to the greatest generalized diagonal product of A, the matrix* $\mathrm{comp}(PA)$ *is a nonsingular M-matrix.*

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